DOI: 10.2478/auom-2014-0026
An. Şt. Univ. Ovidius Constanţa
VERSITA
Vol. 22(2),2014, 5–19

On the Classification of Simple Singularities in Positive Characteristic

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Abstract

The aim of the article is to describe the classification of simple isolated hypersurface sin- gularities over a field of positive characteristic by certain invariants without computing the normal form. We also give a description of the algorithms to compute the classification which we have implemented in the Singular libraries classifyCeq.lib and classifyReq.lib. 1 Int

1 Introduction

Simple hypersurface singularities in charateristic $p > 0$ were classified by Greuel and Kröning [4] with respect to contact equivalence. Greuel and Nguyen Hong Duc [6] classified the simple hypersurface singularities in characteristic $p > 2$ with respect to right equivalence and the simple plane curve singularities in characteristic 2 with respect to right equivalence. We complete the classification of hypersurface singularities in characteristic 2.
Also we describe our implementation of a classifyer for simple singularities with respect to contact equivalence respectively right equivalence in SINGULAR [5],[2]. We use for distinguishing the different cases the blowing up as a new tool.

Key Words: Simple Singularity, Normal form, Contact equivalent, Right equivalent.
2010 Mathematics Subject Classification: Primary 14B05; Secondary 14H20, 14J17.
Received: February 2013
Revised: May 2013
Accepted: June 2013

2 Basic Definitions

Let $K[[x]] = K[[x_1, \dots, x_n]]$ is the local ring of formal power series and m is its maximal ideal, K an algebraically closed field of characteristic $p > 0$.
The following definitions can be found in [3].

Definition 2.1. Let $f \in m - \{0\}$ be a formal power series, then the ideal $j(f) =< f_{x_1}, ..., f_{x_n} >$ is called Jacobian ideal and the $K-$algebra $M_f = K[[x]]/j(f)$ is called Milnor algebra, and $\mu(f) = dim(M_f)$ is called the Milnor number.

Definition 2.2. Let $f \in m - \{0\}$ is a formal power series, then the ideal $tj(f) =< f, f_{x_1}, ..., f_{x_n} >$ is called Tjurina ideal and the $K-$algebra $N_f = K[[x]]/tj(f)$ is called Tjurina algebra, and $\tau(f) = dim(N_f)$ is called the Tjurina number.

Definition 2.3. Let f and $g \in m \subset K[[x]]$
f is called to be right equivalent to g, $f \overset{R}{\sim} g$ if there exists an automorphism ϕ of $K[[x]]$ such that $\phi(f) = g$.

f is called to be contact equivalent to g, $f \overset{c}{\sim} g$ if there exists an automorphism ϕ of $K[[x]]$ such that $< \phi(f) >=< g >$, that is, there exists a unit u such that $\phi(f) = ug$.

Definition 2.4. Let $f \in m \subset K[[x]]$. Then f is called $k - determined$ if all $g \in K[[x]]$ with $f - g \in m^{k+1}$ are right equivalent to f.

Definition 2.5. Let $f \in K[[x]]$ then $k - jet$ of f is the Taylor expansion of f up to degree k terms.

Definition 2.6. Let $f \in m^2 \subset K[[x]]$. Then Hesse matrix of f is defined by

$$H(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right)_{1 \le i,j \le n}$$

The corank of f is defined as $corank(f) = n - rank(H(f))$.

3 Arnold's Classification Of Simple Singularities

Arnold gave a classification of simple singularities with respect to right equivalence over the field of complex numbers [1].These are also the simple singularities with respect to contact equivalence.

Normal forms of simple singularities with respect to the Arnold's classification are given as follows

Name	Normal form
A_k	$x_1^{k+1} + x_2^2 + \cdots + x_n^2,\ k \geq 1$
D_k	$x_1 x_2^2 + x_1^{k-1} + x_3^2 + \cdots + x_n^2,\ k \geq 4$
E_6	$x_1^3 + x_2^4 + x_3^2 + \cdots + x_n^2$
E_7	$x_1^3 + x_1 x_2^3 + x_3^2 + \cdots + x_n^2$
E_8	$x_1^3 + x_2^5 + x_3^2 + \cdots + x_n^2$

Note that for this singularaties the Milnor number is equal to the Tjurina number and given by the index in the notation.

4 Greuel and Kröning's Classification of Simple Singularities

Greuel and Kröning gave the classification of simple singularities with respect to contact equivalence in characteristic $p > 0$ [4].

