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POINTWISE AND GLOBAL SUMS AND NEGATIVES OF TRANSLATION RELATIONS

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Abstract

A relation F on a groupoid X to a set Y is called a pointwise translation relation if $F(y) \subset F(x+y)$ for all $x, y \in X$. Moreover, a relation F on a groupoid X is called a global translation relation if $x + F(y) \subset F(x+y)$ for all $x, y \in X$. After establishing some basic properties of the translation relations, we prove some simple theorems about the pointwise and global sums and negatives of translation relations. For instance, we show that if F is a normal and G is an arbitrary global translation relation on a group, then the global sum of F and G coincides with the composition of G and F. And the global negative of F coincides with the inverse of F. Global translation functions and relations play important roles in the extensions and uniformizations of semigroups and groups, respectively. While the pointwise ones seem to have no such applications despite that they are also closely related to additive relations.

1. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a relation on X to Y. In particular, the relations $\Delta_X = \{(x, x) : x \in X\}$ and $X^2 = X \times X$ are called the identity and universal relations on X, respectively.

Namely, if in particular $F \subset X^2$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$.

Key Words: Relations and groupoids, sums and negatives of relations, translation relations.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : (x, y) \in F \}$ and $F(A) = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(y) = \big\{ x \in X : y \in F(x) \big\}$ for all $y \in Y$.

Moreover, if F is a relation on X to Y and G is a relation on Y to Z, then the composition $G \circ F$ of G and F can be defined such that $(G \circ F)(x) = G\big(F(x)\big)$ for all $x \in X$. Note that thus we have $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

If F is a relation on X to Y, then the sets $D_F = F^{-1}(X)$ and $R_F = F(X)$ are called the domain and the range of F, respectively. If in particular $X = D_F$ (and $Y = R_F$), then we say that F is a relation of X into (onto) Y. A relation F of X into Y is called a function if for each $x \in X$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ instead of $F(x) = \{y\}$.

2. A few basic facts on groupoids

If X is a nonvoid set and $+$ is a function of X^2 into X, then the ordered pair $X(+) = (X, +)$ is called a groupoid. In this case, we may also naturally write $x + y = +(x, y)$ for all $x, y \in X$. Moreover, if X is a groupoid, then we may also naturally write $A + B = \big\{ x + y : x \in A, y \in B \big\}$ for all nonvoid $A, B \subset X$. Thus, the family $P(X)$ of all nonempty subsets of X is also a groupoid. Note that if X is, in particular, a group, then $P(X)$ is, in general, only a semigroup with zero element $\{0\}$. However, we can still naturally use the notations $-A = \big\{-x : x \in A\big\}$ and $A - B = A + (-B)$. A nonepts subset A of a groupoid X is called normal if $A + x \subset x + A$ for all $x \in X$. Note that if A is a normal subset of a group X, then we also have $x + A = x + (A - x) + x \subset x + (-x + A) + x = A + x$ for all $x \in X$. Concerning the operations in $P(X)$, we shall only need here the following two theorems.

Theorem 2.1 *If A and B are nonvoid subsets of a group X, then*

$$-(-A) = A \qquad \textit{and} \qquad -(A + B) = -B - A.$$

Remark 2.2 By using the first assertion of this theorem, we can at once see that $-A \subset B$ implies $A \subset -B$.

Theorem 2.3 *If* $(A_i)_{i\in I}$ *is a family of nonempty subsets of a groupoid* X *and* $\phi \neq B \subset X$, *then*

$$\Big(\bigcup_{i\in I} A_i\Big) + B = \bigcup_{i\in I}(A_i + B) \qquad \textit{and} \qquad B + \Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I}(B + A_i).$$

Proof Define a relation F on X such that $F(x) = x + B$ for all $x \in X$. Then, we obviously have

$$F(A) = \bigcup_{x\in A} F(x) = \bigcup_{x\in A}(x + B) = A + B,$$

for all $A \subset X$. Hence, since unions are preserved under relations, it is clear that

$$\Big(\bigcup_{i\in I} A_i\Big) + B = F\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} F(A_i) = \bigcup_{i\in I}(A_i + B).$$

The second assertion of the theorem can be immediately derived from the first one by making use of the dual operation $\oplus$ defined by $x \oplus y = y + x$, for all $x, y \in X$.

