



On the function $\pi(x)$

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Abstract

Let $\pi(x)$ be the number of primes not exceeding x . We prove that $\pi(x) < \frac{x}{\log x - 1.006789}$ for $x \geq e^{10^{12}}$, and that for sufficiently large x : $\frac{x}{\log x - 1 + (\log x)^{-1.5} + 2(\log x)^{-0.5}} < \pi(x) < \frac{1}{\log x - 1 - 2(\log x)^{-0.5} - (\log x)^{-1.5}}$. We finally prove that for $x \geq e^{10^{12}}$ and $k = 2, 3, \dots, 147297098200000$, the closed interval $[(k-1)x, kx]$ contains at least one prime number, i.e. the Bertrand's postulate holds for x and k as above.

1 Introduction

In 1962, (see [10], page 69, Th.2), B. Rosser and L. Schoenfeld proved the following inequalities, that rely on the computation of the first 25 000 zeros of Riemann's zeta function obtained by D.H. Lehmer, (see [6]):

$$\frac{x}{\log x - \frac{1}{2}} < \pi(x), \quad \text{for } x \geq 67$$
$$\pi(x) < \frac{x}{\log x - \frac{3}{2}}, \quad \text{for } x > e^{\frac{3}{2}},$$

where $\pi(x)$ is the number of prime numbers not exceeding x .

In 1986, J. Van de Lune, H. J. J. Te Riele and D.T. Winter computed a number of 1500000001 of the zeros of zeta function (see [14]). Using this and Rosser-Schoenfeld method, Dusart improved the inequalities above. In this

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respect, the best inequalities involving the function $\pi(x)$ established so far, obtained by P. Dusart (see [4] Theorem 1, p.1-3, and Theorem 10, p.16-20), are:

$$\frac{x}{\log x - 1} < \pi(x) \quad \text{for } x \geq 5393, \quad (1)$$

$$\frac{x}{\log x - 1.1} \geq \pi(x) \quad \text{for } x \geq 60184, \quad (2)$$

while for Chebyshev theta function $\theta(x) = \sum_{p \leq x} \log p$ one has

$$|\theta(x) - x| < 0.006788 \frac{x}{\log x} \quad \text{for } x \geq 10544111, \quad (3)$$

$$|\theta(x) - x| \leq 515 \frac{x}{\log^3 x} \quad \text{for } x > 1, \quad (4)$$

$$|\theta(x) - x| \leq 1717433 \frac{x}{\log^4 x} \quad \text{for } x > 1. \quad (5)$$

In 2000, L. Panaitopol improved the estimates on $\pi(x)$, relying on the inequalities of Rosser-Schoenfeld (see [9], p. 374, Theorem 1). More precisely, he obtained the following inequalities:

$$\pi(x) < \frac{x}{\log x - 1 - (\log x)^{-0.5}} \quad \text{for } x \geq 6, \quad (6)$$

$$\pi(x) > \frac{x}{\log x - 1 + (\log x)^{-0.5}} \quad \text{for } x \geq 59. \quad (7)$$

In 2003, G. Mincu improved the inequalities above, by using the inequalities of Dusart, and proved that (see [8], pag.57-58, Lemma 1 and Lemma 2):

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1.51}{\log x}} \quad \text{for } x \geq 6.22, \quad (8)$$

$$\pi(x) > \frac{x}{\log x - 1 - \frac{0.7}{\log x}} \quad \text{for } x \geq 70111. \quad (9)$$

For several results on $\pi(x)$, and on other related functions, we refer the reader to the monograph [11]. For other connections with inequalities of type (1) and (2), the reader is referred to [3] and [12]. For a solution of a conjecture on a multiplicative property of $\pi(x)$, the reader is referred to [13].

The aim of this paper is to improve the inequality (2) and the inequalities (8) and (9), by using the method described in [9]. As a consequence, several particular cases of the generalized Bertrand's postulate are proven. Recall that the generalized Bertrand's postulate asserts that: for any positive integers n and k with $k = 2, \dots, n$, the interval $[(k-1)n, kn]$ contains a prime number.