Proposition 4.1. Let $p = char(K)$. A plane curve singularity is contact simple if and only if it is contact equivalent to one of the following forms:
(i) $p \neq 2$

Name	Normal form for $f \in K[[x, y]]$	τ	$\tau,\ \ p = 5$	$\tau,\ \ p = 3$
A_k	$x^{k+1} + y^2\ \ k \geq 1$	k if $p \nmid k+1$ $k+1$ if $p \mid k+1$	k if $5 \nmid k+1$ $k+1$ if $5 \mid k+1$	k if $3 \nmid k+1$ $k+1$ if $3 \mid k+1$
D_k	$xy^2 + x^{k-1},\ k \geq 4$	k	k	k
E_6	E_6^0 $x^3 + y^4$	6	6	9
	E_6^1 $x^3 + y^4 + x^2y^2$, in $char = 3$			7
E_7	E_7^0 $x^3 + xy^3$	7	7	9
	E_7^1 $x^3 + xy^3 + x^2y^2$, in $char = 3$			7
E_8	E_8^0 $x^3 + y^5$	8	10	12
	E_8^1 $x^3 + y^5 + x^2y^3$, in $char = 3$			10
	E_8^2 $x^3 + y^5 + x^2y^2$, in $char = 3$			8
	E_8^3 $x^3 + y^5 + xy^4$, in $char = 5$		8	

(ii) $p = 2$

Name	Normal form for $f \in K[[x, y]]$			τ
A_{2m-1}		$x^2 + xy^m$	$m \geq 1$	$2m$ m even $2m-1$ m odd
A_{2m}	A_{2m}^0	$x^2 + y^{2m+1}$	$m \geq 1$	$4m$
	A_{2m}^r	$x^2 + y^{2m+1} + xy^{2m-r}$	$m \geq 1, 1 \leq r \leq m-1$	$4m-2r$ r even $4m-2r-1$ r odd
D_{2m}		$x^2y + xy^m$	$m \geq 2$	$2m$
D_{2m+1}	D_{2m+1}^0	$x^2y + y^{2m}$	$m \geq 2$	$4m$
	D_{2m+1}^r	$x^2y + y^{2m} + xy^{2m-r}$	$m \geq 2, 1 \leq r \leq m-1$	$4m-2r$
E_6	E_6^0	$x^3 + y^4$		8
	E_6^1	$x^3 + y^4 + xy^3$		6
E_7		$x^3 + xy^3$		7
E_8		$x^3 + y^5$		8

Proposition 4.2. Let $p = char(K) > 2$. A hypersurface singularity $f \in m^2 \subset K[[x]]$ is contact simple if and only if it is contact equivalent to one of the following form:

$$f(x_1, \ldots, x_n) = g(x_1, x_2) + x_3^2 + \cdots + x_n^2,$$

where $g \in K[[x_1, x_2]]$ is one of the list in proposition 4.1 part (i).

Proposition 4.3. Let $p = char(K) = 2$. A surface singularity is contact simple if and only if it is contact equivalent to one of the following forms:

Name	Normal form for $f \in K[[x, y, z]]$			τ
A_k		$xy + z^{k+1}$		$k+1$ k *odd* k k *even*
D_{2m}	D_{2m}^0	$z^2 + x^2y + xy^m$	$m \geq 2$	$4m$
	D_{2m}^r	$z^2 + x^2y + xy^m + xy^{m-r}z$	$m \geq 2, 1 \leq r \leq m-1$	$4m - 2r$
D_{2m+1}	D_{2m+1}^0	$z^2 + x^2y + y^mz$	$m \geq 2$	$4m$
	D_{2m+1}^r	$z^2 + x^2y + y^mz + xy^{m-r}z$	$m \geq 2, 1 \leq r \leq m-1$	$4m - 2r$
E_6	E_6^0	$z^2 + x^3 + y^2z$		8
	E_6^1	$z^2 + x^3 + y^2z + xyz$		6
E_7	E_7^0	$z^2 + x^3 + xy^3$		14
	E_7^1	$z^2 + x^3 + xy^3 + x^2yz$		12
	E_7^2	$z^2 + x^3 + xy^3 + y^3z$		10
	E_7^3	$z^2 + x^3 + xy^3 + xyz$		8
E_8	E_8^0	$z^2 + x^3 + y^5$		16
	E_8^1	$z^2 + x^3 + y^5 + xy^3z$		14
	E_8^2	$z^2 + x^3 + y^5 + xy^2z$		12
	E_8^3	$z^2 + x^3 + y^5 + y^3z$		10
	E_8^4	$z^2 + x^3 + y^5 + xyz$		8

Proposition 4.4. Let $p = char(K) = 2$. A hypersurface singularity $f \in m^2 \subset K[[x]]$ is contact simple if and only if it is contact equivalent to the following form:

$$f(x_1, \ldots, x_n) = g(x_1, x_2, x_3) + x_4x_5 + \cdots + x_{2k}x_{2k+1}, n = 2k + 1,$$

where $g \in K[[x_1, x_2, x_3]]$ is one of the list in proposition 4.3., or

$$f(x_1, \ldots, x_n) = g(x_1, x_2) + x_3x_4 + \cdots + x_{2k+1}x_{2k}, n = 2k,$$

where $g \in K[[x_1, x_2]]$ is one of the list in proposition 4.1 part (ii).