Remark 2.4 If $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are families of subsets of a groupoid X, then we can at once see that

$$\bigcup_{i\in I}(A_i + B_i) \subset \bigcup_{i\in I} A_i + \bigcup_{i\in I} B_i.$$

However, if for instance $A_1 = B_2 = \{0\}$ and $A_2 = B_1 = \{1\}$, then

$$(A_1 + B_1) \cup (A_2 + B_2) = \{1\}, \qquad \text{but} \qquad (A_1 \cup A_2) + (B_1 \cup B_2) = \{0, 1, 2\}.$$

Therefore, the corresponding equality is not, in general, true.

3. A few basic facts on sums and negatives of relations

Definition 3.1 If F and G are relations on a set X to groupoid Y and $F+G$ is the relation on X to Y such that

$$(F+G)(x) = F(x) + G(x)$$

for all $x \in X$, then $F+G$ is called the *pointwise sum of F and G.*

While, if F and G are relations on one groupoid X to another Y and

$$F \oplus G = \big\{ (x+z,\, y+w) : \quad (x,\, y) \in F\,, \quad (z,\, w) \in G \big\}\,,$$

then the relation $F \oplus G$ is called *the global sum of F and G.*

Remark 3.2 Thus, we have $D_{F+G} = D_F \cap D_G$ and $D_{F \oplus G} = D_F + D_G$.

The global sum $F \oplus G$ is, in general, quite different from the pointwise one $F+G$, even if $D_{F+G} = D_{F \oplus G}$.

Example 3.3 If X is a group, then $\Delta_X \oplus \Delta_X = \Delta_X$, but $\Delta_X + \Delta_X = \Delta_X$ if and only if $X = \{0\}$.

However, in some very particular cases, the global sum of relations may coincide with the pointwise one.

Example 3.4 Let X be a nonvoid set, and for all $x, y \in X$ define $x+y = x$. Then X is a semigroup such that, for any two relations F and G on X, we have $F \oplus G = F$ whenever $G \neq \emptyset$, and $F+G = F$ whenever $G(x) \neq \emptyset$ for all $x \in D_F$.

Analogously to Definition 3.1, we may also naturally introduce the following

Definition 3.5 If F is a relation on a set X to group Y and $-F$ is the relation on X to Y such that

$$(-F)(x) = -F(x),$$

for all $x \in X$, then $-F$ is called*the pointwise negative of F.* While, if F is a relation on one group X to another Y and

$$\ominus F = \big\{ (-x\,,\, -y) : \quad (x,\, y) \in F \big\}\,,$$

then the relation $\ominus F$ is called the global negative of F.

Remark 3.6 Thus, we have $D_{-F} = D_F$ and $D_{\ominus F} = -D_F$.

The global negative $\ominus F$ is, in general, also quite different from the pointwise one $-F$ even if $D_{\ominus F} = D_{-F}$.

Example 3.7 If X is a group, then $\ominus\Delta_X = \Delta_X$, but $-\Delta_X = \Delta_X$ if and only if $-x = x$ for all $x \in X$.

However, in some very particular cases, the global negative of a relation may coincide with the pointwise one.

Example 3.8 If X is a group such that $-x = x$ for all $x \in X$, then $-F = F$ and $\ominus F = F$ for any relation F on X.

Concerning the images of sets under the relations $-F$, $\ominus F$, $F + G$ and $F \oplus G$, in [1],we have proved the following theorems.