For $k = 2$ the Bertrand's postulate was proved by Chebyshev in 1850. For $k = 3$ the Bertrand's postulate was proved by Bachraoui in 2006, (see [2], Corollary 1.4.), and for $k = 4$ it was proved by Loo in 2011, (see [7], Theorem 2.4). In this paper we will improve the inequalities obtained by Panaitopol and Mincu. Throughout this paper all the functions are defined on the interval $[2, \infty)$.

2 Main results

Theorem 2.1. *For $x \geq 10544111$ and $c = 0.006788$ the following inequality holds:*

$$\pi(x) < \frac{x}{\log x} + (c+1)\frac{x}{\log^2 x} + \frac{x}{\log^{5/2} x}. \quad (10)$$

Proof. Recall the well-known identity $\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(x)}{t \log^2 t} dt$ (see, for example, [1], Theorem 4.3, pages 78-79), and observe that the lower bound for x in the statement of our theorem verifies $e^{16} < 10544111 < e^{17}$. Then, using (3) and integrating it by parts, we obtain:

$$\begin{aligned} \pi(x) &< \frac{x}{\log x} + c \frac{x}{\log^2 x} + \int_2^x \frac{1}{\log^2 t} dt + c \int_2^x \frac{1}{\log^3 t} dt \\ &= \frac{x}{\log x} + c \frac{x}{\log^2 x} + \frac{x}{\log^2 x} - \frac{2}{\log^2 2} + (c+2) \int_2^x \frac{1}{\log^3 t} dt \\ &= \frac{x}{\log x} + (c+1) \frac{x}{\log^2 x} - \frac{2}{\log^2 2} + (c+2) \int_2^x \frac{1}{\log^3 t} dt \\ &= \frac{x}{\log x} + \frac{(c+1)x}{\log^2 x} - \frac{2}{\log^2 2} + (c+2) \int_2^{e^{16}} \frac{1}{\log^3 t} dt \\ &\quad + (c+2) \int_{e^{16}}^x \frac{1}{\log^3 t} dt. \end{aligned}$$

We have therefore proved that

$$\pi(x) < \frac{x}{\log x} + \frac{(c+1)x}{\log^2 x} - \frac{2}{\log^2 2} + (c+2) \left(\int_2^{e^{16}} \frac{1}{\log^3 t} dt + \int_{e^{16}}^x \frac{1}{\log^3 t} dt \right). \quad (11)$$

We search for an upper bound for:

$$(c+2) \int_2^{e^{16}} \frac{1}{\log^3 t} dt = (c+2) \int_2^e \frac{1}{\log^3 t} dt + (c+2) \sum_{k=1}^{15} \int_{e^k}^{e^{k+1}} \frac{1}{\log^3 t} dt.$$

In this respect, we observe that the function $t \mapsto \frac{1}{\log^3 t}$ is strictly convex on $[2, \infty)$. If we apply on each interval $[e^k, e^{k+1}]$, $k = 1, 2, \dots, 15$ and $[2, e]$ the Hermite–Hadamard inequality (see [5]),

$$\int_a^b f(x)dx \leq \frac{b-a}{2}(f(a) + f(b)),$$

we obtain:

$$(c+2) \int_2^{e^{16}} \frac{1}{\log^3 t} dt < 5622. \quad (12)$$

Up to this point, from (11) and (12) we proved that:

$$\pi(x) < \frac{x}{\log x} + \frac{(c+1)x}{\log^2 x} - \frac{2}{\log^2 2} + 5622 + (c+2) \int_{e^{16}}^x \frac{1}{\log^3 t} dt.$$