To distinguish the different singularities in the classificatopn we need detailed information on the resolution of the singularaties.In the next proposition we give the blowing up sequences and the corresponding sequences of the Tjurina numbers. Here the notion $A_3 \leftarrow A_1$ means that the blowing up of A_3 gives A_1.

Proposition 4.5. Assume $f \in K[[x, y]]$ then

(1) $A_{2m-1} \leftarrow A_{2(m-1)-1} \leftarrow \dots A_3 \leftarrow A_1$
is the sequence of blowing ups in the resolution of the A_{2m-1}-singularities. The corresponding sequence of Tjurina numbers is
$(\dots, 2k+5, 2k+4, 2k+1, 2k, 2k-3, \dots, 8, 5, 4, 1)$

(2) $A^0_{2m} \leftarrow A^0_{2(m-1)} \leftarrow \dots A^0_2$
$(4m, 4m-4, \dots, 12, 8, 4)$

(3) $A^r_{2m} \leftarrow A^{r-1}_{2(m-1)} \leftarrow \dots \leftarrow A^1_{2(m-(r-1))} \leftarrow A^0_{2(m-r)} \leftarrow \dots \leftarrow A^0_2$
$(4m-2r, \dots, 4(m-r)+5, 4(m-r)+4, 4(m-r)+1, 4(m-r), \dots, 8, 4)$
if r is *even*.
$(4m-2r-1, \dots, 4(m-r)+5, 4(m-r)+4, 4(m-r)+1, 4(m-r), \dots, 8, 4)$
if r is *odd*.

(4) $D_{2m} \leftarrow A_{2(m-2)-1} \leftarrow \dots \leftarrow A_1$
$(2m, 2m-4, 2m-7, 2m-8, 2m-11, \dots, 1)$ if m is *even*.
$(2m, 2m-5, 2m-6, 2m-9, 2m-10, \dots, 1)$ if m is *odd*

(5) $D^0_{2m+1} \leftarrow A^0_{2(m-2)} \leftarrow \dots \leftarrow A^0_2$
$(4m, 4m-8, 4m-12, \dots, 4)$

(6) $D^r_{2m+1} \leftarrow A^{r-2}_{2(m-2)} \leftarrow A^{r-3}_{2(m-3)} \leftarrow \dots \leftarrow A^1_{2(m-(r-1))} \leftarrow A^0_{2(m-r)} \leftarrow \dots \leftarrow A^0_2$
$(4m-2r, 4m-2r-4, 4m-2r-7, \dots, 4(m-r)+1, 4(m-r), \dots, 8, 4)$
if r is *even*.
$(4m-2r, 4m-2r-5, 4m-2r-6, \dots, 4(m-r)+1, 4(m-r), \dots, 8, 4)$
if r is *odd*.

(7) E^0_6 and E^1_6, the seqence of Tjurina numbers is (8) respectively (6).

(8) $E_7 \leftarrow A_1$, the sequence of Tjurina numbers is $(7, 1)$.

(9) $E_8 \leftarrow A^0_2$, the sequence of Tjurina numbers is $(8, 4)$.

Proposition 4.6. Assume $f \in K[[x, y, z]]$ then

(1) $A_k \leftarrow A_{k-2} \leftarrow \ldots \leftarrow A_1$ k is odd
$(k+1, k-1, \ldots, 2)$
$A_k \leftarrow A_{k-2} \leftarrow \ldots \leftarrow A_2$ k is even
$(k, k-2, \ldots, 2)$

(2) $D^0_{2m} \leftarrow D^0_{2(m-1)} \leftarrow \ldots \leftarrow D^0_4 \leftarrow A_1$
$(4m, 4m-4, \ldots, 8, 2)$

(3) $D^r_{2m} \leftarrow D^{r-1}_{2(m-1)} \leftarrow \ldots \leftarrow, D^1_{2(m-(r-1))} \leftarrow D^0_{2(m-r)} \leftarrow \ldots \leftarrow D^0_4 \leftarrow A_1$
$(4m-2r, 4m-2r-2, \ldots, 4m-4r+2, 4(m-r), \ldots, 8, 2)$

(4) $D^0_{2m+1} \leftarrow D^0_{2m-1} \leftarrow \ldots \leftarrow D^0_5 \leftarrow A_3 \leftarrow A_1$
$(4m, 4(m-1), \ldots, 8, 4, 2)$