Theorem 3.9 *If F is a relation on a set X to a group Y, then for any $A \subset X$ we have*

$$(-F)(A) = -F(A).$$

Theorem 3.10 *If F is a relation on one group ,X to another Y, then for any $A \subset X$ we have*

$$(\ominus F)(A) = -F(-A).$$

Theorem 3.11 *If F and G are relations on a set X to groupoid Y, then for any $A \subset X$ we have*

$$(F + G)(A) \subset F(A) + G(A).$$

Theorem 3.12 *If F and G are relations on one groupoid X to another Y, then for any $A, B \subset X$ we have*

$$F(A) + G(B) \subset (F \oplus G)(A + B).$$

Corollary 3.13 *If F and G are relations on one groupoid X to another Y, then for any subgroupoid A of X we have*

$$F(A) + G(A) \subset (F \oplus G)(A).$$

The $A = \{x\}$ particular case of the following theorem shows that, in contrast to the intersection convolution [5] the union convolution of relations cannot be introduced.

Theorem 3.14 *If F and G are relations on one groupoid X to another Y, then for any $A \subset X$ we have*

$$(F \oplus G)(A) = \bigcup_{u+v \in A} \big(F(u) + G(v) \big).$$

Corollary 3.15 *If F and G are relations on one group X to a groupoid Y, then for any $A \subset X$ we have*

$$(F \oplus G)(A) = \bigcup_{v \in X} \big(F(A - v) + G(v) \big).$$

Corollary 3.16 *If F and G are relations on one group X to a groupoid Y, then for any $A \subset X$ we have*

$$(F \oplus G)(A) = \bigcup_{u \in X} \big(F(u) + G(-u + A) \big).$$

Remark 3.17 By using the preceding results, one can also easily establish some properties of the images of sets under the relations $F - G = F + (-G)$ and $F \ominus G = F \oplus (\ominus G)$.

4. A few basic facts on pointwise translation relations

Definition 4.1 A relation F on a groupoid X to a set Y is called *a pointwise translation relation* if for any $x, y \in X$ we have

$$F(y) \subset F(x+y).$$

Remark 4.2 Thus, we have $X + D_F \subset D_F$. Therefore, $D_F = X + D_F$ whenever X has a zero element 0. Moreover, $D_F = X$ if either $0 \in D_F$ or $D_F \neq \emptyset$ and X is a group.

Remark 4.3 Moreover, it is also worth noticing that, by using the notation $x F y$ instead of $y \in F(x)$, the inclusion $F(y) \subset F(x+y)$ can be expressed by saying that $y F z$ implies $(x+y) F z$.

Example 4.4 If F is a relation on a groupoid X to a set Y such that $F(x) = Z$ for some $Z \subset Y$ and all $x \in X$, then F is a pointwise translation relation on X to Y.

Moreover, we can also easily establish the following

Theorem 4.5 *If F is a relation on a group X to a set Y, then the following assertions are equivalent:*
(1) F is a pointwise translation; (2) $F(x) = F(0)$, for all $x \in X$.

Proof If the assertion (1) holds, then

$$F(0) \subset F(x+0) = F(x) \qquad \text{and} \qquad F(x) \subset F(-x+x) = F(0),$$

for all $x \in X$. Therefore, the assertion (2) also holds. Moreover, by Example 4.4, the converse implication $(2) \Longrightarrow (1)$ is also true.

Remark 4.6 In this respect, it is also worth mentioning that if F is a relation on a group X to a set Y such that $F(x+y) \subset F(y)$ for all $x, y \in X$, then we also have $F(x) = F(0)$ for all $x \in X$. Therefore, F is a pointwise translation relation on X to Y.

Example 4.7 If $X = [\,0\,,\,+\infty\,]$ and F is a relation on X such that $F\,(x) = [\,0\,,\,x\,]$ for all $x \in X$, then F is a pointwise translation (and a total order) relation on X such that $F\,(x) \neq F\,(y)$ for all $x,\,y \in X$ with $x \neq y$.

Example 4.8 More generally, we can also note that if $X = [\,0\,,\,+\infty\,]$ and d is function on Y^2 to X for some nonvoid set Y, and moreover $y \in Y$ and F is a relation on X to Y such that $F\,(x) = \big\{\, z \in Y : \;\; d\,(\,y\,,\,z\,) \leq x \,\big\}$ for all $x \in X$, then F is a pointwise translation relation on X to Y.