To conclude, we will show that for

$$A := \frac{x}{\log x} + \frac{(c+1)x}{\log^2 x} - \frac{2}{\log^2 2} + 5622 + (c+2) \int_{e^{16}}^x \frac{1}{\log^3 t} dt$$

and

$$B := \frac{x}{\log x} + (c+1) \frac{x}{\log^2 x} + \frac{x}{\log^{5/2} x}$$

we have $A < B$ i.e., that

$$(c+2) \int_{e^{16}}^x \frac{1}{\log^3 t} dt - \frac{2}{\log^2 2} - \frac{x}{\log^{5/2} x} + 5622 < 0.$$

The derivative of the function

$$g(x) = (c+2) \int_{e^{16}}^x \frac{1}{\log^3 t} dt - \frac{2}{\log^2 2} - \frac{x}{\log^{5/2} x} + 5622$$

is

$$g'(x) = \frac{-\log x + (c+2)\log^{1/2} x + 2,5}{\log^{7/2} x},$$

and for $\log^{1/2} x > 2.87$ i.e. for $x > e^{8.3}$ we have $g'(x) < 0$, hence for these values of x , g is a decreasing function. Moreover,

$$g(e^{16}) = 5622 - \frac{2}{\log^2 2} - \frac{e^{16}}{1024} \approx -3052 < 0,$$

and consequently, for $x > e^{16}$ we have $g(x) < g(e^{16}) < 0$, which finishes the proof of our theorem. \square

We are now in a position to prove our main result.

Theorem 2.2. *Let $d = 1.006789$. Then for all $x > e^{10^{12}}$ the following inequality holds:*

$$\pi(x) < \frac{x}{\log x - d}. \quad (13)$$

Proof. Note that $d = c + 1 + 10^{-6}$ with $c = 0.006788$. According to Theorem 2.1, it suffices to prove that:

$$\frac{x}{\log x} + (c + 1)\frac{x}{\log^2 x} + \frac{x}{\log^{5/2} x} < \frac{x}{\log x - d}. \quad (14)$$

This is successively equivalent to

$$\begin{aligned} & (\log^{3/2} x + (c + 1)\log^{1/2} x + 1)(\log x - d) < \log^{5/2} x \Leftrightarrow \\ & \log^{5/2} x - d\log^{3/2} x + (c + 1)\log^{3/2} x - d(c + 1)\log^{1/2} x \\ & \quad + \log x - d < \log^{5/2} x \Leftrightarrow \\ & (c + 1 - d)\log^{3/2} x + \log x - d(c + 1)\log^{1/2} x - d < 0 \Leftrightarrow \\ & -10^{-6}\log^{3/2} x + \log x - d(c + 1)\log^{1/2} x - d < 0. \end{aligned}$$

Let $z = \log^{1/2} x$ and let us consider the function:

$$h(z) = -10^{-6}z^3 + z^2 - d(c + 1)z - d = z(-10^{-6}z^2 + z - d(c + 1)) - d$$

Since the greatest root of the equation $-10^{-6}z^2 + z - d(c + 1) = 0$ is 999998.98..., we have $-10^{-6}z^2 + z - d(c + 1) < 0$ for $z \geq 10^6$, hence $h(z) < 0$ for all $z \geq 10^6$, which shows that inequality (13) holds for all $x \geq e^{10^{12}}$. \square

Lemma 2.3. *For sufficiently large x we have the following inequalities:*

$$\theta(x) < x \left(1 + \frac{1}{3(\log x)^{2.5}} \right), \quad (15)$$

$$\theta(x) > x \left(1 - \frac{2}{3(\log x)^{2.5}} \right). \quad (16)$$

Proof. From inequality (4) we deduce that:

$$x - 515\frac{x}{\log^3 x} \leq \theta(x) \leq x + 515\frac{x}{\log^3 x}.$$

Next, from inequality (5) we see that for $x > e^{29831}$ we have

$$\theta(x) \leq x \left(1 + 1717433\frac{1}{\log^4(x)} \right) < x \left(1 + \frac{1}{3(\log^{2.5}(x))} \right).$$

