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On an Arithmetic Inequality

József Sándor

Abstract

We obtain an arithmetic proof and a refinement of the inequality $\varphi(n^k) + \sigma_k(n) < 2n^k$, where $n \geq 2$ and $k \geq 2$. An application to another inequality is also provided.

1 Introduction

If $n \geq 1$ is an integer, then let $\varphi(n)$ denote the classical Euler totient function, and $\sigma_a(n)$ be the sum of ath powers of divisors of n (with a a real number).

Recently [2] H. Alzer and the author have shown that the divisibility

$$n^k | (\varphi(n^k) + \sigma_k(n)) \tag{1}$$

is not solvable for any integers $n \geq 2$ and $k \geq 2$. For $k = 2$ this settled a conjecture of Adiga and Ramaswamy [1].

The proof of our result is based, besides arithmetical properties of φ and σ_k, also on a Weierstrass product-type inequality, whose proof used methods of Mathematical analysis (as differentiability, and convex functions). In fact, the impossibility of (1) for $n \geq 2$ and $k \geq 2$, follows from the inequality

$$\varphi(n^k) + \sigma_k(n) < 2n^k,\ n \geq 2,\ k \geq 2. \tag{2}$$

The aim of this note is to provide a completely arithmetic proof of inequality (2), and in fact to offer an improvement of this inequality.

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We shall use also Dedekind's arithmetical function, defined by

$$\psi(n) = n\prod_{p|n}(1+1/p) \text{ for } n \geq 2,\ \psi(1) = 1.$$

It is clear that ψ, like φ and σ_k, is a multiplicative function, i.e. satisfies $\psi(ab) = \psi(a)\psi(b)$ for $(a,b) = 1$.

2 Lemmas and Main Result

In order to prove inequality (2) we need two auxiliary results.

The first lemma is stated in another form also in [2]; we present here its short proof for the sake of completeness.

Lemma 2.1. *For all integers $n \geq 2$ and $k \geq 2$ we have*

$$\frac{\sigma_k(n)}{n^k} \leq \frac{\sigma_2(n)}{n^2} < \frac{n^2}{\varphi(n)\psi(n)}. \tag{3}$$

Proof. One has

$$\sigma_k(n) = \sum_{d|n} d^k = \sum_{d|n}\left(\frac{n}{d}\right)^k = n^k \sum_{d|n}\frac{1}{d^k},$$

which shows that $\frac{\sigma_k(n)}{n^k}$ is decreasing with respect to k. This leads to the first inequality of (3). Let now

$$n = \prod_{j=1}^{r} p_j^{a_j} \geq 2$$

be the prime factorization of n. Then

$$\frac{\sigma_2(n)}{n^2} = \prod_{j=1}^{r}\frac{p_j^{2a_j+2}-1}{p_j^{2a_j}(p_j^2-1)} = \prod_{j=1}^{r}\left(p_j^2\cdot\frac{1-1/p_j^{2a_j+2}}{p_j^2-1}\right)$$

$$< \prod_{p|n}\frac{p^2}{p^2-1} = \prod_{p|n}\frac{1}{\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right)} = \frac{n^2}{\varphi(n)\psi(n)}.$$

This settles the second inequality in (3). □

Lemma 2.2. *For all $n \geq 1$ one has the inequality*

$$2\frac{\psi(n)}{n} \geq 1 + \frac{n}{\varphi(n)}. \tag{4}$$

Proof. Inequality (4) is stated without proof in [5]. Here we shall provide a complete proof.