(5) $D^r_{2m+1} \leftarrow D^{r-1}_{2(m-1)+1} \leftarrow \ldots \leftarrow D^1_{2(m-(r-1))+1} \leftarrow D^0_{2(m-r)+1} \leftarrow \ldots \leftarrow D^0_5 \leftarrow A_3 \leftarrow A_1$
$(4m-2r, 4m-2r-2, \ldots, 4m-4r+2, 4(m-r), \ldots, 8, 4, 2)$

(6) $E^0_6 \leftarrow A_5 \leftarrow A_3 \leftarrow A_1$ $(8, 6, 4, 2)$
$E^1_6 \leftarrow A_5 \leftarrow A_3 \leftarrow A_1$ $(6, 6, 4, 2)$

(7) $E^0_7 \leftarrow D^0_6 \leftarrow D^0_4 \leftarrow A_1$ $(14, 12, 8, 2)$
$E^1_7 \leftarrow D^0_6 \leftarrow D^0_4 \leftarrow A_1$ $(12, 12, 8, 2)$

(8) $E^2_7 \leftarrow D^1_6 \leftarrow D^0_4 \leftarrow A_1$ $(10, 10, 8, 2)$
$E^3_7 \leftarrow D^2_6 \leftarrow D^1_4 \leftarrow A_1$ $(8, 8, 6, 2)$

(9) $E^0_8 \leftarrow E^0_7 \leftarrow D^0_6 \leftarrow D^0_4 \leftarrow A_1$ $(16, 14, 12, 8, 2)$

(10) $E^1_8 \leftarrow E^0_7 \leftarrow D^0_6 \leftarrow D^0_4 \leftarrow A_1$ $(14, 14, 12, 8, 2)$

(11) $E^2_8 \leftarrow E^1_7 \leftarrow D^0_6 \leftarrow D^0_4 \leftarrow A_1$ $(12, 12, 12, 8, 2)$

(12) $E_8^3 \leftarrow E_7^2 \leftarrow D_6^1 \leftarrow D_4^0 \leftarrow A_1$ $(10, 10, 10, 8, 2)$

(13) $E_8^4 \leftarrow E_7^3 \leftarrow D_6^2 \leftarrow D_4^1 \leftarrow A_1$ $(8, 8, 8, 6, 2)$

Proposition 4.7. Let $f \in (x, y^2)^3 \subseteq K[[x, y]]$, $char(K) = 2$ then $\tau(f) \geq 10$.

Proof. Let $f = x^3 + ax^2y^2 + bxy^4 + cy^6 + h$, h of weighted order at least 7 with respect to the weights 2,1 for x,y. Then $\frac{\partial f}{\partial x} = x^2 + by^4 + \frac{\partial h}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{\partial h}{\partial y}$. Therefore f can be reduced using $\frac{\partial f}{\partial x}$ to $(c - ab)y^6 + k$, k of weighted degree at least 7. This implies that $1, y, \ldots, y^5, x, xy, xy^2, xy^3$ are linearly independent modulo $< f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$. □

Proposition 4.8. Let $f \in (x, y^2)^3 \subseteq K[[x, y]]$, $char(K) > 2$ then $\tau(f) \geq 9$.

Proof. Let $f = g + h$, g weighted homogeneous of degree 6 with respect to the weights 2 resp. 1 for x resp.y, or zero and h of weighted order at least 7. Then $6g = 2x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y}$. This implies if $char(K) \neq 3$ then $< f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > = < h, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$.
But the contribution of h are of weighted degree at least 7. Analyzing the proof of the proposition 5.4 we obtain $\tau(f) \geq 9$. If $char(K) = 3$ then let $g = x^3 + ax^2y^2 + bxy^4 + cy^6$. We have $\frac{\partial f}{\partial x} = 2axy^2 + by^4 + \frac{\partial h}{\partial x}$, and $\frac{\partial f}{\partial y} = 2ax^2y + bxy^3 + \frac{\partial h}{\partial y}$.
In case $a \neq 0$ all s-polys of $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ have weighted order at least 7 (since $2x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 0$). This implies that in case $a \neq 0$, $1, y, \ldots, y^6, x, xy, x^2$ are linearly independent modulo $< f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$.
In case $a = 0$ we obtain that $1, y, y^2, y^3, x, xy, xy^2, x^2, x^2y, x^2y^2$ are linearly independent modulo $< f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$. This implies that $\tau(f) \geq 10$ in this case. □

5 Greuel and Nguyen Hong Duc's Classification of Simple Singularities in positive Characteristic w.r.t. Right Equivalence

The normal forms of simple singularities with respect to right equivalence [6] are given as follows

Proposition 5.1. Let $p = char(K)$. A plane curve singularity is right simple if and only if it is right equivalent to one of the following forms:

(i) $p \neq 2$

Name	Normal form for $f \in K[[x,y]]$		μ
A_k	$x^2 + y^{k+1}$	$1 \leq k \leq p-2$	k
D_k	$x^2 y + y^{k-1}$	$4 \leq k \leq p$	k
E_6	$x^3 + y^4$	$p \neq 3$	6
E_7	$x^3 + xy^3$	$p \neq 3$	7
E_8	$x^3 + y^5$	$p \neq 3,5$	8

(ii) $p = 2$

Name	Normal form for $f \in K[[x,y]]$	μ
A_1	xy	1

Proposition 5.2. Let $p = char(K)$. A hypersurface singularity $f \in m^2 \subset K[[x]]$ is right simple if and only if it is right equivalent to one of the following forms:
(i) $p \neq 2$

$$f(x_1, \dots, x_n) = g(x_1, x_2) + x_3^2 + \dots + x_n^2,$$

where $g \in K[[x_1, x_2]]$ is one of the list in proposition 5.1 part (i).
(ii) $p = 2$

$$f(x_1, \dots, x_n) = x_1 x_2 + x_3 x_4 + \dots + x_{2k+1} x_{2k}, n = 2k.$$

Proof. (i) is proved in [6].
To prove (ii) we use the splitting lemma (proposition 3 of [4]).
Let $f \in m \subset K[[x]]$ and $mult(f) = 2$ then either

$$f \sim x_1^2 + x_2 x_3 +, \dots, + x_{2l} x_{2l+1} + g(x_{2l+2}, \dots, x_n) x_1 + h(x_{2l+2}, \dots, x_n)$$

with $g \in m^2$ and $h \in m^3$ or

$$f \sim x_1 x_2 +, \dots, + x_{2l-1} x_{2l} + h(x_{2l+1}, \dots, x_n)$$

with $h \in m^3$.
In the first case it is easy to see that the Milnor number is not finite i.e. f is not simple.
In the second case we use induction to prove that f is simple if and only if $n = 2l$. This is clear since f is simple if and only if h is simple. □

Proposition 5.3. Let $f \in K[[x]]$ and $\mu(f) = 1$ then f defines A_1- singularity.

Proof. If $char(K) > 2$ then the splitting lemma implies that $f \sim x_1^2 + \cdots + x_l^2 + h(x_{l+1}, \ldots, x_n)$ with $h \in < x_{l+1}, \ldots, x_n >^3$ then $\mu(f) = \mu(h)$. But $h \in < x_{l+1}, \ldots, x_n >^3$ implies $< f_{x_l}, ..., f_{x_n} > \subseteq < x_{l+1}, \ldots, x_n >^2$. This implies $\mu(h) > 1$. We obtain $l = 1$ and f defines A_1- singularity.
If $char(K) = 2$ then either

$$f \sim x_1^2 + x_2x_3 +, \ldots, + x_{2l}x_{2l+1} + g(x_{2l+2}, \ldots, x_n)x_1 + h(x_{2l+2}, \ldots, x_n)$$

with $g \in m^2$ and $h \in m^3$ or

$$f \sim x_1x_2 +, \ldots, + x_{2l-1}x_{2l} + h(x_{2l+1}, \ldots, x_n)$$

with $h \in m^3$.
In the first case it is easy to see that the Milnor number is not finite. The second case can be settled as above. □

Proposition 5.4. Let $f \in (x, y^2)^3 \subseteq K[[x, y]]$, $char(K) > 2$ then $\mu(f) > 8$.

Proof. Let $f = g + h$, g weighted homogeneous of degree 6 with respect to the weights 2 respectively 1 for x respectively y or zero and h of weighted order at least 7. If $g = 0$ then the leading monomials of a standard basis of $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ with respect to the local weighted ordering have degree at least 5. This implies that $1, y, y^2, y^3, y^4, x, xy, xy^2, x^2$ are linearly independent modulo $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$. This implies that $\mu(f) \geq 9$ in this case.
If $g \neq 0$ then 3 cases are possible. We may assume that $g = x^3 + ax^2y^2 + bxy^4 + cy^6$. Over the algebraic closure of K g splits, $g = (x + \alpha_1 y^2)(x + \alpha_2 y^2)(x + \alpha_3 y^2)$. If $\alpha_1 = \alpha_2 = \alpha_3$, we may assume (after the transformation $x \to x - \alpha_1 y^2$) that $g = x^3$. Then $\frac{\partial f}{\partial x} = 3x^2 + \frac{\partial h}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{\partial h}{\partial y}$. If $char(K) = 3$ then we are in the same situation as the previous case and obtain $\mu(f) \geq 9$. If $char(K) \neq 3$ then a minimal standard basis of $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ contains $\frac{\partial f}{\partial x}$ with leading monomial x^2. $\frac{\partial f}{\partial y}$ is of the weighted order at least 6. This implies that $1, y, \ldots, y^5, x, xy, xy^2, xy^3$ are linearly independent modulo $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ and therefore $\mu(f) \geq 10$ in this case.
If two of the α_i are equal, we may assume $f = x^3 + x^2 y^2$. Then $\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 + \frac{\partial h}{\partial x}$ and $\frac{\partial f}{\partial y} = 2x^2y + \frac{\partial h}{\partial y}$. If $char(K) = 3$ then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are in the standard basis and all the other elements have weighted order at least 6. This implies that $1, y, \ldots, y^5, x, xy, x^2$ are linearly independent modulo $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ and $\mu(f) \geq 9$.
If $char(K) \neq 3$ then we can reduce $\frac{\partial f}{\partial y}$ using $\frac{\partial f}{\partial x}$ to $a\,x\,y^3 + \varphi$, where $deg(\varphi) \geq 6$. If $a = 0$ then with the same argument as in the case $g = x^3$ we obtain