Using the same inequality for $x > e^{18793}$, we deduce that

$$\theta(x) \geq x \left(1 - 1717433 \frac{1}{\log^4(x)} \right) > x \left(1 - \frac{2}{3(\log^{2.5}(x))} \right).$$

which completes the proof. \square

Theorem 2.4. *For sufficiently large x the following inequalities hold:*

$$\pi(x) < \frac{1}{\log x - 1 - 2(\log x)^{-0.5} - (\log x)^{-1.5}}, \quad (17)$$

$$\pi(x) > \frac{x}{\log x - 1 + (\log x)^{-1.5} + 2(\log x)^{-0.5}}. \quad (18)$$

Proof. We use the identity (see [1], Th. 4.3, p.78):

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt.$$

From inequality (15), after integrating by parts, we deduce that:

$$\begin{aligned} \pi(x) < \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{1}{3 \log^{2.5} x} + \frac{2}{\log^2 x} \right) - \frac{2e^{29831}}{29831^3} - \frac{e^{29831}}{29831^3} \\ + \int_2^{e^{29831}} \frac{\theta(t)}{t \log^2 t} dt + 6 \int_{e^{29831}}^x \frac{dt}{\log^4 t} + \frac{1}{3} \int_{e^{29831}}^x \frac{dt}{\log^{4.5} t}. \end{aligned}$$

Since

$$-\frac{2e^{29831}}{29831^3} - \frac{e^{29831}}{29831^3} + \frac{1}{3} \int_{e^{29831}}^x \frac{dt}{\log^{4.5} t} < \frac{1}{3} \int_{e^{29831}}^x \frac{dt}{\log^4 t}$$

we deduce that

$$\pi(x) < \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{1}{3 \log^{2.5} x} + \frac{2}{\log^2 x} \right) + \int_2^{e^{29831}} \frac{\theta(t)}{t \log^2 t} dt + \frac{19}{3} \int_{e^{29831}}^x \frac{dt}{\log^4 t}.$$

We define now the function $f : [e^{29831}, \infty) \rightarrow \mathbb{R}$ by

$$f(x) = \frac{2}{3} \cdot \frac{x}{(\log x)^{3.5}} - \frac{19}{3} \int_{e^{29831}}^x \frac{dt}{\log^4 t} - \int_2^{e^{29831}} \frac{\theta(t)}{t \log^2 t} dt.$$

We observe that the derivative of f is

$$f'(x) = \frac{2 \log^{1.5} x - 7 \log^{0.5} x - 19}{(\log x)^{3.5}} > 0,$$

so f is an increasing function and, for sufficiently large x we have $f(x) > 0$. Therefore we have

$$\begin{aligned}\pi(x) &< \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{1}{\log^{2.5} x} \right) \\ &< \frac{1}{\log x - 1 - 2(\log x)^{-0.5} - (\log x)^{-1.5}}.\end{aligned}$$

If we apply the same method to prove inequality (18), we successively obtain:

$$\begin{aligned}\pi(x) &> \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} - \frac{2}{3 \log^{3.5} x} \right) + \int_2^{e^{18793}} \frac{\theta(t)}{t \log^2 t} dt \\ &\quad - \frac{e^{18793}}{18793^2} - \frac{2e^{18793}}{18793^3} + 6 \int_{e^{18793}}^x \frac{dt}{\log^4 t} - \frac{2}{3} \int_{e^{19873}}^x \frac{dt}{\log^{4.5} t} \\ &> \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} - \frac{2}{3 \log^{3.5} x} \right) \\ &> \frac{x}{\log x - 1 + (\log x)^{-1.5} + 2(\log x)^{-0.5}}.\end{aligned}$$

□

3 Applications

Theorem 3.1. *For $x \geq e^{10^{12}}$ and $k = 2, 3, \dots, 147297098200000$, the closed interval $[(k-1)x, kx]$ contains at least one prime number, i.e. the Bertrand's postulate holds for these x and k .*

Proof. The inequality (1) and Theorem 2.2 show that:

$$\pi(kx) - \pi((k-1)x) > \frac{kx}{\log kx - 1} - \frac{(k-1)x}{\log(k-1)x - 1.006789}. \quad (19)$$