Theorem 4.9 *If F is a pointwise translation relation on a groupoid X to a set Y, then for any $A\,,\,B \subset X$, with $A \neq \emptyset$, we have*

$$F\,(B) \subset F\,(\,A + B\,)\,.$$

Moreover, if X is a group, then the corresponding equality is also true.

textbfProof If $z \in F\,(B)$, then there exists $y \in B$ such that $z \in F\,(y)$. Moreover, by choosing $x \in A$, we can see that $F\,(y) \subset F\,(\,x + y\,) \subset F\,(\,A + B\,)$. Therefore, $z \in F\,(\,A + B\,)$ is also true. Consequently, first statement of the theorem is true. To prove the second statement, note that if X is a group, then by Theorem 4.5, for any nonvoid subset C of X, we have

$$F\,(C) = \bigcup_{x \in C} F\,(x) = \bigcup_{x \in C} F\,(0) = F\,(0)\,.$$

Theorem 4.10 *If F is a relation on a groupoid X to a set Y, then the following assertions are equivalent:*
(1) *F is a pointwise translation;*
(2) *$x + F^{-1}(y) \subset F^{-1}(y)$ for all $x \in X$ and $y \in Y$.*

Proof If $x \in X$, $y \in Y$ and $z \in F^{-1}(y)$, then $y \in F\,(z)$. Hence, if the assertion (1) holds, it follows that $y \in F\,(\,x+z\,)$. Therefore, $x+z \in F^{-1}(y)$, and thus the assertion (2) also holds. On the other hand, if $x,\,y \in X$ and $z \in F\,(y)$, then $y \in F^{-1}(z)$. Hence, if the assertion (2) holds, it follows that $x + y \in F^{-1}(z)$. Therefore, $z \in F\,(\,x + y\,)$, and thus the assertion (1) also holds.

Corollary 4.11 *If* F *is a relation on set* X *to a groupoid* Y, *then* F^{-1} *is a pointwise translation relation on* Y to X *if and only if* $y+F(x) \subset F(x)$ *for all* $x \in X$ *and* $y \in Y$.

Theorem 4.12 *If* F *is a relation on one group* X *to another* Y, *then the following assertions are equivalent:*
(1) F *and* F^{-1} *are pointwise translations;*
(2) $F(x) = \emptyset$ *for all* $x \in X$ *or* $F(x) = Y$ *for all* $x \in X$.

Proof If the assertion (1) holds, then by Theorem 4.5, we have $F(x) = F(0)$ for all $x \in X$. Moreover, by Corollary 4.11, we have $y + F(0) \subset F(0)$ for all $y \in Y$. Hence, we can see that $F(0) = \emptyset$ if $0 \notin F(0)$ and $F(0) = Y$ if $0 \in F(0)$. Therefore, we also have $F(x) = \emptyset$ for all $x \in X$ if $0 \notin F(0)$ and $F(x) = Y$ for all $x \in X$ if $0 \in F(0)$. Thus, the assertion (2) also holds. Moreover, by Example 4.4, the converse implication $(2) \Longrightarrow (1)$ is also true.

Theorem 4.13 *If* F *is a pointwise translation relation on a groupoid* X *to a set* Y *and* G *is a an arbitrary relation on* Y *to a set* Z, *then* $G \circ F$ *is a pointwise translation relation on* X *to* Z.

Proof Namely, for any $x, y \in X$, we have

$$(G \circ F)(y) = G\big(F(y)\big) \subset G\big(F(x+y)\big) = (G \circ F)(x+y).$$

5. Sums and negatives of pointwise translation relations

Theorem 5.1 *If* F *and* G *are pointwise translation relations on one groupoid* X *to another* Y, *then* $F+G$ *is also a pointwise translation relation on* X *to* Y.