$\mu(f) \geq 10$. If $a \neq 0$ then we have x^2 and $x y^3$ in the leading ideal of $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$. The next element in the standard basis has weighted order at least 6 i.e $1, y, \ldots, y^5, x, xy, xy^2$ are linearly independent, i.e. $\mu(f) \geq 9$.
Now assume that all the roots are different. Then (after some transformation) we may assume $g = x(x^2+axy^2+y^4)$ and $a \neq \pm 2$. Then $\frac{\partial f}{\partial x} = 3x^2+2axy^2+\frac{\partial h}{\partial x}$, and $\frac{\partial f}{\partial y} = 2ax^2y + 4xy^3 + \frac{\partial h}{\partial y}$. If $char(K) \neq 3$ we use $f_1 := \frac{\partial f}{\partial x}$ to reduce $\frac{\partial f}{\partial y}$ to $f_2 := (a^2-3)xy^3 + \frac{a}{2}y^5 + h_1$, h_1 of weighted order at least 6. If $a^2 = 3$ then $\frac{\partial f}{\partial x}$, $\frac{a}{2}y^5 + h_1$ is a standard basis of $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ and $\mu(f) = 10$. If $a^2 \neq 3$ then we obtain as standard basis $\{\frac{\partial f}{\partial x}, f_2, y^7\}$ and also $\mu(f) = 10$. If $char(K) = 3$ we conclude similar to the case $g = x^3 + x^2y^2$ that $\mu(f) \geq 9$.

□

6 Description of the Classifier

In this section we give the description of our classifier and its implementation in SINGULAR. This classifier is used for computing the type and normal forms of the simple singularities in characteristic $p > 0$ with respect to contact and right equivalence.
Our classifier is based on two main algorithms
Algorithm : **1** ; $classifyCeq$
Algorithm : **2** ; $classifyReq$
First algorithm of our classifier is $classifyCeq$, which classifies the simple singularities in characteristic $p > 0$ with respect to the contact equivalence. This algorithm further consist on two algorithms $classifyC_1eq$ and $classifyC_2eq$. $classifyC_1eq$, classifies the simple singularities with respect to contact equivalence when characteristic $p \neq 2$ and $classifyC_2eq$, classifies the simple singularities with respect to the contact equivalence when characteristic $p = 2$.
The second algorithm of our classifier $classifyReq$, classifies the simple singularities in characteristic $p > 0$ with respect to the right equivalence.

Now we explain our classifier in detail.

6.1 The Algorithm(classifyCeq)

In the first case we consider $char(K) \neq 2$, n=number of variables= 2. If Tjurina number of f is not finite then f is not an isolated singularity and therefore not simple.
For this case our classifier computes the $corank(f)$. If $corank(f) = 0$ then it gives A_1- singularity and if $corank(f) = 1$ then it gives A_k-singularities, $k > 1$. It makes a single blow-up of f at the origin and computes the difference of the