We need to prove that

$$\frac{kx}{\log kx - 1} - \frac{(k-1)x}{\log(k-1)x - 1.006789} > 0,$$

which is equivalent to:

$$\begin{aligned}&k \log(k-1) + k \log x - k \cdot 1.006789 - (k-1)(\log k + \log x - 1) > 0 \\ \Leftrightarrow &\log \left(\frac{k-1}{k} \right)^k - 0.006789k + \log k - 1 + \log x > 0 \\ \Leftrightarrow &x > \frac{\left(\frac{k}{k-1} \right)^k e^{1+0.006789k}}{k}.\end{aligned}$$

Since we have $\left(\frac{k}{k-1}\right)^k = \left(1 + \frac{1}{k-1}\right)^{k-1} \left(1 + \frac{1}{k-1}\right) < 2e$, in order to prove our last inequality, it is sufficient to prove that the following inequality is true:

$$x \geq \frac{2e \cdot e^{1+0.006789k}}{k} = \frac{2e^{2+(d-1)k}}{k}. \quad (20)$$

Since $x \geq e^{10^{12}}$, if $2 \leq k \leq \frac{10^{12}-2}{d-1}$, we have:

$$x \geq e^{10^{12}} \geq e^{2+(d-1)k} \geq \frac{2}{k} e^{2+(d-1)k}$$

so (20) holds. We conclude that Theorem 3.1, i.e. the Bertrand's postulate is true for any $x \geq e^{10^{12}}$ and for any k with $2 \leq k \leq \frac{10^{12}-2}{d-1} = 147297098200000$. \square

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References

- [1] T. APOSTOL, Introduction to Analytic Number Theory, *Springer-Verlag*, 1976.
- [2] M.EL BACHRAOUI, Primes in the Interval $[2n, 3n]$, *Int. J. Contemp. Math. Sciences*, Vol. 1 (2006), no. 13, 617–621.
- [3] D. BURDE, A remark on an inequality for the prime counting function, *Math.Ineq.Appl.* 10 (2007), no.1, 9–13.
- [4] P. DUSART, Sharper bounds for ϕ , θ , π , p_n , *Rapport de recherche 1998*, Université de Limoges.
- [5] J. HADAMARD, Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, *Journal de Mathématiques Pures et Appliquées*, volume 58 (1893), 171–215.
- [6] D.H. LEHMER, On the roots of the Riemann zeta-functions, *Acta Math.* 95 (1956), 291–298.
- [7] A. LOO, On the Primes in the Interval $(3n, 4n)$, *International Journal of Contemporary Mathematical Sciences* 6 (38): 1871–1882, (2011)

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- [8] G. MINCU, A few inequalities involving $\pi(x)$, *Analele Universității București, Matematică, Anul LII, Nr.1 (2003)*, 55–64.
- [9] L. PANAITOPOL, Inequalities concerning the function $\pi(x)$: Applications, *Acta Arithmetica, XCIV (2000), no.4*, 317–324.
- [10] J.B.ROSSER AND L.SCHOENFELD, Approximate formulas for some functions of prime numbers, *Illinois J. Math. 6 (1962)*, 64–94.
- [11] J. SÁNDOR, D.S. MITRINOVIC AND B. CRSTICI, Handbook of number theory I, *first ed. 1996, by Kluwer Acad. Publ., 2nd printing 2006 by Springer Verlag*.
- [12] J. SÁNDOR, On some inequalities of Dusart and Panaitopol on the function $\pi(x)$, *Octogon Math.Mag. vol.14 (2006), no.2*, 592–594.
- [13] J. SÁNDOR, On a conjecture of Miliakos, *Octogon Math.Mag., vol.14, (2006), no.1*, 450–451.
- [14] J. VAN DE LUNE, H. J. J. TE RIELE AND D.T. WINTER, On the Zeros of the Riemann Zeta Function in the Critical Strip.IV, *Math. Of Computation 46 (1986), no 174*, 667–681.

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