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DIFFERENT SPECTRA FOR NONLINEAR OPERATORS

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To Professor Dan Pascali, at his 70's anniversary

Abstract

We approach the spectra for nonlinear operators and we show that the Neuberger spectrum is always nonempty in complex Banach spaces and may be unbounded or not closed; the Kachurovskij spectrum is always compact; the Rhodius and Dörfner spectra aren't necessary to be bounded or closed; the FMV-spectrum and the Feng spectrum are always closed but may be unbounded.

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1. Introduction

The last 30 years have represented an opportunity to study several spectra for nonlinear operators, such as : the Kachurovskij spectrum for Lipschitz continuous operators [8], the Neuberger spectrum for C^1-operators [9], the Rhodius spectrum for continuous operators, the Dörfner spectrum for linear bounded operators [4].These spectra are modelled on the familiar spectrum for bounded linear operators in Banach spaces. In 1978 a nonlinear spectrum constructed in a different way was introduced by Furi, Martelli and Vignoli [3].The Furi-Martelli-Vignoli spectrum (denoted by FMV-spectrum) of a continuous nonlinear operator F in a Banach space X is based on solvability properties of the operator equation in X :

(1.1) $\quad F(x) = G(x)$,

where G is a compact operator. Feng [2] defined another spectrum which is built on solvability properties of equation (1.1), in a similarly way, for G

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satisfying boundary conditions on spheres. If we choose $F(x) = x$ in (1.1), the existence results are reduced to classical fixed point theorems. We apply the classical fixed point methods for a problem of the form :

$$Lx = F(x) \ , \tag{1.2}$$

where F is a nonlinear operator and L is a linear operator with trivial nullspace. In this case, the equation (1.2) is equivalent with the fixed point problem $x = L^{-1}F(x)$.

2. The Rhodius and Neuberger spectra

Throughout the paper, X will be a Banach space over K ($\mathbb{R}$ or $\mathbb{C}$) and $M(X)$ the class of continuous nonlinear operators $F : X \to X$ with $F(0) = 0$. We denote the identity operator by I. We also define the *resolvent set of* $F \epsilon M(X)$ by

$$\rho(F) = \left\{\lambda \epsilon K : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \epsilon M(X)\right\} \tag{2.1}$$

and the *spectrum of* $F \epsilon M(X)$ by

$$\sigma(F) = K - \rho(F). \tag{2.2}$$

For $\lambda \epsilon \rho(F)$ we denote by

$$R(\lambda; F) = (\lambda I - F)^{-1} : X \to X \tag{2.3}$$

the nonlinear *resolvent operator* of F.

Moreover, the *spectral radius of* F defined by

$$r(F) = \sup\{|\lambda| : \lambda \epsilon \sigma(F)\} \tag{2.4}$$

may be calculated by Gel'fand formula :

$$r(F) = \lim_{n \to \infty} \sqrt[n]{\|F^n\|}, \tag{2.5}$$

where F^n denotes the n^{th} iterate of F.

We set the class $M(X) = C(X)$ of all continuous operators on X. The *Rhodius resolvent set* is given by

$$\rho_R(F) = \{\lambda \epsilon K : \lambda I - F \text{ is bijective and } R(\lambda; F) \epsilon C(X)\} \tag{2.6}$$

and the *Rhodius spectrum* by

$$\sigma_R(F) = K - \rho_R(F). \tag{2.7}$$

Remark 1. *A point $\lambda \epsilon K$ belongs to $\rho_R(F)$ if and only if $\lambda I - F$ is a homeomorphism on X.*

The relation (2.7) gives the definition of the usual spectrum in the case of a bounded linear operator. If we want to find more properties of the linear spectrum by restricting the operator class $M(X)$, a choice is the class $C^1(X)$ of all continuously Fréchet differentiable operators on X which leads to the *Neuberger resolvent set*

$$\rho_N(F) = \left\{\lambda \epsilon K : \lambda I - F \text{ is bijective and } R(\lambda; F)\, \epsilon C^1(X)\right\} \tag{2.8}$$

and the *Neuberger spectrum*

$$\sigma_N(F) = K - \rho_N(F). \tag{2.9}$$

Remark 2. *A point $\lambda \epsilon K$ belongs to $\rho_N(F)$ if and only if $\lambda I - F$ is a diffeomorphism on X.*

The relations (2.8) and (2.9) give the classical resolvent set and the classical spectrum, if F is linear.

