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NEW UNKNOWNS ON THE MIDSURFACE OF A NAGHDI'S SHELL MODEL

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To Professor Dan Pascali, at his 70's anniversary

Abstract

In this work, we revisit Naghdi's model for linearly elastic shells, introducing as new primary unknowns for the corresponding quadratic minimization problem, the linearized change of metric, the linearized transverse shear strain tensor, and Naghdi's linearized change of curvature tensor, associated with the displacement and linearized rotation fields of the middle surface of the shell.

1 Introduction

The mathematical analysis of two commonly used two-dimensional linear shell models, viz., the Koiter and Naghdi shell models, concerning existence, uniqueness, and regularity results, is essentially based on a "displacement approach". A key ingredient in proving the existence of a solution for the corresponding minimization problems is a fundamental lemma of J.L. Lions. The positive-definiteness of the two-dimensional elasticity tensor and Korn's inequality on a surface then allow to apply the Lax-Milgram lemma.

Recently, another, and in a sense more realistic, formulation was proposed and studied in [4], [5], [10], intuitively seen as a "deformation approach" yielding "intrinsic equations". On the theoretical side, the first step was made in [4], where the authors considered from this new perspective the pure traction problem of linearized three-dimensional elasticity, with the linearized strain tensor as the primary unknown instead of the displacement itself. Their justification for a new weak version of the St Venant compatibility relations crucially hinges on an H^{-2} -version of a classical theorem of Poincaré, which replaces the lemma of J.L. Lions as the keystone in the classical study.

Key Words: linearly elastic shells, Naghdi's shell model, quadratic minimization problems

The second step undertaken in [5], consisted in an extension to a surface of these three-dimensional compatibility conditions. As such, it was a challenging attempt, since all the mathematical "passages" from three dimensions to two dimensions require some care. This new approach to linear shell theory, more precisely to Koiter's model, proposed as the new unknowns the linearized change of metric and change of curvature tensors, instead of the displacement field.

In addition to its mathematical novelty, such a method could release significant engineering applications. Since the constitutive equations of linear shell theories are invertible, the new minimization problems recast with the new more realistic unknowns can be easily written as minimization problems with the stress resultants and bending moments as the only unknowns, which are of high interest from the mechanical and computational perspectives.

In the present work, we focus on Naghdi's model for linearly elastic thin shells, by using this more versatile "deformation" approach. This article is organized as follows. In Section 2, we recall some necessary notions of differential geometry and mathematical elasticity, and we describe Naghdi's model. Section 3 is devoted to the classical approach, with the primary unknowns as the displacement field of the middle surface S and the linearized rotation field of the unit normal vector along S.

Our basic idea, introduced in Section 4, consists in considering the new primary unknowns as the linearized change of metric, linearized transverse shear strain tensor, and Naghdi's linearized change of curvature tensor, associated with displacement and linearized rotation fields of the middle surface of the shell.

2 Naghdi's shell model

In Naghdi's approach [9], the shell is identified with a one-director Cosserat surface, i.e., a surface endowed with a director field. P.M. Naghdi based his derivation on two *a priori* assumptions (see [9]), one of a mechanical nature about the stresses inside the shell, which is the same as in Koiter's approach, and one of a geometrical nature, different from the Kirchhoff-Love assumption adopted by Koiter.

The mechanical assumption states that if the thickness is small enough, then the state of stress is "approximately" planar and the stresses parallel to the middle surface S vary "approximately linearly" across the thickness, at least "away from the lateral face".

The geometrical assumption states that the points situated on a line normal to S should remain on a line and the lengths are unmodified along this line after the deformation has taken place (as in Koiter's derivation), but this line need no longer remain normal to the deformed middle surface.

This model takes into account membrane deformation and bending of the middle surface, altogether with transverse shear deformations.

The unknowns of the linearized version of this problem are the displacement of the points of the middle surface and the rotation field of the normal vector to the middle surface.

To begin with, we recall some notations and definitions that will be subsequently needed. As is customary in mathematical elasticity theory, Greek indices or exponents: α, β, μ , etc. take their values in the set {1,2}, while Latin indices or exponents: i, j, k, etc. take their values in the set {1,2,3}, and the summation convention on repeated indices and exponents is used; for instance,

$$a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}\gamma_{\alpha\beta} = \sum_{\alpha,\beta,\sigma,\tau=1,2} a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}\gamma_{\alpha\beta} \text{ and } p^i\eta_i = \sum_{i=1,2,3} p^i\eta_i \ .$$

Let $(\mathbf{e}_1\,,\,\mathbf{e}_2\,,\,\mathbf{e}_3)$ be the canonical orthonormal basis of the three-dimensional Euclidean space identified with $\mathbf{R}^3$. We note $\mathbf{a}\cdot\mathbf{b}$ the inner-product of $\mathbf{a},\mathbf{b}\in\mathbf{R}^3$, $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$ the associated Euclidean norm of $\mathbf{a}\in\mathbf{R}^3$, and $\mathbf{a}\wedge\mathbf{b}$ the exterior product of $\mathbf{a},\mathbf{b}\in\mathbf{R}^3$.