It is easy to see that for $n = 1$ and $n = p$ - prime, inequality (4) holds true; i.e. $2\frac{p+1}{p} \geq 1 + \frac{p}{p+1}$ is valid, with equality only for $p = 2$. Since

$$\frac{\psi(p^a)}{p^a} = \frac{\psi(p)}{p} \text{ and } \frac{\varphi(p^a)}{p^a} = \frac{\varphi(p)}{p}$$

for any primes p and integers $a \geq 1$, clearly it is sufficient to prove (4) when n is squarefree number, i.e. a product of distinct primes. Let us assume that n is the least squarefree integer, for which (4) is false, and let p be the greatest prime factor of n. Then n can be written as $n = p \cdot m$, where $(p, m) = 1$. Let q denote the greatest prime factor of m. Then $q < p$. On the other hand, remark that

$$\frac{m}{\varphi(m)} = \prod_{s|m,\ s \text{ prime}} \frac{s}{s-1} \leq \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{s}{s-1} \cdots \frac{q}{q-1} = q,$$

so

$$\frac{m}{\varphi(m)} \leq q. \tag{5}$$

Now, by the definition of n one has

$$2\frac{\psi(n)}{n} < 1 + \frac{n}{\varphi(n)},$$

i.e.

$$2\frac{p+1}{p} \cdot \frac{\psi(m)}{m} < 1 + \frac{p}{p-1} \cdot \frac{m}{\varphi(m)}. \tag{6}$$

Since $m < n$ and m squarefree, by definition of n one has

$$2\frac{\psi(m)}{m} \geq 1 + \frac{m}{\varphi(m)}. \tag{7}$$

Now multiplying both sides of (7) with $\frac{p+1}{p}$, and by taking into account of (6) we can write

$$1 + \frac{p}{p-1} \cdot \frac{m}{\varphi(m)} > \frac{p+1}{p} + \frac{p+1}{p} \cdot \frac{m}{\varphi(m)},$$

i.e.

$$\frac{1}{p(p-1)} \cdot \frac{m}{\varphi(m)} > \frac{1}{p}. \tag{8}$$

From (8) we get

$$p - 1 < \frac{m}{\varphi(m)} \leq q$$

by relation (5). Since $q < p$, we get the contradiction $p - 1 < q < p$. This proves Lemma 2.2. □

Theorem 2.1. *For all $n \geq 2$ and $k \geq 2$ one has the inequality*

$$\frac{\varphi(n^k)}{n^k} + \frac{\sigma_k(n)}{n^k} < \frac{\varphi(n)}{n} + \frac{n^2}{\varphi(n)\psi(n)} \leq \phi + \frac{2}{1+\phi} < 2, \tag{9}$$

where $\phi = \frac{\varphi(n)}{n} < 1$.

Proof. The first inequality of (8) follows by the remark that

$$\varphi(n^k)/n^k = \varphi(n)/n,$$

and by Lemma 2.1. For the second inequality use Lemma 2.2 in the form

$$\frac{n}{\psi(n)} \leq \frac{2}{1 + n/\varphi(n)}. \tag{10}$$

Finally, the last inequality is equivalent to

$$\left(\frac{\varphi(n)}{n}\right)^2 < \frac{\varphi(n)}{n},$$

i.e. $\varphi(n) < n$, which is well-known. This concludes the proof of the theorem. □

Remark 1. By the methods applied here, we have obtained a completely arithmetic study of problem (1) (see [2]).

An application. Let $d(n)$ denote the number of distinct divisors of n. The following theorem gives an improvement of a result from [3]:

Theorem 2.2. *For all $n \geq 2$ not a prime number and $k \geq 2$ one has the inequalities*

$$\frac{\sigma_k(n)}{n^k} < \frac{2n}{n + \varphi(n)} < \frac{d(n)}{2}. \tag{11}$$

Proof. The first inequality follows by a combination of relations (3) and (10). As the second inequality may be written as $nd(n) + d(n)\varphi(n) > 4n$, remark that this is true for $d(n) \geq 3$, since by a well known inequality of R. Sivaramakrishnan [4] one has $d(n)\varphi(n) > n$ for all $n > 1$. Clearly, $d(n) = 2$ only if n is a prime, so the result follows. □

Remark 2. The weaker inequality of (11) , in case when n has at least two distinct prime factors, appears in paper [3], as a corollary to more general results.

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József Sándor
Department of Mathematics
Babeş-Bolyai University
400084 Cluj-Napoca, Romania
email: jsandor@math.ubbcluj.ro