Since $C^1(X) \subseteq C(X)$, we have the inclusions :

$$\rho_N(F) \subseteq \rho_R(F) \text{ and } \sigma_N(F) \supseteq \sigma_R(F), \text{ for } F \epsilon C^1(X). \tag{2.10}$$

Theorem 1. (Neuberger [9]) *The spectrum $\sigma_N(F)$ is nonempty in case $K = \mathbb{C}$.*

Remark 3. *It is not necessary for the Neuberger spectrum to be bounded or closed. The Neuberger spectrum is defined for continuously differentiable operators and it might be expressed through the spectra of the Fréchet derivatives $F'(x)$ of F.*

Definition 1. A continuous operator F on a Banach space X is called *proper* if the preimage $F^{-1}(K)$ of any compact set $K \subset X$ is compact.

This notion plays an important role in the existence and uniqueness results for solutions of nonlinear operator equations.

Theorem 2. (Appell and Dorfner [4]) *Given $F \epsilon C^1(X)$, denote by $\pi(F)$ the set of all elements $\lambda \epsilon K$ such that the operator $\lambda I - F$ is not proper. Then the formula :*

$$\sigma_N(F) = \pi(F) \bigcup \left(\bigcup_{x \epsilon X} \sigma\left(F'(x)\right)\right) \tag{2.11}$$

holds, where $\sigma(L)$ denotes the usual spectrum of a bounded linear operator L. In particular, $\sigma_N(F) \neq \emptyset$ in case $K = \mathbb{C}$.

Corollary. *If X is an infinite-dimensional Banach space and $F : X \to X$ is compact, then F cannot be proper and thus $0 \epsilon \pi(F) \subseteq \sigma_N(F)$.*

3. The Kachurovskij and Dörfner spectra

Let X be a Banach space over K ($\mathbb{R}$ or $\mathbb{C}$). We write $F \epsilon Lip(X)$ if F is Lipschitz continuous on X, i.e.

(3.1) $$[F]_{Lip} = \sup_{x \neq y} \frac{\|F(x)-F(y)\|}{\|x-y\|} < \infty.$$

If $F(0) = 0$, the number (3.1) is a norm on the linear space $Lip(X)$ which makes it a Banach space.

Let us take $M(X) = Lip(X)$, then we get to the *Kachurovskij resolvent set*

(3.2) $$\rho_K(F) = \{\lambda \epsilon K : \lambda I - F \text{ is bijective and } R(\lambda; F) \epsilon Lip(X)\}$$

and the *Kachurovskij spectrum*

(3.3) $$\sigma_K(F) = K - \rho_K(F).$$

Remark 4. *A point $\lambda \epsilon K$ belongs to $\rho_K(F)$ if and only if $\lambda I - F$ is a lipeomorphism on X, i.e. $\lambda I - F$ is bijective on X and satisfies:*

$c\|x-y\| \leq \|\lambda(x-y) - F(x) + F(y)\| \leq C\|x-y\|$, $x, y \epsilon X$, *for some* $C, c > 0$.

We write $F \epsilon B(X)$, if F is linearly bounded on X, i.e.

(3.4) $$[F]_B = \sup_{x \neq 0} \frac{\|F(x)\|}{\|x\|} < \infty .$$

The linear space $B(X)$ with norm (3.4) is a Banach space and $[F]_B \leq [F]_{Lip}$.

