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A Krull-Schmdit type theorem for coherent sheaves

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Abstract

Let X be projective variety over an algebraically closed field k and G be a finite group with $g.c.d.(char(k), |G|) = 1$. We prove that any representations of G on a coherent sheaf, $\rho : G \longrightarrow End(\mathcal{E})$, has a natural decomposition $\mathcal{E} \simeq \bigoplus V \otimes_k \mathcal{F}_V$, where G acts trivially on $\mathcal{F}_V$ and the sum run over all irreducible representations of G over k.

1 Introduction

Let X be a complete variety over an algebraically closed field k and let G be a finite group with $char(k)$ and $|G|$ coprimes, thus the k-algebra $k[G]$ is semisimple (not necessary commutative) of finite dimension $|G|$ over k. Let $\mathcal{O}_X[G] := \mathcal{O}_X \otimes_k k[G]$. Thus, we can define the category $\mathbf{A}$ where the objects are $\mathcal{O}_X[G]$-modules, which are defined as pairs consisting of an $\mathcal{O}_X$-module $\mathcal{E}$ together with a k-morphism of rings $\rho : k[G] \to End(\mathcal{E})$ and the morphisms are defined in the natural way. Clearly this an Abelian category. We say that an $\mathcal{O}_X[G]$-module $\mathcal{E}$ is indecomposable if every direct decomposition of $\mathcal{E}$ into $\mathcal{O}_X[G]$-modules is trivial. From the Krull-Schmidt Theorem proved by Atiyah in [1], on a complete variety X every non-zero $\mathcal{O}_X[G]$-module $\mathcal{E}$ has a direct sum decomposition into indecomposable $\mathcal{O}_X[G]$-modules and this decomposition is unique up to permutations. The objective of this paper is to prove the next structure theorem for $\mathcal{O}_X[G]$-modules.

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Theorem 1. *Let X be a complete variety over an algebraically closed field k. Let $\mathcal{E}$ be a coherent $\mathcal{O}_X[G]$-module and suppose there is a surjective G-morphism*

$$\mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

with $\mathcal{F}$ torsion free. Let $V_0, \ldots V_r$ be the irreducible representations of G over k. Then, there is a natural $\mathcal{O}_X[G]$-isomorphism

$$\mathcal{E} \simeq (V_0 \otimes_k \mathcal{E}_{V_0}) \oplus ... \oplus (V_r \otimes_k \mathcal{E}_{V_r})$$

where the action of G on $\mathcal{E}_{V_i}$ is trivial for all i. In particular, if X is a projective variety, this is true for any coherent $\mathcal{O}_X[G]$-module.

2 Proof of Theorem 1

Lemma 1. *Let k be an algebraically closed field and G be a finite group. Then, for any field extension K of k, any K-representation of G is of the form $V \otimes_k K$ with V a k-representation of G, unique up to G-isomorphisms.*

Proof. It is sufficient to prove the lemma for irreducible representations. By the corollary 3.61 in [2] page 68, any representation of the form $V \otimes_k K$ is irreducible over K if V is irreducible over k, and by theorem 30.15 in [3] page 214 all those are different irreducible representations, then those are all the irreducible representations. □

Definition 1. Let X be an integral scheme over an algebraically closed field k, K be the function field of X and ϵ be the generic point. Let $\mathcal{E}$ be a torsion free coherent $\mathcal{O}_X[G]$-module, and $\mathcal{E}_\epsilon \simeq V \otimes_k K$ be the representation of G at the generic point, we define the type of the representation of G on $\mathcal{E}$ as the isomorphisms class of the representation V. Also, if $\mathcal{G}$ is any $\mathcal{O}_X[G]$-module , we define the isotypical decomposition of $\mathcal{G}$ by the decomposition

$$\mathcal{G} = e_0\mathcal{G} \oplus ... \oplus e_r\mathcal{G}$$

where $\{e_i\}$ are the respective idempotents of $k[G]$. Notice that e_V define an exact functor from $\mathcal{A}$ to $\mathcal{A}$.