Proof Namely, for any $x, y \in X$, we have

$$(F+G)(y) = F(y) + G(y) \subset F(x+y) + G(x+y) = (F+G)(x+y).$$

Theorem 5.2 *If F is a pointwise translation and G is an arbitrary relation on a semigroup X to a groupoid Y, then $F \oplus G$ is also a pointwise translation relation on X to Y.*

Proof Suppose that $x, y \in X$ and $z \in (F \oplus G)(y)$. Then, by Theorem 3.14, we also have $z \in \bigcup_{y=u+v} \big(F(u) + G(v)\big)$. Therefore, there exist $u, v \in X$, with $y = u + v$, such that $z \in F(u) + G(v)$. Hence, by defining $\omega = x+u$ and using the inclusing $F(u) \subset F(\omega)$, we can see that $x+y = \omega+v$ and $z \in F(\omega) + G(v)$. Therefore, $z \in \bigcup_{x+y=\omega+v} \big(F(\omega) + G(v)\big)$. Hence, again by Theorem 3.14, it follows that $z \in (F \oplus G)(x + y)$. Therefore, $(F \oplus G)(y) \subset (F \oplus G)(x + y)$, and thus the required assertion is true.

Theorem 5.3 *If F is a pointwise translation and G is an arbitrary relation on a group X to a groupoid Y, then for any nonvoid subset A of X we have*

$$(F \oplus G)(A) = F(0) + G(X).$$

Proof Namely, by Corollary 3.15 and Theorems 4.5 and 2.3,

Corollary 5.4 *If F and G are pointwise translation relations on a group X to a groupoid Y, then $F \oplus G = F + G$.*

Proof By Theorems 5.3 and 4.5, for any $x \in X$, we have

$$(F \oplus G)(x) = F(0) + G(X) = F(x) + G(x) = (F + G)(x).$$

Analogously to Theorems 5.2 and 5.3, we can easily prove the following two theorems.

Theorem 5.5 *If F is an arbitrary and G is a pointwise translation relation on a commutative semigroup X to a groupoid Y, then $F \oplus G$ is also a pointwise translation relation on X to Y.*

Theorem 5.6 *If F is an arbitrary and G is a pointwise translation relation on a group X to a groupoid Y, then for any nonvoid subset A of X we have*

$$(F \oplus G)(A) = F (X) + G (0) .$$

Corollary 5.7 *If F and G are as in Theorem 5.6,then $F \oplus G$ is also a pointwise translation relation on X to Y.*

Example 5.8 Let X be a set, with $(X) > 1$, and for all $x, y \in X$ define $x + y = x$. Then X is a semigroup such that $\Delta_X \oplus G$ is not a pointwise translation relation on X for any nonvoid relation G on X and $\Delta_X + G$ is not a pointwise translation relation on X for any relation G of X. Namely, under the above assumptions, we have $\Delta_X \oplus G = \Delta_X$ and $\Delta_X + G = \Delta_X$. Moreover, Δ_X is not a pointwise translation relation on X.

Remark 5.9 In contrast to this, note that if X is a nonvoid set and for all $x, y \in X$ we define $x + y = y$, then X is a semigroup such that every relation F on X is a pointwise translation relation.

Finally, we note that, as immediate consequence of Definition 4.1 or Theorem 4.13, we can also state

Theorem 5.10 *If F is a pointwise translation relation on a groupoid X to a group Y, then $-F$ is also a pointwise translation relation on X* to *Y.*

Moreover, by using Theorems 4.5 and and 3.10, we can also easily prove

Theorem 5.11 *If F is a pointwise translation relation on one group X to another Y, then $\ominus F$ is a pointwise translation relation on X to Y such that $\ominus F = -F$.*

Proof By Theorem 4.5, for any $x \in X$, we have $F (x) = F (0)$. Hence, by using Theorem 3.10, we can see that

$$(\ominus F)(x) = -F (-x) = -F (0) = -F (x) = (-F)(x) .$$

Therefore, $\ominus F = -F$, and thus, by Theorem 5.10, $\ominus F$ is also a pointwise translation relation on X to Y.

6. A few basic facts on global translation relations

Definition 6.1 A relation F on a groupoid X is called *a global translation relation* if for any $x, y \in X$ we have

$$x + F(y) \subset F(x + y).$$

Remark 6.2 Thus, we again have $X + D_F \subset D_F$. Therefore, the corresponding assertions of Remark 4.2 can be repeated.

Remark 6.3 Moreover, it is also worth mentioning that, by using the notation $x F y$ instead of $y \in F(x)$, the inclusion $x + F(y) \subset F(x + y)$ can be expressed by saying that $y F z$ implies $(x + y) F (x + z)$.