Let us have $M(X) = B(X)$. Then we get to the *Dörfner resolvent set*

(3.5) $$\rho_D(F) = \{\lambda \epsilon K : \lambda I - F \text{ is bijective and } R(\lambda; F) \epsilon B(X)\}$$

and the *Dörfner spectrum*

(3.6) $\sigma_D(F) = K - \rho_D(F)$, introduced in [4].

Remark 5 . *A point $\lambda \epsilon K$ belongs to $\rho_D(F)$ if and only if $\lambda I - F$ is a homeomorphism on X satisfying :*

(3.7) $$c\|x\| \leq \|\lambda x - F(x)\| \leq C\|x\| , \ x \epsilon X, \textit{ for some } C, c > 0.$$

Since $Lip(X) \subseteq B(X) \subseteq C(X)$, we have the following inclusions :

(3.8) $$\rho_K(F) \subseteq \rho_D(F) \subseteq \rho_R(F) \text{ and}$$
$$\sigma_K(F) \supseteq \sigma_D(F) \supseteq \sigma_R(F), \text{ for } F \epsilon Lip(X).$$

For $F\epsilon Lip(X) \cap C^1(X)$, we have a relation between the spectra $\sigma_K(F)$ and $\sigma_N(F)$, given by the following theorem.

Theorem 3. *For $F\epsilon Lip(X)\cap C^1(X)$ the inclusions $\rho_K(F) \subseteq \rho_N(F)$ and $\sigma_K(F) \supseteq \sigma_N(F)$ are true.*

Proof. Fix $x\epsilon X$. From $\lambda\epsilon\sigma\left(F'(x)\right)$, it follows that $\lambda I - F'$ is not bijective and hence $\lambda I - F$ cannot be a lipeomorphism. This shows that

$[F]_B = \sup\limits_{x\neq 0}\frac{\|F(x)\|}{\|x\|} < \infty$.

Moreover, $\lambda\epsilon\rho_K(F)$ implies that $\lambda I - F$ is proper and $\lambda \notin \pi(F)$. We conclude that $\pi(F) \subseteq \sigma_K(F)$.

Theorem 1 is not true for the Kachurovskij and Dörfner spectra. On the other hand, the Kachurovskij spectrum has another property, as we will see in the next theorem.

Theorem 4. (Maddox and Wickstead [10]) *The spectrum $\sigma_K(F)$ is compact.*

Remark 6. *The Rhodius, Kachurovskij and Dörfner spectra may be empty. If we impose additional conditions on the operator F and the underlying space X, we may force these spectra to be nonempty.*

Theorem 5. *Suppose that $\dim X = \infty$ and $F\epsilon C(X)$ is compact. Then $0\epsilon\sigma_R(F)$.*

Proof. The fact that $\lambda = 0$ belongs to one of the indicated spectra implies that F is a compact homeomorphism. In particular, the identity $I = F^{-1}F$ would be compact on X, contradicting the assumption that X is infinitely dimensional.

4. The Furi-Martelli-Vignoli spectrum

Let X be a Banach space over K. *The measure of noncompactness* of a bounded set $M \subset X$ is defined by :

(4.1) $\alpha(M) = \inf\{\varepsilon > 0 : M \text{ has a finite } \varepsilon-net \text{ in } X\}$.

$\alpha(M) = 0$ if and only if M is precompact.

Given a continuous operator $F : X \to X$, the number

(4.2) $[F]_A = \inf\{k > 0 : \alpha(F(M)) \leq k\alpha(M), M \subset X \text{ bounded}\}$

is called *the measure of noncompactness of F*.

