Irreducibility criteria for compositions and multiplicative convolutions of polynomials with integer coefficients

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Dedicated to Professor Mirela Ştefănescu

Abstract

We provide irreducibility criteria for multiplicative convolutions of polynomials with integer coefficients, that is, for polynomials of the form $h^{\deg f} \cdot f(g/h)$, where f, g, h are polynomials with integer coefficients, and g and h are relatively prime. The irreducibility conditions are expressed in terms of the prime factorization of the leading coefficient of the polynomial $h^{\deg f} \cdot f(g/h)$, the degrees of f, g, h, and the absolute values of their coefficients. In particular, by letting $h = 1$ we obtain irreducibility conditions for compositions of polynomials with integer coefficients.

1 Introduction

To decide whether the sum of two relatively prime polynomials is an irreducible polynomial, or in general, if a linear combination of two relatively prime polynomials is irreducible, is by no means an easy problem, and no general answer in this respect seems to be available. This problem may be raised for various classes of polynomials, like for instance polynomials with coefficients in an arbitrary unique factorization domain, or for multivariate

polynomials over an arbitrary field. However, for some particular classes of such linear combinations of polynomials of the form $n_1 f + n_2 g$ some progress was recorded by using some information on the canonical decomposition of n_1 and n_2, or on the canonical decomposition of the coefficients of f and g, or on their absolute values. Inspired by the work of Fried [8] and Langmann [9] in connection with Hilbert's irreducibility theorem, Cavachi [6] investigated the problem of the irreducibility of polynomials of the form $f(X) + pg(X)$, with p prime, $f(X), g(X)$ relatively prime polynomials with rational coefficients, and $\deg f < \deg g$. Given two relatively prime polynomials $f(X), g(X) \in \mathbb{Q}[X]$ with $\deg f < \deg g$, an explicit bound p_0 was provided in [7] such that for all prime numbers $p > p_0$, the polynomial $f(X) + pg(X)$ is irreducible over $\mathbb{Q}$. In [4], explicit upper bounds have been derived for the number of factors over $\mathbb{Q}$ of a linear combination $n_1 f(X) + n_2 g(X)$, covering also the case $\deg f = \deg g$. In [5] the same methods along with a Newton polygon argument have been used to find irreducibility conditions for linear combinations of the form $f(X) + p^k g(X)$. In [2] similar methods have been employed to study the irreducibility of some classes of compositions of polynomials, while in [3] the study focused on the irreducibility of some classes of multiplicative convolutions of polynomials, which offer considerably more flexibility to such irreducibility results, as they include linear combinations and compositions of polynomials as well.

Given two polynomials $g(X) = b_0 + b_1 X + \cdots + b_n X^n$, $h(X) = c_0 + c_1 X + \cdots + c_l X^l \in \mathbb{Z}[X]$, $b_n c_l \neq 0$, by *a multiplicative convolution of g and h* we understand any polynomial of the form

$$\sum_{i=0}^{m} a_i g(X)^i h(X)^{m-i},$$

with $a_0, a_1, \ldots, a_m \in \mathbb{Z}$, $m \geq 1$, $a_0 a_m \neq 0$. If we associate to $a_0, a_1, \ldots, a_m$ the polynomial $f(X) = a_0 + a_1 X + \cdots + a_m X^m$, and assume that $h \neq 0$, then

$$\sum_{i=0}^{m} a_i g(X)^i h(X)^{m-i} = h(X)^m f\left(\frac{g(X)}{h(X)}\right).$$

The aim of this paper is to complement some of the results in [3] and [2], by providing similar irreducibility conditions for the case when a_m is divisible by a sufficiently large prime power. The irreducibility results that we will obtain for this kind of convolutions will be expressed in terms of the prime factorization of the leading coefficient of the polynomial $h^{\deg f} f(g/h)$, the degrees of f, g, h, and the absolute value of their coefficients. We use the same notation that was used in [3]. Given a polynomial $f(X) = a_0 + \cdots + a_m X^m \in \mathbb{Z}[X]$ of degree $m \geq 0$, we let

$$H(f) = \max\{|a_0|, \ldots, |a_m|\} \quad \text{and} \quad L(f) = \sum_{i=0}^{m} |a_i|,$$

and if $m > 0$ we let

$$H_1(f) = \max\{|a_0|, \ldots, |a_{m-1}|\} \quad \text{and} \quad L_1(f) = \sum_{i=0}^{m-1} |a_i|.$$