Now, the type of the representation and the isotypical decomposition are related by

Lemma 2. *Let X be a variety over an algebraically closed field k, K be its function field and ϵ be the generic point. Let $\mathcal{E}$ be a torsion free coherent $\mathcal{O}_X[G]$-module of type V. Then*

$$e_i(\mathcal{E}_\epsilon) = e_i(V \otimes_k K) = (e_i\mathcal{E})_\epsilon$$

where e_i is the idempotent element on $k[G]$ corresponding to the representation V_i. Furthermore, the isotytpical decomposition of the $K[G]$-module $\mathcal{E}_\epsilon$ is given by

$$\mathcal{E}_\epsilon = V \otimes_k K = \bigoplus_{i=0}^{r} (e_i\mathcal{E})_\epsilon$$

Proof. The first part of the theorem is an immediately consequence of the properties of the stalk at a point. The second part follows from lemma 1. □

Lemma 3. ***(Characterization of free torsion $\mathcal{O}_X[G]$-modules)*** *Let X be a complete variety over an algebraically closed field k, let $\mathcal{E}$ be an indecomposable torsion free coherent $\mathcal{O}_X[G]$-module of W type. Then $\mathcal{E} \simeq V \otimes_k \mathcal{F}$ with $\mathcal{F}$ an indecomposable $\mathcal{O}_X$-module and $W \simeq V^{\oplus rank \mathcal{F}}$, with V an irreducible representation.*

In particular, if $\mathcal{G}$ is a torsion free coherent $\mathcal{O}_X[G]$-module of type $V_0^{n_0} \oplus ... \oplus V_r^{n_r}$. Then the isotypical decomposition of $\mathcal{G}$ is given by

$$\mathcal{G} \simeq (V_0 \otimes_k \mathcal{G}_{V_0}) \oplus ... \oplus (V_r \otimes_k \mathcal{G}_{V_r})$$

where $\mathcal{G}_{V_i}$ is an $\mathcal{O}_X$-module of rank n_i.

Proof. As X is a complete variety, we can apply the Krull-Schmidt theorem for coherent sheaves proved in [1]. Thus, let $\mathcal{E} = \mathcal{F}_1^{\oplus n_1} \oplus ... \oplus \mathcal{F}_r^{\oplus n_r}$ be the unique decomposition of $\mathcal{E}$ into indecomposable $\mathcal{O}_X$-modules with $\mathcal{F}_i \neq \mathcal{F}_j$, if $i \neq j$. Hence, if $g \in G$, $g\mathcal{E} = g\mathcal{F}_1^{\oplus n_1} \oplus ... \oplus g\mathcal{F}_r^{\oplus n_r}$ imply $g\mathcal{F}_i = \mathcal{F}_j$ for some j, then $\mathcal{F}_i \simeq \mathcal{F}_j$ and $i = j$. Now, any $\mathcal{F}_i^{\oplus n_i}$ must be G-invariant but $\mathcal{E}$ is an indecomposable $\mathcal{O}_X[G]$-module so $r = 1$ and $\mathcal{E} = \mathcal{F}^{\oplus n}$ with $\mathcal{F}$ an indecomposable $\mathcal{O}_X$-module.

Let W the type of the representation, therefore, $W \simeq V^{\oplus s}$ with V irreducible. Thus, the next step is to show that $s = rank \, \mathcal{F}$ and $dim \, V = n$. For this, we consider the part of type V_0 of $V^\vee \otimes_k \mathcal{E}$, where $V^\vee$ is the dual representation of V, this is a direct summand of $V^\vee \otimes_k \mathcal{E} \simeq \mathcal{F}^{\oplus n \cdot dim V}$, so this component must be $\mathcal{F}^{\oplus i}$ for some i. Now, taking the composition of the canonical inclusion of $\mathcal{O}_X[G]$-module