tjurina number of f before and after the blowing-up. If the difference is less or equal to two then it gives A_k-singularities such that $\tau(f) = k$, otherwise it gives A_k- singularities such that $\tau(f) = k+1$.
Now if $corank(f) = 2$ then it computes $j^3(f)$. If $j^3(f) = 0$ then it gives f is not simple. If $j^3(f)$ has only one factor then it is transformed to x , it gives either f is not simple or E-singularities. If $f \in (x, y^2)^3$, then f is not simple. This can be detected by $\tau(f) \geq 9$, cf. Proposition.4.8. When $char(K) \neq 3, 5$ and $\tau(f) \leq 8$ then for $\tau(f) = 6, 7, 8$ it gives E_6^0, E_7^0 and E_8^0 singularities respectively. When $char(K) = 5$ then for $\tau(f) = 6, 7, 10, 8$ it gives E_6^0, E_7^0 , E_8^0 and E_8^1 singularities respectively. When $char(K) = 3$ then for $\tau(f) = 12, 10, 8$ it gives E_8^0, E_8^1 and E_8^2 singularities respectively. If $char(K) = 3$ and $\tau(f) = 7$ then it makes a single blow-up of f at the origin. If f is smooth after this blow up then it gives E_6^1, otherwise it gives E_7^1. If $char(K) = 3$ and $\tau(f) = 9$ then again it makes a single blow-up of f at the origin. If f is smooth after this blow up then it gives E_6^0, otherwise it gives E_7^0. And if $j^3(f)$ has two or three factors then for $\tau(f) = k$ it gives D_k-singularities.
If n=number of variables > 2 then our classifier splits f as explained is section-4 and reduce the case into two variables.

Now we consider $char(K) = 2$, n=number of variables= 2. If Tjurina number of f is not finite then f is not an isolated singularity and therefore not simple.
In this case our classifier checks the order of f, if $ord(f) = 2$ then it gives A_k singularities. To determine k, it performs a blow-up of f at the origin and computes the difference of the tjurina number of f before and after the blowing-up. If the difference is equal to 4, then it gives the A_{2k}^0 singularities, where $k = \tau(f)/4$ if $\tau(f)$. Otherwise it makes successive blow-ups and computes the Tjurina number after each blow-up untill it gets the Tjurina number 0. If the Tjurina number of the last singularity in the resolution is 4 then it gives A_{2k}^r singularities, where $k = (\tau(f) + 2r)/4$ if $\tau(f)$ is even and $k = (\tau(f) + 1 + 2r)/4$ if $\tau(f)$ is odd. And for the other case it give A_{2k-1} singularities, where $k = \tau(f)/2$ if $\tau(f)$ is even and $k = (\tau(f) + 1)/2$ if $\tau(f)$ is odd. To find value of r it makes successive blow ups and find the difference of tjurina numbers for every two consecutive blow ups. Let $t - 1$ and t are consecutive blow ups such that difference of tjurina numbers of these blow ups becomes 4 then it gives $r = t - 1$.

Now if $ord(f) = 3$ then it computes $j^3(f)$. If $j^3(f)$ has only one factor then it is transformed to x. If $f \in (x, y^2)^3$ then f is not simple. This can be detected by $\tau(f) \geq 10$, cf. Proposition.4.7.If $\tau(f) \leq 9$ then it gives E-singularities. For $\tau(f) = 6, 7$ it gives E_6^1 and E_7 respectively. If $\tau(f) = 8$ then it makes a single blow-up of f at the origin. If f is smooth after this blow up then it gives E_6^0, otherwise it gives E_8. If $j^3(f)$ has 3 factors then it gives the D_4-singularity, and if $j^3(f)$ has two factors then it makes a blow-up and computes the difference of tjurina numbers before and after blow-up, if the difference is 8 then it gives D_{2k+1}^0. Otherwise it makes successive blow-ups and computes the tjurina number after each blow-up untill it gets Tjurina number 0. If Tjurina number of the last singularity in the resolution is 4 then it gives D_{2k+1}^r-singularities with $k = (\tau(f) + 2r)/4$ and for the other case it gives D_{2k}-singularities with $k = \tau(f)/2$. To find value of r it makes successive blow ups and find the difference of Tjurina numbers for every two consecutive blow ups. Let $t-1$ and t are consecutive blow ups such that difference of tjurina numbers of these blow ups becomes 4 then it gives $r = t$. If $ord(f) \geq 4$ then the singularity is not simple.