Our main results refer to the case $\deg h < \deg g$:

Theorem 1.1. *Let* $f(X) = a_0 + a_1 X + \cdots + a_m X^m$, $g(X) = b_0 + b_1 X + \cdots + b_n X^n$ *and* $h(X) = c_0 + c_1 X + \cdots + c_l X^l \in \mathbb{Z}[X]$ *be polynomials of degree* m, n *and* l *respectively, with* $m \geq 1$, $n > l$, $a_0 \neq 0$, *and* g, h *relatively prime. Put*

$$d = \max\{i \,:\, i < m \text{ and } a_i \neq 0\} \text{ and } \beta = 1 + [H_1(g) + H(h)]/|b_n|.$$

Assume that $a_m = p^k q$ *with* p *a prime number,* q *a non-zero integer,* $p \nmid qa_d b_n c_l$ *and* k *is a positive integer prime to* $(m - d)(n - l)$. *If*

$$|a_m| > \sum_{i=0}^{d} |a_i| \cdot [|q|^n |b_n|^{mn} L(h(\beta X))]^{m-i},$$

then the polynomial $h^m \cdot f(g/h)$ *is irreducible over* $\mathbb{Q}$. *The same conclusion holds in the wider range*

$$|a_m| > \sum_{i=0}^{d} |a_i| \cdot [|q|^{\frac{n}{m}} |b_n|^n L(h(\beta X))]^{m-i},$$

provided that f *is irreducible over* $\mathbb{Q}$.

We also obtain the following irreducibility criterion, by replacing the hypothesis "f irreducible" by a simple numerical condition.

Corollary 1.2. *Let* $f(X) = a_0 + a_1 X + \cdots + a_m X^m$, $g(X) = b_0 + b_1 X + \cdots + b_n X^n$ *and* $h(X) = c_0 + c_1 X + \cdots + c_l X^l \in \mathbb{Z}[X]$ *be polynomials of degree* m, n *and* l *respectively, with* $m \geq 1$, $n > l$, $a_0 \neq 0$, *and* g, h *relatively prime. Put*

$$d = \max\{i \,:\, i < m \text{ and } a_i \neq 0\} \text{ and } \beta = 1 + [H_1(g) + H(h)]/|b_n|.$$

Assume that $a_m = p^k q$ with p a prime number, q a non-zero integer, $p \nmid q a_d b_n c_l$ and k is a positive integer prime to $(m-d)(n-l)$. If

$$|a_m| > \max\left\{\sum_{i=0}^{d} |a_i| \cdot |q|^{m-i}, \ \sum_{i=0}^{d} |a_i| \cdot [|q|^{\frac{n}{m}} |b_n|^n L(h(\beta X))]^{m-i}\right\},$$

then the polynomial $h^m \cdot f(g/h)$ is irreducible over $\mathbb{Q}$.

One of the corollaries of the main result in [2] is the following irreducibility criterion for compositions of polynomials with integer coefficients.

Corollary 1.3. ([2, Corollary 4]) *Let $f(X) = \sum_{i=0}^{m} a_i X^i$ and $g(X) = \sum_{i=0}^{n} b_i X^i$ in $\mathbb{Z}[X]$ be non-constant polynomials of degree m and n respectively, with $a_0 \neq 0$. If $a_m = p \cdot q$ with p a prime satisfying*

$$p > \max\left\{|q|^{m-1} L_1\left(f\left(\frac{X}{|q|}\right)\right), \ |q|^{n-1}|b_n|^{mn} L_1\left(f\left(\frac{X}{|q|^{n/m}|b_n|^n}\right)\right)\right\},$$

then the polynomial $f \circ g$ is irreducible over $\mathbb{Q}$.