Let ω be a domain in $\mathbf{R}^2$, i.e., an open, bounded, connected subset with a Lipschitz-continuous boundary $\gamma = \partial\omega$, the set ω being locally on one side of γ . Let $y = (y_\alpha)$ denote a generic point in the closed set $\bar{\omega}$. The area element in ω is dy and the partial derivatives with respect to the variable y are denoted $\partial_\alpha := \frac{\partial}{\partial y_\alpha}$ and $\partial_{\alpha\beta} := \frac{\partial^2}{\partial y_\alpha \partial y_\beta}$.

Let $\boldsymbol{\theta} : \bar{\omega}\subset\mathbf{R}^2\to\mathbf{R}^3$ be an injective and smooth enough mapping, such that the two vectors $\mathbf{a}_\alpha(y) := \partial_\alpha\boldsymbol{\theta}\,(y)$ are linearly independent at all points $y = (x_1, x_2)\in\bar{\omega}$. They then form the covariant basis of the tangent plane to the surface $S := \boldsymbol{\theta}(\bar{\omega})$ at the point $\boldsymbol{\theta}\,(y)$. The two vectors $\mathbf{a}^\alpha(y)$, defined by the relations $\mathbf{a}^\alpha(y)\cdot\mathbf{a}_\beta(y) = \delta^\alpha_\beta$, form the contravariant basis of the same tangent plane.

At each point $\boldsymbol{\theta}\,(y)$, we also introduce a third vector, normal to S at the point $\boldsymbol{\theta}\,(y)$, with Euclidean norm one, defined by

$$\mathbf{a}_3(y) = \mathbf{a}^3(y) := \frac{\mathbf{a}_1(y)\wedge\mathbf{a}_2(y)}{|\mathbf{a}_1(y)\wedge\mathbf{a}_2(y)|} \ .$$

The triple $\left(\mathbf{a}^1(y)\,,\,\mathbf{a}^2(y)\,,\,\mathbf{a}^3(y)\right)$ is the contravariant basis at $\boldsymbol{\theta}\,(y)$ and $(\mathbf{a}_1(y),\,\mathbf{a}_2(y),\,\mathbf{a}_3(y))$ is the covariant basis at the same point.

A general shell structure can be fully represented by a middle surface geometry $S := \boldsymbol{\theta}(\bar{\omega})$ and the thickness at each point of its middle surface. We deal only with shells of constant thickness 2ε . Note that Koiter's equations are often preferred for the numerical simulations of shells with “small” thickness, while Naghdi's equations are generally favored for those of shells with "moderate" thickness. For a detailed study of shells and of the required differential geometry, we refer to [3].

The reference configuration of a shell is the three-dimensional set $\mathbf{\Theta}(\bar{\Omega})$, where $\Omega = \omega \times]-\varepsilon, \varepsilon[\subset \mathbf{R}^3$, and the mapping $\mathbf{\Theta} : \bar{\Omega} \to \mathbf{R}^3$ is defined by

$$\mathbf{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \mathbf{a}_3(y).$$

The area element along the surface $S = \boldsymbol{\theta}(\bar{\omega})$ is $\sqrt{a}dy$ where $a := \det(a_{\alpha\beta}(y))$, and

$$a_{\alpha\beta}(y) := \mathbf{a}_\alpha(y) \cdot \mathbf{a}_\beta(y) = \partial_\alpha \boldsymbol{\theta}\,(y) \cdot\, \partial_\beta \boldsymbol{\theta}\,(y)$$

are the covariant components of the *metric tensor* of the surface S (also named the first fundamental form of S). The contravariant components of the metric tensor of S are defined by $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$.

Note that the matrix $(a_{\alpha\beta}(y))$ is positive-definite since the vectors $\mathbf{a}_\alpha(y)$ are assumed to be linearly independent. In particular, there exists a positive constant a_0 such that $0 < a_0 \leq a(y)$, for all $y \in \bar{\omega}$.