Example 6.4 Clearly, the identity function Δ_X of a groupoid X is a global translation relation on X.
Moreover, the oder relation $\leq$ of a left-ordered group X [3, p.127] is a global translation relation on X.

Example 6.5 More generally, we can also note that if Y is a subset of a semigroup X and F is a relation on X such that $F(x) = x + Y$ for all $x \in X$, then F is a global translation relation on X.

Moreover, we can also easily establish the following

Theorem 6.6 *If F is a relation on a group X, then the following assertions are equivalent:*
(1) *F is a global translation;* (2) $F(x) = x + F(0)$, for all $x \in X$.

Proof If the assertion (1) holds, then $x + F(0) \subset F(x + 0) = F(x)$ and

$$F(x) = x - x + F(x) \subset x + F(-x + x) = x + F(0),$$

for all $x \in X$. Therefore, the assertion (2) also holds. Moreover, by Example 6.5, the converse implication $(2) \Longrightarrow (1)$ is also true.

Remark 6.7 In this respect, it is also worth noticing that if F is a relation on a group X such that $F(x+y) \subset x+F(y)$ for all $x, y \in X$, then we also have $F(x) = x + F(0)$ for all $x \in X$. Therefore, F is a global translation relation on X.

Example 6.8 If X and F are as in Example 4.7, then F is a global translation relation on X such that $F(x) \neq x + F(0)$ for all $x \in X \setminus \{0\}$.

Example 6.9 More generally, we can also note that if p is a function on a group X to $[0, +\infty]$, and moreover $r \in [0, +\infty]$ and F is a relation on X such that $F(x) = \{ y \in X : \ p(-x+y) \leq r \}$ for all $x \in X$, then F is a global translation relation on X. Therefore, $F(x) = x + F(0)$, for all $x \in X$.

Concerning global translation relations, we shall also need the following theorems which have been mostly proved in [6]

Theorem 6.10 *If F is a global translation relation on a groupoid X, then for any $A, B \subset X$ we have*

$$A + F(B) \subset F(A+B).$$

Moreover, if X is a group, then the corresponding equality is also true.

Now, by calling a global translation relation F on a groupoid X to be *normal* if $F(0)$ is a normal subset of X, we can also prove

Corollary 6.11 *If F is a normal global translation relation on a group X, then for any $A, B \subset X$ we have*

$$F(A+B) = F(A) + B.$$

Proof By Theorem 6.10 and the normality of $F(0)$, we evidently have

$$F(A+B) = A + B + F(0) = A + F(0) + B = F(A) + B.$$

Theorem 6.12 *If F is a global translation relation on a groupoid X, then F^{-1} is also a global translation relation on X.*

Theorem 6.13 *If F is a normal global translation relation on a group X, then for any $A \subset X$ we have*

$$F^{-1}(A) = -F(-A).$$

Remark 6.14 The equality $F^{-1}(0) = -F(0)$ is true even if the global translation relation F is not normal.

Theorem 6.15 *If F and G are global translation relations on a groupoid X, then $G \circ F$ is also a global translation relation on X.*

Theorem 6.16 *If F is a normal and G is an arbitrary global translation relation on a group X, then for any $A, B \subset X$ we have*

$$(G \circ F)(A + B) = F(A) + G(B).$$

Remark 6.17 The equality $(G \circ F)(0) = F(0) + G(0)$ is true even if F is an arbitrary relation on X. Moreover, the equality $(G \circ F)(A) = F(A) + G(0)$ is true even if the global translation relation F is not normal.

Moreover, in addition to Theorem 6.16, we can also prove

Corollary 6.18 *If F and G are as in Theorem 6.16, then $F \circ G = G \circ F$.*

Proof By Corollary 6.11 and Theorem 6.16, for any $x \in X$, we have

$$(F \circ G)(x) = F\big(G(x)\big) = F(0) + G(x) = (G \circ F)(x).$$

7. Sums and negatives of global translation relations

Theorem 7.1 *If* F *is a relation on a groupoid* X, *then the following assertions are equivalent* :
(1) F *is a global translation* , (2) $\Delta_X \oplus F \subset F$.