By taking $h(X) = 1$ in Corollary 1.2 we obtain the following irreducibility criterion for compositions of polynomials, that complements Corollary 4 in [2].

Corollary 1.4. *Let $f(X) = a_0 + a_1 X + \cdots + a_m X^m$ and $g(X) = b_0 + b_1 X + \cdots + b_n X^n \in \mathbb{Z}[X]$ be polynomials of degree $m \geq 1$ and $n \geq 1$ respectively, $a_0 \neq 0$. Put*

$$d = \max\{\, i \,:\, i < m \text{ and } a_i \neq 0\}$$

and assume that $a_m = p^k q$ with p a prime number, q a non-zero integer, $p \nmid q a_d b_n$ and k is a positive integer prime to $(m-d)n$. If

$$|a_m| > \max\left\{\sum_{i=0}^{d} |a_i| \cdot |q|^{m-i}, \ \sum_{i=0}^{d} |a_i| \cdot [|q|^{\frac{n}{m}} |b_n|^n]^{m-i}\right\},$$

then the polynomial $f \circ g$ is irreducible over $\mathbb{Q}$.

Perhaps a comparison with [3] is in place. For the sake of convenience, we quote a similar irreducibility criterion derived from the main result of [3].

Corollary 1.5. ([3, Corollary 1]) *Let $f(X) = a_0 + a_1 X + \cdots + a_m X^m$, $g(X) = b_0 + b_1 X + \cdots + b_n X^n$ and $h(X) = c_0 + c_1 X + \cdots + c_l X^l \in \mathbb{Z}[X]$ be polynomials of degree m, n and l respectively, with $m \geq 1$, $n > l$, $a_0 \neq 0$, f*

irreducible over $\mathbb{Q}$, and g, h relatively prime. Let $\beta = 1 + [H_1(g) + H(h)]/|b_n|$.
If $a_m = pq$ with p a prime satisfying

$$|a_m| > \sum_{i=0}^{m-1} |a_i| \cdot [|q|^n |b_n|^{mn} L(h(\beta X))]^{m-i},$$

then the polynomial $h^m \cdot f(g/h)$ is irreducible over $\mathbb{Q}$.

It is apparent that in Corollary 1.5 the hypothesis is less restrictive with regard to p, the requirement $p \nmid q a_d b_n c_l$ being added in Theorem 1.1. The condition that k is coprime to $(m - d)(n - l)$ is automatically satisfied when $k = 1$, which is the case dealt with in Corollary 1 from [3]. However, as the cofactor q can be much bigger when applying Corollary 1.5 than when using Theorem 1.1, the latter result is applicable to polynomials f having the leading coefficient a_m somewhat smaller.

2   Proof of the main results

For the proof of Theorem 1.1 we need the following lemma from [1], which extends Capelli's Theorem to multiplicative convolutions of polynomials.

Lemma 2.1. *Let K be a field, $f, g, h \in K[X]$, f irreducible over K, g and h relatively prime, and $f(\alpha) = 0$. If*

$$g - \alpha h \overset{can}{\underset{K(\alpha)}{=}} const \cdot \prod_{i=1}^{r} \phi_i(X)^{e_i},$$

then

$$h^{\deg f} \cdot f(g/h) \overset{can}{\underset{K}{=}} const \cdot \prod_{i=1}^{r} N_{K(\alpha)/K} \phi_i(X)^{e_i}.$$

In particular, the degree of every irreducible factor of $h^{\deg f} \cdot f(g/h)$ must be a multiple of $\deg f$.

Here $F \overset{can}{\underset{K}{=}} const \cdot \prod_{i=1}^{r} \phi_i(X)^{e_i}$ means that the ϕ_i's are irreducible over K and prime to each other. For a proof of this result in the case when $char(K) = 0$, which is relevant here, we refer the reader to [3].

Another result that will be needed in the proof of Theorem 1.1 is the following lemma in [5], whose proof relies on Newton's polygon method.