Every metric notion on a surface, such as lengths of arcs, angles between curves, and surface areas can be expressed in terms of its metric tensor, but a surface is not determined by the three functions $a_{\alpha\beta}$. In addition, one needs to compute curvatures of curves drawn along it. The second fundamental form provides this valuable information.

The covariant and mixed components of the *curvature tensor* of S (also named the second fundamental form of the surface) are respectively defined by

$$b_{\alpha\beta} = \mathbf{a}^3 \cdot \partial_\beta \mathbf{a}_\alpha \text{ and } b_\alpha^\beta = a^{\beta\sigma} b_{\sigma\alpha} \; .$$

Finally, the *Christoffel symbols* of the surface S are defined by $\Gamma^\sigma_{\alpha\beta} = \mathbf{a}^\sigma \cdot \partial_\beta \mathbf{a}_\alpha$.

We assume for simplicity that the shell is made of an *homogeneous isotropic* material and that the reference configuration $\mathbf{\Theta}(\bar{\Omega})$ is a *natural state*, i.e., stress-free. Hence, the material is characterized by its two *Lamé constants* $\lambda > 0$ and $\mu > 0$. The contravariant components $a^{\alpha\beta\sigma\tau}$ of the "*two-dimensional" shell elasticity tensor*, also named the constitutive tensor in engineering literature, are then given by

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}) \; .$$

This tensor is uniformly positive-definite: there exists a constant $C > 0$ such that

$$C \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq a^{\alpha\beta\sigma\tau}(y)\, t_{\sigma\tau}\, t_{\alpha\beta} \ ,$$

for all $y \in \bar{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ of order two.

Assume that the shell is subjected to *applied forces* acting only in its interior and on its "upper" and "lower" faces $\mathbf{\Theta}(\bar{\omega} \times \{\varepsilon\})$ and $\mathbf{\Theta}(\bar{\omega} \times \{-\varepsilon\})$, whose resultant after integration across the thickness of the shell has contravariant components $p^i \in L^2(\omega)$. Assume that the *lateral face of the shell is free*, i.e., the displacement is not subjected to any boundary condition there. This means that we are dealing only with the *pure traction problem* for a linearly elastic shell.

For a Naghdi shell, constant shear deformations are allowed across the thickness of the shell, in the sense that the displacement of the point $(\boldsymbol{\theta}(y) + x_3\mathbf{a}^3(y))$ is given by the vector $\eta_i(y)\,\mathbf{a}^i(y) + x_3 r_\alpha(y)\mathbf{a}^\alpha(y)$, where $\eta_i : \bar{\omega} \to \mathbf{R}^3$ are the covariant components of the displacement field $\eta_i\,\mathbf{a}^i$ of the middle surface S and $r_\alpha : \bar{\omega} \to \mathbf{R}$ are the covariant components of the linearized rotation field $r_\alpha\,\mathbf{a}^\alpha$ of the unit normal vector $\mathbf{a}^3$ along S. Therefore, in this model, there are *five unknowns defined on the middle surface of the shell*: the three functions η_i (as in Koiter's model) and in addition, the two functions r_α .

Given a surface $S = \boldsymbol{\theta}(\bar{\omega})$ and a displacement field $\boldsymbol{\eta} = \eta_i\mathbf{a}^i$ of S with smooth enough covariant components $\eta_i : \bar{\omega} \to \mathbf{R}$, the covariant components of the *linearized change of metric tensor* are given by:

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}]^{lin} = \frac{1}{2}\left(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\beta + \partial_\beta\boldsymbol{\eta}\cdot\mathbf{a}_\alpha\right) .$$

The covariant components of *Naghdi's linearized change of curvature tensor* are given by:

$$\chi_{\alpha\beta}(\boldsymbol{\eta},\mathbf{r}) := [b_{\alpha\beta}(\boldsymbol{\eta},\mathbf{r}) - b_{\alpha\beta}]^{lin} =$$
$$\frac{1}{2}\left(\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\mathbf{a}_3 + \partial_\beta\boldsymbol{\eta}\cdot\partial_\alpha\mathbf{a}_3 + \partial_\alpha\mathbf{r}\cdot\mathbf{a}_\beta + \partial_\beta\mathbf{r}\cdot\mathbf{a}_\alpha\right) .$$