In case of $char(K) = 2$ and n=number of variables= 3. Our classifier checks the multiplicity of f. If the multiplicity of f is 2 then it factorizes $j^2(f)$. If $j^2(f)$ has only one factor of multiplicity 1 then it gives the A_1-singularity. If $j^2(f)$ has two different factors then we have A_k-singularities. In order to obtain the k we resolve the singularity. If the last singularity before becoming smooth is A_2 than we have $k = \tau(f)$ otherwise $k = \tau(f) - 1$.
At this step our classifier makes some coordinate change and transform $j^2(f)$ to z^2. Then we have $f = z^2 + h(x,y) + h_1(x,y)z + z^2\phi + \psi$, where $\phi \in m, \psi \in m^4$ and h, h_1 are homogeneous polynomials of degree 3 and 2. Now it factorizes $g = h(x,y) + h_1(x,y)z$. If it is irreducible then it gives E_6^0 or E_6^1 corresponding to the $\tau(f) = 8$ or 6 respectively. If g has one linear and one quadratic factor of multiplicity 1 then it gives D_5^1 corresponding to $\tau(f) = 6$ and D_5^0, E_7^3 or E_8^4 corresponding to $\tau(f) = 8$. To distinguish the D_5^0, E_7^3 and E_8^4 singularities, it performs blow ups of f at origin to compute the sequence of the tjurina number: $(8,4,2)$ for D_5^0, $(8,8,6,2)$ for E_7^3 and $(8,8,8,6,2)$ for E_8^4. Now if g has three different factors then it gives D_4^0 and D_4^1 corresponding to the $\tau(f) = 8$, 6 respectively. If g has one factor of multiplicity 1 and one factor of multiplicity 2 then it gives D_k singularities. To determine k, it performs a blow-up and computes the difference of Tjurina numbers before an after blow-up. If the difference is 4 then it gives D_{2k+1}^0 or D_{2k}^0 singularities with $k = \tau(f)/4$ and if the difference is 2 then it gives D_{2k+1}^r or D_{2k}^r singularities with $k = (\tau(f) + 2r)/4$, where r is the number of that blow up after which either the difference of the Tjurina numbers of blow ups becomes 4 (i.e $r \neq k-2$) or if the difference is not 4 then $r = k-2$. The resolution sequence separates the cases: D_{2m+1}^r , D_{2m+1}^0 resolution have $\ldots \leftarrow A_3 \leftarrow A_1$ at the end. And D_{2m}^r, D_{2m}^0 resolution have $\ldots \leftarrow D_4^1 \leftarrow A_1$ respectively. $\ldots D_4^0 \leftarrow A_1$ at the end cf. Proposition 4.6.

Now if g has only one factor of multiplicity 3 then it gives E_k singularities. If $\tau(f) = 10, 12, 14$ then it gives E_7^2 or E_8^3, E_7^1 or E_8^2 and E_7^0 or E_8^1 respectively. These cases can be separated by the length of the Tjurina sequence i.e 5 for E_8 and 4 for E_7 cf.proposition 4.6. And if $\tau(f) = 16$ then it gives E_8^0.
If n=number of variables > 3 then our classifier splits f [7] and reduces the case into two or three variables.

6.2 The Algorithm(classifyReq)

We consider $char(K) \neq 2$, n=number of variables= 2. If the Milnor number of f is not finite then f is not simple. And if the Milnor number of f is finite then our classifier computes $corank(f)$. If $corank(f) \leq 1$ then it gives A_k-singularities, where $1 \leq k = \mu(f) \leq char(K) - 2$. And if $corank(f) = 2$ then it computes $j^3(f)$. If $j^3(f)$ has only one factor and $\mu(f) \leq 8$ then it gives E-singularities. If the factor is transformed to x and $f \in (x, y^2)^3$ then f is not simple. This can be detected by $\mu(f) \geq 9$, cf. proposition 5.4. If $char(K) \neq 3$ and $\mu(f) = 6, 7$ then it gives E_6, E_7 respectively. If $char(K) \neq 3, 5$ and $\mu(f) = 8$ then it gives E_8. And if $j^3(f)$ has two or three factors then it gives D_k-singularities, where $4 \leq k = \mu(f) \leq char(K)$.
If $char(K) \neq 2$ and n=number of variables > 2 then our classifier splits f as described in proposition-5.2 part (i) and reduces the case into two variables. For the case when $char(K) = 2$, our classifier gives A_1-singularity for $\mu(f) = 1$. This is the only normal form in this case.

7 Singular Examples

```
LIB"classifyCeq.lib";

ring R=2,(x,y,z),ds;
poly f1 = x2+y2+z2+x3+x2y+xy2+y3+y5;
classifyCeq(f1);
> E_8^0 = z2+x3+y5


ring R=2,(x,y),ds;
poly f2 = x2+y2+x12+x11y+x21;
classifyCeq(f2);
> A_2k^r
> A_18^9 = x2+y21+xy11
```

```
ring R=5,(x,y),ds;
poly f3 = x3-2x2y-2xy2+y3+x4+x3y;
classifyCeq(f3);
> E_7^0 = x3+xy3


LIB"classifyReq.lib";

ring R = 7,(x,y),ds;
poly f4 = x3+2x2y-xy2-y3+xy3+3y4+3x3y2+2x2y3+3x5y+2x4y2+x7+3x6y
         +3x2y5+2xy6-x4y4-3x3y5+3x6y3+2x5y4+3x3y7+2x2y8+3x5y6
         +2x4y7+x4y9+3x3y10;
classifyReq(f4);
> E7 = x3+xy3


ring R=2,(x,y),ds;
poly f5 = x2+xy;
classifyReq(f5);
> A1 = xy
```

Acknowledgements We would like to thank Gerhard Pfister for insightful conversations.

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