Lemma 2.2. *Let u, $v \in \mathbb{Z}[X]$ be two polynomials with $\deg v = n$ and $\deg u = n - d$, $d \geq 1$. Let also p be a prime number that divides none of the leading coefficients of u and v, and let k be any positive integer prime to d. If $u(X) + p^k v(X)$ may be written as a product of two non-constant polynomials with integer coefficients, say f_1 and f_2, then one of the leading coefficients of f_1 and f_2 must be divisible by p^k.*

We will also use the following basic lemma.

Lemma 2.3. *Let $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{C}[X]$ be a polynomial of degree n. If for a positive real δ we have*

$$\frac{|a_n|}{\delta^n} > L_1\left(f\left(\frac{X}{\delta}\right)\right),$$

then all the roots of f lie in the disk $|z| < \frac{1}{\delta}$.

Proof: Assume that f has a root $\theta \in \mathbb{C}$ with $|\theta| \geq \frac{1}{\delta}$. Then we have

$$0 = \frac{|f(\theta)|}{|\theta^n|} \geq |a_n| - \frac{|a_{n-1}|}{|\theta|} - \cdots - \frac{|a_0|}{|\theta|^n} \geq |a_n| - |a_{n-1}|\delta - \cdots - |a_0|\delta^n. \quad (1)$$

On the other hand, according to our hypothesis, we have $\frac{|a_n|}{\delta^n} > L_1\left(f\left(\frac{X}{\delta}\right)\right)$, so in fact we have $|a_n| > |a_{n-1}|\delta + \cdots + |a_0|\delta^n$, which contradicts (1). $\square$

We will now proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $f(X)$, $g(X)$ and $h(X)$ be as in the statement of the theorem. Let us assume to the contrary that $h^m \cdot f(g/h)$ factors as $h^m \cdot f(g/h) = F_1 \cdot F_2$, with $F_1, F_2 \in \mathbb{Z}[X]$, and $\deg F_1 \geq 1, \deg F_2 \geq 1$. Let $t_1, t_2 \in \mathbb{Z}$ be the leading coefficients of F_1 and F_2, respectively. By comparing the leading coefficients in the equality

$$h^m \cdot f(g/h) = a_0 h^m + \cdots + a_d g^d h^{m-d} + a_m g^m = F_1 \cdot F_2$$

one finds that

$$p^k q b_n^m = t_1 \cdot t_2. \quad (2)$$

By Lemma 2.2 with $a_0 h(X)^m + \cdots + a_d g(X)^d h(X)^{m-d}$ instead of $u(X)$, and $qg(X)^m$ instead of $v(X)$, since $p \nmid q a_d b_n c_l$ and k is prime to $(m-d)(n-l)$, we deduce that one of the leading coefficients of F_1 and F_2, say that of F_2, will be divisible by p^k. Then, in view of (2), it follows that t_1 will divide qb_n^m. In particular we have

$$|t_1| \leq |q| \cdot |b_n|^m. \quad (3)$$

Let now $\overline{f} = h^m \cdot f(g/h) - a_m g^m$. Then $h^m \cdot f(g/h) = \overline{f} + a_m g^m$, and since $a_0 \neq 0$, the polynomials $\overline{f}$ and g^m are algebraically relatively prime.

Next, we will estimate the resultant $R(g^m, F_1)$. Since g^m and F_1 are also algebraically relatively prime, $R(g^m, F_1)$ must be a non-zero integer number, so in particular we must have

$$|R(g^m, F_1)| \geq 1. \tag{4}$$

Let $r = \deg F_1 \geq 1$, and consider the decomposition of F_1, say

$$F_1(X) = t_1(X - \theta_1) \cdots (X - \theta_r),$$

with $\theta_1, \ldots, \theta_r \in \mathbb{C}$. Then

$$|R(g^m, F_1)| = |t_1|^{mn} \prod_{1 \leq j \leq r} |g^m(\theta_j)|. \tag{5}$$

Since each root θ_j of F_1 is also a root of $h^m \cdot f(g/h)$, we have

$$g^m(\theta_j) = -\frac{\overline{f}(\theta_j)}{a_m} \tag{6}$$

and moreover, since $\overline{f}$ and g^m are relatively prime, $\overline{f}(\theta_j) \neq 0$ and $g^m(\theta_j) \neq 0$ for any $j \in \{1, \ldots, r\}$. Combining (5) and (6) we obtain