The *linearized transverse shear strain tensor* which is specific to this type of shells has the following covariant components:

$$\delta_{\alpha3}(\boldsymbol{\eta},\mathbf{r}) := \frac{1}{2}\left(\partial_\alpha\eta_3 + b^\sigma_\alpha\eta_\sigma + r_\alpha\right) = \frac{1}{2}\left(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_3 + \mathbf{r}\cdot\mathbf{a}_\alpha\right) .$$

3 The classical approach

The "classical" unknowns $(\boldsymbol{\eta}\,,\mathbf{r}) = ((\eta_i),(r_\alpha))$ belong to the space

$$\mathbf{V}(\omega) := \mathbf{H}^1(\omega) = [H^1(\omega)]^5 \ ,$$

equipped with the norm:

$$\| (\boldsymbol{\eta}\,, \mathbf{r}) \|_{\mathbf{V}(\omega)} := \{ \sum_i \| \eta_i \|^2_{1,\omega} + \sum_\alpha \| r_\alpha \|^2_{1,\omega} \}^{1/2} \;.$$

Obviously, the space $\mathbf{V}(\omega)$ is a Hilbert space.

The energy functional $j_N : \mathbf{V}(\omega) \to \mathbf{R}$ is defined by

$$\begin{aligned} j_N(\boldsymbol{\eta}\,, \mathbf{r}) := \quad & \tfrac{1}{2} \int_\omega [\varepsilon\, a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \;+ \tfrac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \chi_{\sigma\tau}(\boldsymbol{\eta}\,, \mathbf{r}) \chi_{\alpha\beta}(\boldsymbol{\eta}\,, \mathbf{r}) \\ & + 8\varepsilon\, \mu\, a^{\alpha\beta}\, \delta_{\alpha 3}(\boldsymbol{\eta}\,, \mathbf{r}) \delta_{\beta 3}(\boldsymbol{\eta}\,, \mathbf{r})] \sqrt{a}\, \mathrm{d}y - \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}y, \end{aligned}$$

where the functions $\mathbf{p} = (p^i) : \omega \to \mathbf{R}^3$ take into account the given applied body and surface forces acting on the shell, viewed as a three-dimensional body.

The two-dimensional shell equations proposed by Naghdi [1963] then take the form of a quadratic minimization problem:

$$\text{Find } (\boldsymbol{\eta}^*\,, \mathbf{r}^*) = ((\eta_i^*), (r_\alpha^*)) \in \mathbf{V}(\omega) \text{ such that } j(\boldsymbol{\eta}^*, \mathbf{r}^*) = \inf_{(\boldsymbol{\eta}, \mathbf{r}) \in \mathbf{V}(\omega)} j(\boldsymbol{\eta}, \mathbf{r}\,) \;.$$

Define the *space of infinitesimal rigid displacements of the surface S*:

$$\mathbf{Rig}(\omega) := \{\; (\boldsymbol{\eta}\,, \mathbf{r}) \in \mathbf{V}(\omega) \;;\; \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \chi_{\alpha\beta}(\boldsymbol{\eta}, \mathbf{r}) = \delta_{\alpha 3}(\boldsymbol{\eta}, \mathbf{r}) = 0 \text{ in } \mathrm{L}^2(\omega)\}.$$

As shown by Coutris [7] (see also [1, Lemma 3.4]), the space $\mathbf{Rig}(\omega)$ is also given by:

$$\begin{gathered} \mathbf{Rig}(\omega) := \{\; (\boldsymbol{\eta}\,, \mathbf{r}) \in \\ \mathbf{V}(\omega);\; \eta_i \mathbf{a}^i = \mathbf{a} + \mathbf{b} \wedge \boldsymbol{\theta},\; r_\alpha \mathbf{a}^\alpha = \mathbf{b} \wedge \mathbf{a}_3,\; \mathbf{a} \in \mathbf{R}^3,\, \mathbf{b} \in \mathbf{R}^3\}. \end{gathered}$$

We will assume that the linear form $L(\boldsymbol{\eta}, \mathbf{r}) = \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}y$ associated with the applied forces satisfies the compatibility conditions:

$$L(\boldsymbol{\eta}, \mathbf{r}) = 0 \text{ for all } (\boldsymbol{\eta}, \mathbf{r}) \in \mathbf{Rig}(\omega) \;,$$

since these are clearly necessary for the existence of a minimizer of the energy functional j_N over the space $\mathbf{V}(\omega)$. Then, the above minimization problem becomes well-posed over the quotient space $\dot{\mathbf{V}}(\omega) := \mathbf{V}(\omega)/\mathbf{Rig}(\omega)$, viz.,

$$\text{Find } (\dot{\boldsymbol{\eta}}^*, \dot{\mathbf{r}}^*) \in \dot{\mathbf{V}}(\omega) \text{ such that } j_N(\dot{\boldsymbol{\eta}}^*, \dot{\mathbf{r}}^*) = \inf_{(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \in \dot{\mathbf{V}}(\omega)} j(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}\,) \;.$$