Remark 7. $[F]_A = 0$ *if and only if* F *is a compact operator.*

For the measure of noncompactness of F, let be the number

(4.3) $$[F]_a = \sup\{k > 0 : \alpha(F(M)) \geq k\alpha(M), M \subset X \text{ bounded}\}.$$

Definition 2. We call an operator $F\epsilon C(X)$ *quasibounded*, if

(4.4) $$[F]_Q = \lim_{\|x\|\to\infty} \sup \frac{\|F(x)\|}{\|x\|} < \infty.$$

We also consider the number :

(4.5) $$[F]_q = \lim_{\|x\|\to\infty} \inf \frac{\|F(x)\|}{\|x\|.}$$

From the definition and the assumption $F(0) = 0$, it follows that :

(4.6) $$[F]_Q \leq [F]_B \leq [F]_{Lip}, [F]_A \leq [F]_{Lip},$$

where each of these inequalities may be strict.

If F is linear, we have $[F]_Q = [F]_B = [F]_{Lip} = \|F\|$.

Definition 3. A continuous operator $F : X \to X$ is called *stably solvable* if, given any compact operator G in X with $[G]_Q = 0$, the equation $F(X) = G(X)$ has a solution $x\epsilon X$.

Remark 8. *Each stably solvable operator is surjective. For linear operators, surjectivity is equivalent to stable solvability [7].*

Definition 4. An operator $F\epsilon C(X)$ is called $FMV-regular$ if F is stably solvable, $[F]_a > 0$, and $[F]_q > 0$.

Definition 5. Given $F\epsilon C(X)$, the set

(4.7) $$\rho_{FMV}(F) = \{\lambda\epsilon K : \lambda I - F \text{ is FMV-regular}\}$$

is called the $Furi - Martelli - Vignoli$ *resolvent set* and

(4.8) $$\sigma_{FMV}(F) = K - \rho_{FMV}(F)$$

the $Furi - Martelli - Vignoli$ *spectrum* of F.

Lemma 1. *Let* $F\epsilon C(X)$ *be stably solvable,* $B \subseteq X$ *be a closed subset, and* $H : B \to X$ *be a continuous operator. Assume that*

(4.9) $$F^{-1}(\bar{co}\, H(B)) \subseteq B$$

and that the equality

(4.10) $$\alpha(F(M)) = \alpha(H(M)) \; (M \subseteq X)$$

implies the precompactness of M. *Then the equation* $F(x) = H(x)$ *has a solution* $x\epsilon X$.

Lemma 1 contains the fixed point theorems of Schauder, Darbo and

Sadovskij, by choosing $F = I$ and B closed, bounded and convex.

Lemma 2. *Let $F, G \epsilon C(X)$ with F being $FMV - regular$. Suppose that $[G]_A < [F]_a$ and $[G]_Q < [F]_q$. Then $F + G$ is $FMV - regular$.*

Theorem 6. *The spectrum $\sigma_{FMV}(F)$ is closed.*

Proof. Fix $\lambda \epsilon\, \sigma_{FMV}(F)$, and let $0 < \varepsilon < \min\left\{[\lambda I - F]_a, [\lambda I - F]_q\right\}$.

We apply Lemma 2 to show that $\mu \epsilon\ \sigma_{FMV}(F)$ for $|\mu - \lambda| < \varepsilon$.

In fact, from $[(\mu - \lambda) I]_A = |\mu - \lambda| < [\lambda I - F]_a$ and

$[(\mu - \lambda) I]_Q = |\mu - \lambda| < [\lambda I - F]_q$, it follows that

$\mu I - F = (\lambda I - F) + (\mu - \lambda) I$ is FMV-regular.

This shows that λ is an interior point of $\sigma_{FMV}(F)$, and thus $\sigma_{FMV}(F)$ is open in K.

Theorem 7. *Suppose that $F \epsilon C(X)$ satisfies $[F]_A < \infty$ and $[F]_Q < \infty$. Then the spectrum $\sigma_{FMV}(F)$ is bounded.*

Proof. For $\lambda \epsilon K$ with $|\lambda| > \max\left\{[F]_A, [F]_Q\right\}$, we have $[\lambda I - F]_q > 0$ and $[\lambda I - F]_a \geq |\lambda| - [F]_A > 0$.