$$|R(g^m, F_1)| = \frac{|t_1|^{mn}}{|a_m|^r} \cdot \prod_{1 \leq j \leq r} |\overline{f}(\theta_j)|. \tag{7}$$

We now proceed to find an upper bound for $|\overline{f}(\theta_j)|$. In order to do this, we have to find an upper bound for the moduli of the roots of f. To this end, we first fix a positive real number δ such that $|a_m|/\delta^m > L_1(f(X/\delta))$. Later on we shall specify how to choose a convenient value of δ. By Lemma 2.3 we see that all the roots λ_i of f will verify

$$|\lambda_i| < \frac{1}{\delta}. \tag{8}$$

Let now $\theta_1, \ldots, \theta_{mn}$ be the roots of $h^m \cdot f(g/h)$. Since g and h are relatively prime, one has $h(\theta_j) \neq 0$ and $f(g(\theta_j)/h(\theta_j)) = 0$ for $j = 1, \ldots, mn$. Thus, for a given θ_j, there exists $i_j \in \{1, \ldots, m\}$ such that $g(\theta_j)/h(\theta_j) = \lambda_{i_j}$, a root of f. By (8) we then have

$$\left| \frac{g(\theta_j)}{h(\theta_j)} \right| < \frac{1}{\delta}. \tag{9}$$

Recall that $\overline{f} = a_0 h^m + a_1 g h^{m-1} + \cdots + a_d g^d h^{m-d}$. Using (9), we derive that

$$\left| \overline{f}(\theta_j) \right| = \left| \sum_{i=0}^{d} a_i g(\theta_j)^i h(\theta_j)^{m-i} \right| \leq |h(\theta_j)|^m \cdot L_1(f(X/\delta)). \qquad (10)$$

Combining now (7), (10) and (3) we deduce the following upper bound for $|R(g^m, F_1)|$:

$$|R(g^m, F_1)| \leq |q|^{mn} |b_n|^{m^2 n} \cdot \left[\frac{|h(\theta_j)|^m L_1(f(X/\delta))}{|a_m|} \right]^r. \qquad (11)$$

The inequality (9) allows us to find also an upper bound for $|h(\theta_j)|$, as follows. We rewrite (9) as

$$\delta |b_0 + b_1 \theta_j + \cdots + b_n \theta_j^n| < |c_0 + c_1 \theta_j + \cdots + c_l \theta_j^l|,$$

which further gives

$$\begin{aligned} \delta |b_n| \cdot |\theta_j|^n \quad &< \quad (|c_0| + \delta|b_0|) + (|c_1| + \delta|b_1|)|\theta_j| + \cdots + (|c_l| + \delta|b_l|)|\theta_j|^l \\ &\qquad + \delta|b_{l+1}| \cdot |\theta_j|^{l+1} + \cdots + \delta|b_{n-1}| \cdot |\theta_j|^{n-1} \\ &< \quad (H(h) + \delta H_1(g)) \cdot (1 + |\theta_j| + \cdots + |\theta_j|^{n-1}). \end{aligned}$$

Hence, for $|\theta_j| > 1$ we obtain

$$\delta |b_n| \cdot |\theta_j|^n < (H(h) + \delta H_1(g)) \cdot \frac{|\theta_j|^n - 1}{|\theta_j| - 1} < (H(h) + \delta H_1(g)) \cdot \frac{|\theta_j|^n}{|\theta_j| - 1},$$

that is

$$|\theta_j| < 1 + \frac{H(h) + \delta H_1(g)}{\delta |b_n|}. \qquad (12)$$

Note that this inequality is trivially satisfied when $|\theta_j| \leq 1$. Denoting by γ the right-hand side of (12), we find that for any root θ_j of $h^m \cdot f(g/h)$ it holds

$$|h(\theta_j)| < L(h(\gamma X)),$$

which combined with (11) yields

$$|R(g^m, F_1)| \leq |q|^{mn} |b_n|^{m^2 n} \cdot \left[\frac{L(h(\gamma X))^m L_1(f(X/\delta))}{|a_m|} \right]^r. \qquad (13)$$