Proof By Theorem 3.14, we have

$$(\Delta_X \oplus F)(x) = \bigcup_{x=u+v} \big(\Delta_X(u) + F(v) \big) = \bigcup_{x=u+v} \big(u + F(v) \big)$$

for all $x \in X$. Therefore, if the assertion (1) holds, then we have

$$(\Delta_X \oplus F)(x) = \bigcup_{x=u+v} \big(u + F(v) \big) \subset \bigcup_{x=u+v} F(u+v) \subset F(x)$$

for all $x \in X$. Thus, the assertion (2) also holds. While if the assertion (2) holds, then we have

$$x + F(y) \subset \bigcup_{x+y=u+v} \big(u + F(v) \big) = \big(\Delta_X \oplus F \big)(x+y) \subset F(x+y)$$

for all $x, y \in X$. Thus, the assertion (1) also holds.

Corollary 7.2 *If* F *is a global translation relation on a groupoid* X *with a zero element, then* $F = \Delta_X \oplus F$.

Proof Namely, we have $F = \{(0, 0)\} \oplus F \subset \Delta_X \oplus F$ for any relation F on X.

By Theorem 4.5, it is clear that we also have the following

Theorem 7.3 *If* F *is a relation on a group* X, *then the following assertions are equivalent* :
(1) F *is a global translation* , (2) $F = \Delta_X + X \times F(0)$.

A simple application of Theorem 7.1 gives the following

Theorem 7.4 *If* F *is a global translation and* G *is an arbitrary relation on* *a semigroup* X, *then* $F \oplus G$ *is also a global translation relation on* X.

Proof By Theorem 7.1, we have

$$\Delta_X \oplus (F \oplus G) = (\Delta_X \oplus F) \oplus G \subset F \oplus G .$$

Therefore, again by Theorem 7.1, the required assertion is true.

Remark 7.5 If X is a group, then Δ_X is a global translation relation on X, but $\Delta_X + \Delta_X$ is a global translation relation on X if and only if $X = \{0\}$.

However, as an immediate consequence of the corresponding definitions, we can also state the following

Theorem 7.6 *If* F *is a global and* G *is a pointwise translation relation on a semigroup* X, *then* $F + G$ *is also a global translation relation on* X.

Proof Namely, for any $x, y \in X$, we have

$$x + (F + G)(y) = x + F(y) + G(y) \subset F(x+y) + G(x+y) = (F + G)(x+y) .$$

Theorem 7.7 *If* F *is a normal and* G *is an arbitrary global translation relation on a group* X, *then*

$$F \oplus G = G \circ F .$$

Proof By Corollary 3.15 and Theorem 6.16, we have

$$(F \oplus G)(x) = \bigcup_{v \in X} \big(F (x - v) + G (v) \big) = \bigcup_{v \in X} (G \circ F)(x) = (G \circ F)(x)$$

for all $x \in X$. Therefore, the required equality is also true.

Remark 7.8 If F and G are as in Theorem 7.7, then we also have

$$(F * G)(x) = \bigcap_{v \in X} \big(F (x - v) + G (v) \big) = \bigcap_{v \in X} (G \circ F)(x) = (G \circ F)(x)$$

for all $x \in X$. Therefore, $F * G = G \circ F$ is also true.

Now, as an immediate consequence of Theorems 7.7 and 6.16, we can also state the following addition to Theorem 3.12.

Theorem 7.9 *If F is a normal and G is an arbitrary global translation relation on a group X, then for any $A, B \subset X$ we have*

$$(F \oplus G)(A + B) = F(A) + G(B).$$

Moreover, as an immediate consequence of of Theorems 3.10, 6.13 and 6.12, we can also state the following

Theorem 7.10 *If F is a normal global translation relation on a group X, then $\ominus F$ is a global translation relation on X such that $\ominus F = F^{-1}$.*

Remark 7.11 If X is a vector space, then Δ_X is a global translation relation on X, but $-\Delta_X$ is a global translation relation on X if and only if $X = \{0\}$.

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