In order to establish the existence and uniqueness of a minimizer of the energy functional over the space $\dot{\mathbf{V}}(\omega)$, it suffices, thanks to the positive-definiteness of the two-dimensional shell elasticity tensor, to show that the mapping

$$(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \in \dot{\mathbf{V}}(\omega) \to \| (\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}),\ \boldsymbol{\chi}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}),\ \boldsymbol{\delta}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}})\,) \|_{0,\omega}\ ,$$

where $\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}) = (\gamma_{\alpha\beta}(\dot{\boldsymbol{\eta}}))$, $\boldsymbol{\chi}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) = (\chi_{\alpha\beta}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}))$, $\boldsymbol{\delta}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) = (\delta_{\alpha 3}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}))$, is a norm over the quotient space $\dot{\mathbf{V}}(\omega)$, equivalent to the quotient norm $\| \,.\, \|_{\dot{\mathbf{V}}(\omega)}$.

Define the norms:

$$\| (\boldsymbol{\gamma},\ \boldsymbol{\chi}, \boldsymbol{\delta})(\boldsymbol{\eta}, \mathbf{r}) \|_{0,\omega} = \\ \{ \sum_{\alpha,\beta} \| \gamma_{\alpha\beta}(\boldsymbol{\eta}) \|^2_{0,\omega} + \sum_{\alpha,\beta} \| \chi_{\alpha\beta}(\boldsymbol{\eta}, \mathbf{r}) \|^2_{0,\omega} + \sum_{\alpha} \| \delta_{\alpha 3}(\boldsymbol{\eta}, \mathbf{r}) \|^2_{0,\omega} \}^{1/2}\ ,$$

for all $(\boldsymbol{\gamma},\ \boldsymbol{\chi}, \boldsymbol{\delta}) \in \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2(\omega)$,

$\| (\boldsymbol{\eta}\,, \mathbf{r}) \|_{\mathbf{V}(\omega)} := \{ \sum_i \| \eta_i \|^2_{1,\omega} + \sum_\alpha \| r_\alpha \|^2_{1,\omega} \}^{1/2}$ for all $(\boldsymbol{\eta}, \mathbf{r}) \in \mathbf{V}(\omega)$,

$\| (\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \|_{\dot{\mathbf{V}}(\omega)} := \inf_{(\boldsymbol{\xi}, \mathbf{s}) \in \mathbf{Rig}(\omega)} \| (\boldsymbol{\eta}\,, \mathbf{r}) + (\boldsymbol{\xi}, \mathbf{s}) \|_{\mathbf{V}(\omega)}$
for all $(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \in \dot{\mathbf{V}}(\omega) = \mathbf{V}(\omega)/\mathbf{Rig}(\omega)$.

The first stage is to establish a basic *Korn inequality on a surface* due to [1] over the space $\mathbf{V}(\omega)$:

Theorem 1. Let there be given a domain $\omega \subset \mathbf{R}^2$ and an immersion $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbf{R}^3)$. Then there exists a constant $c = c(\omega, \boldsymbol{\theta}) > 0$ such that

$$\| (\boldsymbol{\eta}, \mathbf{r}) \|_{\mathbf{V}(\omega)} \le c \{ \sum_i \| \eta_i \|^2_{0,\omega} + \sum_\alpha \| r_\alpha \|^2_{0,\omega} + \| (\boldsymbol{\gamma},\ \boldsymbol{\chi}, \boldsymbol{\delta})(\boldsymbol{\eta}, \mathbf{r}) \|^2_{0,\omega} \}^{1/2}$$

for all $(\boldsymbol{\eta}, \mathbf{r}) \in \mathbf{V}(\omega)$.