The next step is to choose δ so that inequalities (4) and (13) contradict each other. Since $\deg F_1 = r \geq 1$, all it remains to prove is that our assumption

on the size of $|a_m|$ will imply on the one hand $|a_m| > \delta^m L_1(f(X/\delta))$ for a suitable $\delta > 0$, and on the other hand will force

$$|q|^{mn}|b_n|^{m^2n} \cdot \frac{L(h(\gamma X))^m L_1(f(X/\delta))}{|a_m|} < 1,$$

or equivalently

$$|a_m| > |q|^{mn}|b_n|^{m^2n} L(h(\gamma X))^m L_1(f(X/\delta)).$$

It will be therefore sufficient to have $|a_m| > \delta^m L_1(f(X/\delta))$ for a $\delta > 0$ as small as possible satisfying $\delta^m \geq |q|^{mn}|b_n|^{m^2n} L(h(\gamma X))^m$, that is $\delta \geq |q|^n |b_n|^{mn} L(h(\gamma X))$. Recalling the definition of γ, the last inequality reads

$$\delta \geq |q|^n |b_n|^{mn} \sum_{i=0}^{l} |c_i| \left(1 + \frac{H_1(g)}{|b_n|} + \frac{H(h)}{\delta|b_n|}\right)^i.$$

A suitable candidate for δ is

$$\delta_0 = |q|^n |b_n|^{mn} \sum_{i=0}^{l} |c_i| \left(1 + \frac{H_1(g) + H(h)}{|b_n|}\right)^i \geq 1,$$

because

$$1 + \frac{H_1(g) + H(h)}{|b_n|} \geq 1 + \frac{H_1(g)}{|b_n|} + \frac{H(h)}{|b_n|\delta_0}.$$

By the definition of β, the positive real number δ_0 just found coincides with $|q|^n |b_n|^{mn} L(h(\beta X))$, which proves that for

$$
\begin{aligned}
|a_m| \quad &> \quad [|q|^n |b_n|^{mn} L(h(\beta X))]^m \cdot L_1\left(f\left(\frac{X}{|q|^n |b_n|^{mn} L(h(\beta X))}\right)\right) \\
&= \quad \sum_{i=0}^{d} |a_i| \cdot [|q|^n |b_n|^{mn} L(h(\beta X))]^{m-i}
\end{aligned}
$$

we actually have $|R(g^m, F_1)| < 1$, a contradiction. Therefore $h^m \cdot f(g/h)$ must be irreducible over $\mathbb{Q}$, and this proves the first part of the theorem.

Assuming now that f is irreducible over $\mathbb{Q}$, the proof goes as in the first part, except that according to Lemma 2.1, the degree of every irreducible factor of $h^m \cdot f(g/h)$ must be a multiple of m, so in particular we must have $\deg(F_1) = r \geq m$. In this case we need to prove that our assumption on the size of p^k implies again $|a_m| > \delta^m L_1(f(X/\delta))$ for a suitable $\delta > 0$, and at the same time it forces the inequality

$$|q|^n |b_n|^{mn} \cdot \frac{L(h(\gamma X))^m L_1(f(X/\delta))}{|a_m|} < 1,$$

or equivalently

$$|a_m| > |q|^n |b_n|^{mn} L(h(\gamma X))^m L_1(f(X/\delta)),$$

which in view of (13) will contradict (4). It will be therefore sufficient to find a $\delta > 0$ such that $\delta^m \geq |q|^n |b_n|^{mn} L(h(\gamma X))^m$, that is $\delta \geq |q|^{n/m} |b_n|^n L(h(\gamma X))$, which, recalling the definition of γ, reads

$$\delta \geq |q|^{\frac{n}{m}} |b_n|^n \sum_{i=0}^{l} |c_i| \left(1 + \frac{H_1(g)}{|b_n|} + \frac{H(h)}{\delta |b_n|} \right)^i.$$