We claim that $\lambda I - F$ is stably solvable for such λ. Now, if $G : X \to X$ is compact with $[G]_Q = 0$, then the operator :

$H = (1/\lambda)(F + G)$ satisfies both the inequalities:

$[H]_A \leq (1/|\lambda|)[F]_A < 1$ and $[H]_Q \leq (1/|\lambda|)[F]_Q < 1$.

From Darbo's fixed point theorem, it follows that H has a fixed point $x \epsilon X$, which is a solution of the equation $\lambda x - F(x) = G(x)$.

We have proved that the Furi-Martelli-Vignoli spectral radius

$$r_{FMV}(F) = \sup\{|\lambda| : \lambda \epsilon \sigma_{FMV}(F)\} \tag{4.11}$$

satisfies the upper estimate:

$$r_{FMV}(F) \leq \max\left\{[F]_A, [F]_Q\right\}. \tag{4.12}$$

Remark 9. *The spectrum $\sigma_{FMV}(F)$ may be unbounded if $[F]_Q = \infty$.*

Theorem 3 is analogous to the following statement :

If $\dim X = \infty$ and $F \epsilon C(X)$ is compact, then $0 \epsilon \sigma_{FMV}(F)$.

The FMV-spectrum is one of the most useful nonlinear spectrum from the point of view of applications. The class of stably solvable operators, which is basic in the definition of this spectrum has been introduced in [7]. This spectrum is based on the notion of stable solvability and has several applications of topological character.

5. The Feng spectrum

The notion of F-regularity may be used to define a new spectrum in rather the same way as defining of FMV-spectrum by means of FMV-regularity.

Given $F \epsilon C(X)$, we call the set

(5.1) $\rho_F(F) = \{\lambda \epsilon K : \lambda I - F \text{ is } F - regular\}$

the *Feng resolvent set*, and its complement

(5.2) $\sigma_F(F) = K - \rho_F(F)$

the *Feng spectrum* of F.

Remark 10. *A bounded linear operator on a Banach space is F-regular if and only if it is an isomorphism [2].*

Moreover, we have the following theorem :

Theorem 8.(Feng [2]) *The spectrum $\sigma_F(F)$ is closed.*

Theorem 9. (Feng [2]) *Suppose that $F \epsilon C(X)$ satisfies $[F]_A < \infty$ and $[F]_B < \infty$. Then the spectrum $\sigma_F(F)$ is bounded.*

The proof of Theorem 9 provides, as that of Theorem 7, the upper estimate

(5.3) $r_F(F) \leq \max\{[F]_A, [F]_B\}$

for the *Feng spectral radius*

(5.4) $r_F(F) = \sup\{|\lambda| : \lambda \epsilon \sigma_F(F)\}$.

Remark 11. *The Feng spectrum may be unbounded if $[F]_B = \infty$.*

The next theorem shows that the Feng spectrum has an intermediate role between the Dörfner spectrum and the Furi-Martelli-Vignoli spectrum.

Theorem 10. *For $F \epsilon B(X)$, the inclusions*

(5.5) $\rho_D(F) \subseteq \rho_F(F) \subseteq \rho_{FMV}(F)$ *and* $\sigma_D(F) \supseteq \sigma_F(F) \supseteq \sigma_{FMV}(F)$ *are true.*

Proof. The inclusion $\rho_F(F) \subseteq \rho_{FMV}(F)$ was proved in [2], so we have to prove that $\rho_D(F) \subseteq \rho_F(F)$, the two-sided estimate (3.7) is true which implies that $[\lambda I - F]_b \geq c > 0$ and $[\lambda I - F]_a \geq c > 0$.

Moreover, for any $r > 0$, the set $(\lambda I - F) B_r$ is opened in X, by the continuity of $(\lambda I - F)^{-1}$.

The operator $\lambda I - F$ is k-epi on B_r for $0 < k < c$. Then $v(\lambda I - F) > 0$ and hence $\lambda \epsilon \rho_F F$.

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