Proof. The essence of this inequality is that the two Hilbert spaces $\mathbf{V}(\omega)$ and

$$\mathbf{W}(\omega) := \{ (\boldsymbol{\eta}, \mathbf{r}) = ((\eta_i), (r_\alpha)) \in \mathbf{L}^2(\omega) = [L^2(\omega)]^5; \\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega),\ \chi_{\alpha\beta}(\boldsymbol{\eta}, \mathbf{r}) \in L^2(\omega),\ \ \delta_{\alpha 3}(\boldsymbol{\eta}, \mathbf{r}) \in L^2(\omega) \}$$

coincide. The keystone of the proof is a fundamental Lemma of J.L. Lions (see [8]):

Let Ω be a domain in $\mathbf{R}^n$ and let v be a distribution on Ω . Then

$$\left\{ v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega),\ 1 \leq i \leq n \right\} \ \Rightarrow\ v \in L^2(\Omega).$$

The second stage consists in establishing another basic Korn inequality on a surface, this time over the quotient space $\dot{\mathbf{V}}(\omega)$:

Theorem 2. Let there be given a domain $\omega \subset \mathbf{R}^2$ and an immersion $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbf{R}^3)$. Then there exists a constant $\dot{c} = \dot{c}(\omega, \boldsymbol{\theta})$ such that

$$\| \dot{\boldsymbol{\eta}}, \dot{\mathbf{r}} \|_{\dot{\mathbf{V}}(\omega)} \leq \dot{c} \, \| (\boldsymbol{\gamma},\ \boldsymbol{\chi}, \boldsymbol{\delta})(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \|_{0,\omega}$$

for all $(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \in \dot{\mathbf{V}}(\omega) = \mathbf{V}(\omega)/\mathbf{Rig}(\omega)$.

Thanks to the Korn inequality and the positive-definiteness of the elasticity tensor, the existence and the uniqueness of a solution to the Naghdi equations follow.

Remarks. (a) In [2], Blouza and Le Dret extended these results to Naghdi's equations for shells whose middle surface has little regularity, in the sense that the mapping $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbf{R}^3)$ need only be in the space $W^{2,\infty}(\omega; \mathbf{R}^3)$, hereby allowing the middle surface to have discontinuous curvatures. Their main idea was to consider the unknowns of the problem as vector-valued functions instead of identifying them with their covariant or contravariant components, as is usually done in the classical approach.

(b) Non-homogeneous and anisotropic linearly elastic materials are likewise amenable to Naghdi's approach.

(c) Other shell models that also include constant shear deformations differ from Naghdi's model only by some strictly positive factor appearing in front of the "shear strain part" $\int_\omega a^{\alpha\beta}\, \delta_{\alpha 3}(\boldsymbol{\eta}\,, \mathbf{r})\delta_{\beta 3}(\boldsymbol{\eta}\,, \mathbf{r})\sqrt{a}\, \mathrm{d}y$, and the analysis made in [1] applies as well to these cases.

4 Formulation of the existence problem with the new unknowns

Our main objective is to introduce the following *new unknowns*:

$\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ = covariant components of the linearized change of metric tensor,

$\chi_{\alpha\beta}(\boldsymbol{\eta}, \mathbf{r}) \in L^2(\omega)$ = covariant components of the linearized change of curvature tensor,

$\delta_{\alpha 3}(\boldsymbol{\eta}, \mathbf{r}) \in L^2(\omega)$ =covariant components of the linearized transverse shear strain tensor.

The new energy functional will be then written as

$$E_N(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta}) :=$$
$$\frac{1}{2}\int_\omega [\varepsilon\, a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}\gamma_{\alpha\beta} + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}\chi_{\sigma\tau}\chi_{\alpha\beta} + 8\varepsilon\,\mu\, a^{\alpha\beta}\,\delta_{\alpha 3}\delta_{\beta 3}]\sqrt{a}\,\mathrm{d}y - L(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta})\ ,$$

and the new space for the "admissible" unknowns is

$$\mathbf{T}(\omega) := \{(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta}) \in \mathbf{L}^2_{sym}(\omega)\times\mathbf{L}^2_{sym}(\omega)\times\mathbf{L}^2(\omega)\ ;\ \mathbf{R}(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta}) = \mathbf{0} \text{ in } \mathbf{H}^{-2}(\hat{\Omega})\}\ ,$$

where $\mathbf{R}(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta}) = \mathbf{0}$ are *ad-hoc compatibility conditions.* Accordingly, the new minimization problem reads:

Find $(\boldsymbol{\gamma}^*,\boldsymbol{\chi}^*,\boldsymbol{\delta}^*) \in \boldsymbol{T}(\omega)$ such that $E_N(\boldsymbol{\gamma}^*,\boldsymbol{\chi}^*,\boldsymbol{\delta}^*) = \inf_{(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta})\in\mathbf{T}(\omega)} E_N(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta})$
.

To find the compatibility relations $\mathbf{R}(\boldsymbol{\gamma},\boldsymbol{\chi},\boldsymbol{\delta}) = \mathbf{0}$, we will use two basic tools, namely, the weak versions of a classical theorem of Poincaré and of St Venant compatibility conditions.