A suitable candidate for δ in this case is

$$\delta_1 = |q|^{\frac{n}{m}} |b_n|^n \sum_{i=0}^{l} |c_i| \left(1 + \frac{H_1(g) + H(h)}{|b_n|} \right)^i = |q|^{\frac{n}{m}} |b_n|^n L(h(\beta X)) \geq 1,$$

so the contradiction $|R(g^m, F_1)| < 1$ follows now if

$$
\begin{aligned}
|a_m| &> [|q|^{\frac{n}{m}} |b_n|^n L(h(\beta X))]^m \cdot L_1 \left(f \left(\frac{X}{|q|^{\frac{n}{m}} |b_n|^n L(h(\beta X))} \right) \right) \\
&= \sum_{i=0}^{d} |a_i| \cdot [|q|^{\frac{n}{m}} |b_n|^n L(h(\beta X))]^{m-i}.
\end{aligned}
$$

This completes the proof of the theorem. $\square$

Proof of Corollary 1.2 We apply Theorem 1.1 twice. On taking $g(X) = X$ and $h(X) = 1$, we first note that if $f(X) = a_0 + a_1 X + \cdots + a_m X^m \in \mathbb{Z}[X]$ is a polynomial of degree $m \geq 1$, with $a_0 \neq 0$, and $a_m = p^k q$ with p a prime, q a non-zero integer, $p \nmid q a_d$, and k is a positive integer prime to $m - d$ and such that $p^k > |q|^{m-1} L_1(f(X/|q|))$, then f is irreducible over $\mathbb{Q}$. Now by the second part of Theorem 1.1, the polynomial $h^m \cdot f(g/h)$ must be irreducible over $\mathbb{Q}$. So, if p satisfies the hypothesis of Corollary 1.2, then both polynomials f and $h^m \cdot f(g/h)$ will be irreducible over $\mathbb{Q}$.

References

[1] A.I. Bonciocat, N.C. Bonciocat, *A Capelli type theorem for multiplicative convolutions of polynomials*, Math. Nachr. 281 (2008), no. 9, 1240–1253.

[2] A.I. Bonciocat, A. Zaharescu, *Irreducibility results for compositions of polynomials with integer coefficients*, Monatsh Math. 149 (2006), no. 1, 31–41.

[3] A.I. Bonciocat, N.C. Bonciocat, A. Zaharescu, *On the number of factors of convolutions of polynomials with integer coefficients*, Rocky Mountain J. Math. 38 (2008), no. 2, 417–431.

[4] N.C. Bonciocat, *Upper bounds for the number of factors for a class of polynomials with rational coefficients*, Acta Arith. 113. 2 (2004), 175–187.

[5] N.C. Bonciocat, Y. Bugeaud, M. Cipu, M. Mignotte, *Irreducibility criteria for sums of two relatively prime polynomials*, Int. J. Number Theory 9 (2013), no. 6, 1529–1539.

[6] M. Cavachi, *On a special case of Hilbert's irreducibility theorem*, J. Number Theory 82 (2000), no. 1, 96–99.

[7] M. Cavachi, M. Vâjâitu, A. Zaharescu, *A class of irreducible polynomials*, J. Ramanujan Math. Soc. 17 (2002), no. 3, 161–172.

[8] M. Fried, *On Hilbert's irreducibility theorem*, J. Number Theory 6 (1974), 211–231.

[9] K. Langmann, *Der Hilbertsche Irreduzibilitätssatz und Primzahlfragen*, J. Reine Angew. Math. 413 (1991), 213–219.

[10] A. Schinzel, *Polynomials with Special Regard to Reducibility*, in Encyclopedia of Mathematics and its Applications, Cambridge University Press (2000).

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