$$V \otimes_k \mathcal{F}^{\oplus i} \simeq (V \otimes_k \mathcal{F})^{\oplus i} \longrightarrow (V \otimes_k V^\vee) \otimes_k \mathcal{E}$$

with the canonical map

$$(V \otimes_k V^\vee) \otimes_k \mathcal{E} \longrightarrow \mathcal{E}$$

we have an $\mathcal{O}_X[G]$-morphism

$$\alpha : (V \otimes_k \mathcal{F})^{\oplus i} \longrightarrow \mathcal{E} \simeq \mathcal{F}^{\oplus n}$$

that is a $K[G]$-isomorphism at the generic point, so $n = i \cdot dim(V)$. Now, $\mathcal{F}$ is torsion free, so α must be injective and we can apply the last corollary in [1], this corollary claim that, over a complete variety, an injective endomorphism is an isomorphism, but we are supposing that $\mathcal{E}$ is an indecomposable $\mathcal{O}_X[G]$-module, then $i = 1$ and $\mathcal{E} \simeq V \otimes_k \mathcal{F}$ and the theorem follows.

□

Now we have

Proof of Theorem 1 From the hypotheses we have and exact sequence of $\mathcal{O}_X[G]$-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0,$$

with $\mathcal{F}$ a coherent torsion free sheaf, thus we have that $\mathcal{K}$ is torsion free.

On the other hand, for each $T \in \mathcal{I}$ we have an exact sequence

$$0 \longrightarrow e_T(\mathcal{K}) \longrightarrow e_T(\mathcal{F}) \longrightarrow e_T(\mathcal{E}) \longrightarrow 0$$

and from Lemma 2 above, there are unique sheaves $\mathcal{K}_T$ and $\mathcal{F}_T$ such that $e_T(\mathcal{K}) \simeq T \otimes \mathcal{K}_T$ and $e_T(\mathcal{F}) \simeq T \otimes \mathcal{F}_T$, thus we have the exact sequence

$$0 \longrightarrow T \otimes_k \mathcal{K}_T \longrightarrow T \otimes_k \mathcal{F}_T \longrightarrow e_T(\mathcal{E}) \longrightarrow 0 \quad (1)$$

applying the exact functor $e_0(T^\vee \otimes_k \_)$ we obtain

$$0 \longrightarrow \mathcal{K}_T \longrightarrow \mathcal{F}_T \longrightarrow e_0(T^\vee \otimes_k e_T(\mathcal{E})) \longrightarrow 0$$

and applying $T \otimes_k \_$

$$0 \longrightarrow T \otimes_k \mathcal{K}_T \longrightarrow T \otimes_k \mathcal{F}_T \longrightarrow T \otimes_k e_0(T^\vee \otimes_k e_T(\mathcal{E})) \longrightarrow 0 \quad (2)$$

but the first morphism in sequences 1 and 2 are the same, then both have the same cokernel, then $e_T(\mathcal{E}) \simeq T \otimes_k e_0(T^\vee \otimes_k e_T(\mathcal{E}))$ and the first part of the Theorem is proved.

On the other hand, if X is projective, for any coherent sheaf $\mathcal{E}$ there is an exact sequence

$$\widehat{\mathcal{F}} \xrightarrow{\widehat{\phi}} \mathcal{E} \longrightarrow 0$$

of $\mathcal{O}_X$-modules with $\widehat{\mathcal{F}}$ locally free (see [5] II 5.18). Therefore, when $\mathcal{E}$ is an $\mathcal{O}_X[G]$-module, we can construct an exact sequence of $\mathcal{O}_X[G]$-modules

$$\mathcal{F} \xrightarrow{\phi} \mathcal{E} \longrightarrow 0$$

with $\mathcal{F} = k[G] \otimes \widehat{\mathcal{F}}$, $\phi(\sum a \cdot g) \otimes f \mapsto \sum a \cdot g\widehat{\phi}(f)$. Then the Theorem follows. □

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