In [4], P.G. Ciarlet and P. Ciarlet Jr. consider the linearized strain tensor $\mathbf{e}\in\mathbf{L}^2_{sym}(\Omega)$ as the primary unknown instead of the displacement, for the pure traction problem of linearized three-dimensional elasticity. Their main objective was to characterize those symmetric 3×3 matrix fields with components $e_{ij}\in L^2_{sym}(\Omega)$ that can be written as

$$e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)\ ,$$

for some $v = (v_i) \in H^1(\Omega)$, where Ω is a domain in $\mathbf{R}^3$. Their main results are the following, the second one being obtained as a consequence of the first one:

Theorem 3 (Poincaré 's Theorem in weak form). Let Ω be a simply-connected domain in $\mathbf{R}^3$. Let $h_k\in H^{-1}(\Omega)$ be distributions that satisfy $\partial_l h_k = \partial_k h_l$ in $H^{-2}(\Omega)$. Then there exists a function $p\in L^2(\Omega)$, unique up to an additive constant, such that $h_k = \partial_k p$ in $H^{-1}(\Omega)$.

Theorem 4 (weak form of St Venant compatibility conditions). Let Ω be a simply-connected domain in $\mathbf{R}^3$. If $\mathbf{e} = (e_{ij}) \in \mathbf{L}^2_{sym}(\Omega)$ satisfy the compatibility conditions:

$$R_{ijkl}(\mathbf{e}) := \partial_{lj}e_{ik} + \partial_{ki}e_{jl} - \partial_{li}e_{jk} - \partial_{kj}e_{il} = 0 \text{ in } H^{-2}(\Omega)\ ,$$

then there exists $\mathbf{v} = (v_i)\in\mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v})$ in $\mathbf{L}^2_{sym}(\Omega)$, or

$$e_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j) \ ,$$

and all other solutions differ by infinitesimal rigid displacements.

Remark. For smooth functions, these conditions go back to St Venant [1864].

In our case, the analog of Theorem 4 takes the following form:

Theorem 5. Let $\omega \subset \mathbf{R}^2$ be a simply-connected domain and an injective immersion $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbf{R}^3)$. There exist $\varepsilon_0 > 0$ and a linear continuous mapping

$$\mathbf{R} : \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2(\omega) \to \mathbf{H}^{-2}(\hat{\Omega}),$$

where $\hat{\Omega} = \omega \times]-\varepsilon_0, \varepsilon_0[$ and $\mathbf{H}^{-2}(\hat{\Omega}) := (H^{-2}(\hat{\Omega}))^6$, such that a pair of symmetric matrix $[\boldsymbol{\gamma}, \boldsymbol{\chi}] \in S^2_{sym}(\omega) \times S^2_{sym}(\omega)$ and vector $[\boldsymbol{\delta}] \in \mathbf{R}^2$ fields satisfy

$$\mathbf{R}((\gamma_{\alpha\beta}), (\chi_{\alpha\beta}), (\delta_{\alpha 3})) = \mathbf{0} \text{ in } \mathbf{H}^{-2}(\hat{\Omega}),$$

if and only if there exists a vector field $(\boldsymbol{\eta}, \mathbf{r}) \in \mathbf{V}(\omega)$ such that

$$\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}) \ , \ \chi_{\alpha\beta} = \chi_{\alpha\beta}(\boldsymbol{\eta}, \mathbf{r}\,) \ , \ \delta_{\alpha 3} = \delta_{\alpha 3}(\boldsymbol{\eta}, \mathbf{r}\,) \text{ in } L^2(\omega).$$

Then any other solution differs by an infinitesimal rigid displacement.

Surprisingly, this approach provides "as by-products" new proofs of the Korn inequalities on a surface of Theorem 1. First, one proves:

Theorem 6. Let there be given a simply-connected domain $\omega \subset \mathbf{R}^2$ and an immersion $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbf{R}^3)$. Define the space

$$\mathbf{T}(\omega) := \{(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta}) \in \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2(\omega);\ \mathbf{R}(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta}) = \mathbf{0} \text{ in } \mathbf{H}^{-2}(\hat{\Omega})\},$$

Given any element $(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta}) \in \mathbf{T}(\omega)$, there exists a unique equivalent class $(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}})$ in the quotient space $\dot{\mathbf{V}}(\omega)$ such that

$$\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}\,) = \boldsymbol{\gamma}\,, \boldsymbol{\chi}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) = \boldsymbol{\chi} \text{ in } \mathbf{L}^2_{sym}(\omega)\,, \text{ and } \boldsymbol{\delta}(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) = \boldsymbol{\delta} \text{ in } \mathbf{L}^2(\omega).$$

Then the mapping

$$\mathbf{H} : \mathbf{T}(\omega) \to \dot{\mathbf{V}}(\omega) \ , \quad \mathbf{H}(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta}) := (\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}})$$

is an isomorphism between the Hilbert spaces $\mathbf{T}(\omega)$ and $\dot{\mathbf{V}}(\omega)$.

The Korn inequalities are then obtained as simple corollaries of this theorem:

Theorem 7. Let there be given a simply-connected domain $\omega \subset \mathbf{R}^2$ and an immersion $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbf{R}^3)$. The fact that the mapping $\mathbf{H} : \mathbf{T}(\omega) \to \dot{\mathbf{V}}(\omega)$ is an isomorphism implies both Korn's inequalities on a surface, i.e.,

$$\| (\boldsymbol{\eta}, \mathbf{r}) \|_{\mathbf{V}(\omega)} \leq c \{ \sum_i \| \eta_i \|^2_{0,\omega} + \sum_\alpha \| r_\alpha \|^2_{0,\omega} + \| (\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta})(\boldsymbol{\eta}, \mathbf{r}) \|^2_{0,\omega} \}^{1/2}$$

for all $(\boldsymbol{\eta}, \mathbf{r}) \in \mathbf{V}(\omega)$, and

$$\| (\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \|_{\dot{\mathbf{V}}(\omega)} \leq \dot{c} \, \| (\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta})(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \|_{0,\omega} \ \text{ for all } (\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \in \dot{\mathbf{V}}(\omega) = \mathbf{V}(\omega)/\mathbf{Rig}(\omega).$$

We are now in a position to answer the main question addressed here, at least for the *pure traction problem for a linearly elastic shell modeled by Naghdi's equations.* Recall that in this case, the quadratic functional j_N is to be minimized over the whole space $\mathbf{V}(\omega)$, since we did not impose boundary conditions.

Theorem 8. Given a simply-connected domain $\omega \subset \mathbf{R}^2$ and an immersion $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbf{R}^3)$, define the quadratic functional $E_N : \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2_{sym}(\omega) \times \mathbf{L}^2(\omega) \to \mathbf{R}$ by

$$E_N(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta}) :=$$
$$\frac{1}{2} \int_\omega [\varepsilon \, a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau} \gamma_{\alpha\beta} + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} \chi_{\sigma\tau} \chi_{\alpha\beta} + 8\varepsilon \, \mu \, a^{\alpha\beta} \, \delta_{\alpha 3} \delta_{\beta 3}] \sqrt{a} \, \mathrm{d}y - L(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta}) \ .$$

Then the minimization problem

$$\text{Find } (\boldsymbol{\gamma}^*, \boldsymbol{\chi}^*, \boldsymbol{\delta}^*) \in \mathbf{T}(\omega) \text{ such that } E_N(\boldsymbol{\gamma}^*, \boldsymbol{\chi}^*, \boldsymbol{\delta}^*) = \inf_{(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta}) \in \mathbf{T}(\omega)} E_N(\boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\delta})$$

has one and only one solution. Furthermore,

$$(\boldsymbol{\gamma}^*, \boldsymbol{\chi}^*, \boldsymbol{\delta}^*) = (\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}}^*), \boldsymbol{\chi}(\dot{\boldsymbol{\eta}}^*, \dot{\mathbf{r}}^*), \boldsymbol{\delta}(\dot{\boldsymbol{\eta}}^*, \dot{\mathbf{r}}^*)) \ ,$$

where $(\dot{\boldsymbol{\eta}}^*, \dot{\mathbf{r}}^*)$ is the unique solution of the "classical" minimization problem:

$$\text{Find } (\dot{\boldsymbol{\eta}}^*, \dot{\mathbf{r}}^*) \in \dot{\mathbf{V}}(\omega) := \mathbf{V}(\omega)/\mathbf{Rig}(\omega) \text{ such that}$$
$$j_N(\dot{\boldsymbol{\eta}}^*, \dot{\mathbf{r}}^*) = \inf_{(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \in \dot{\mathbf{V}}(\omega)} j(\dot{\boldsymbol{\eta}}, \dot{\mathbf{r}}) \ .$$

The detailed proofs of the results presented here can be found in [6].

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