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A CLASS OF DIRICHLET FORMS GENERATED BY PSEUDO DIFFERENTIAL OPERATORS

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Abstract

We prove that under suitable assumptions it is possible to construct some Dirichlet forms starting with pseudo differential operators.

1. Introduction

The notion of Dirichlet form had been introduced by A. Beurling and J. Deny with the aim to generalize Hilbert spaces methods from classical potential theory to more general situations. To any Dirichlet form one can associate a self-adjoint operator, its generator. All properties of the form must be reflected in properties of this generator. If the form is local, this generator is a closed extension of a differential operator. In the case of translation invariant Dirichlet forms one has a complete description of all forms and their generators. One of the most important tool to get these results is the Fourier transform. In the case of a non-local form, it is only little known about non-local generators of Dirichlet forms. N. Jacob and co-workers have shown that under suitable assumptions it is possible to construct Dirichlet forms associated with pseudo differential operators. Starting from [2] and [4], we done a class of non-local Dirichlet forms generated by pseudo differential operators.

2. Sobolev spaces generated by a negative definite function

We denote by $C_0^\infty(\mathbb{R}^n)$ and $C_\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions which have compact support, respective the set of all continuous functions which vanish at infinity on $\mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform:

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x.$$

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The well known formula of Parseval takes place:

$$\int_{\mathbb{R}^n} \varphi\overline{\psi}\mathrm{d}x = \int_{\mathbb{R}^n} \overset{\wedge}{\varphi}\, \overline{\overset{\wedge}{\psi}}\mathrm{d}\xi.$$

If $u \in L^2(\mathbb{R}^n)$ then $\hat{u} \in L^2(\mathbb{R}^n)$ and

$$\|u\|_0 = \|\widehat{u}\|_0,$$

where $\| \ \|_0$ is the norm in $L^2(\mathbb{R}^n)$.

The general form of a *pseudo-differential operator* is

$$p\left(x, D\right)\varphi\left(x\right) = \left(2\pi\right)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\widehat{\varphi}(\xi)\mathrm{d}\xi,$$

for $\varphi \in C_0^\infty\left(\mathbf{R}^n\right)$. $p\left(x,\xi\right)$ is called the *symbol* of this operator.

A function $a : \mathbb{R}^n \to \mathbb{C}$ is said to be *negative definite* if for all $m \in \mathbb{N}$ and $\left(x^1, x^2, ..., x^m\right)$, $x^j \in \mathbb{R}^n$, $1 \leq j \leq m$ and for all m-tuple $(c_1, c_2, ..., c_m) \in \mathbb{C}^m$ we have

$$\sum_{i,j=1}^{m} \left[a\left(x^i\right) + \overline{a\left(x^j\right)} - a\left(x^i - x^j\right)\right] c_i\overline{c_j} \geq 0.$$

Let a be a continuous negative definite function.Then exists $C > 0$ such that

$$| \ a(\xi) \ | \leq C\left(1 + | \ \xi \ |^2\right),$$

holds for all $\xi \in \mathbb{R}^n$ ([1]). If a is a continuous negative definite function then the same holds for $a^{1/2}$.

Definition. Let a be a continuous negative definite function on $\mathbb{R}^n$. We define

$$H_a^{1/2}\left(\mathbb{R}^n\right) := \left\{u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \left(1 + a(\xi)\right) | \ \widehat{u}(\xi) \ |^2 \ \mathrm{d}\xi < \infty\right\}$$
$$\|u \ \|_{1/2,a}^2 := \int_{\mathbb{R}^n} \left(1 + a(\xi)\right) | \ \widehat{u}(\xi) \ |^2 \ \mathrm{d}\xi$$

For $a(\xi) = | \ \xi \ |^{2s}$ we obtain the usual Sobolev spaces. For more details see [6], [7], [8].

PROPOSITION 2.1. Let a be as previous. Then:

1) $H_a^{1/2}\left(\mathbb{R}^n\right)$ is a Hilbert space with the norm $\|u \ \|_{1/2,a}$;

2) for all u from $H_a^{1/2}\left(\mathbb{R}^n\right)$,

$$\|u \ \|_{1/2,a}^2 = \|u\|_0^2 + \|a^{1/2}\left(D\right)u\|_0^2,$$

where

$$a^{1/2}(D)\,u(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} a^{1/2}(\xi)\widehat{u}(\xi)\mathrm{d}\xi;$$

3) $C_0^\infty(\mathbb{R}^n)$ is a dense subspace of $H_a^{1/2}(\mathbb{R}^n)$.

Proposition 2.2. Let a, b be continuous negative definite functions on $\mathbb{R}^n$ and suppose that exist ρ, $C > 0$ such that

$$a(\xi) \geq Cb(\xi),$$

for all ξ with $|\xi| \geq \rho$. Then $H_a^{1/2}(\mathbb{R}^n)$ is continuously embedded into $H_b^{1/2}(\mathbb{R}^n)$. In particular,

$$H^1(\mathbb{R}^n) \subset H_a^{1/2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n).$$

Definition. A *Dirichlet form* on $L^2(\mathbb{R}^n)$, is a closed symmetric non-negative bilinear form B with the domain $D(B)$ such that $u \in D(B)$ implies that $u^+ \wedge 1 \in D(B)$ and

$$B\left(u^+ \wedge 1, u^+ \wedge 1\right) \leq B(u,u).$$

The pair $(D(B), B)$ is called *Dirichlet space*.

Theorem 2.3. (Beurling-Deny)

Let B be a translation invariant symmetric Dirichlet form on $\mathbb{R}^n$. Then there exists a continuous real valued negative definite function a with $a^{-1} \in L^1_{loc}(\mathbb{R}^n)$ such that for all $u, v \in D(B) \cap C_0(\mathbb{R}^n)$ we have

$$B(u,v) = \int_{\mathbb{R}^n} a(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\mathrm{d}\xi.$$

Conversely, any continuous negative definite function with the property stated above defines a translation invariant symmetric Dirichlet form B with domain

$$D(B) = \left\{ u \in L^2(\mathbf{R}^n) \mid \int_{\mathbb{R}^n} a(\xi) \mid \widehat{u}(\xi) \mid^2 \mathrm{d}\xi < \infty \right\}.$$

Assume that $C_0^\infty(\mathbb{R}^n) \subset D(B)$. We define for $u \in C_0^\infty(\mathbb{R}^n)$ a pseudo-differential operator by

$$a(D)\,u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(\xi)\widehat{u}(\xi)\mathrm{d}\xi.$$

We observe

$$\|a(D)u\|_0^2 = \int_{\mathbb{R}^n} a^2(\xi) \mid \widehat{u}(\xi) \mid^2 \mathrm{d}\xi \le C \int_{\mathbb{R}^n} (1+ \mid \xi \mid^2)^2 \mid \widehat{u}(\xi) \mid^2 \mathrm{d}\xi < \infty.$$

Hence if $u \in C_0^\infty(\mathbb{R}^n)$ then $a(D)u \in L^2(\mathbb{R}^n)$. From Plancherel's theorem we obtain:

$$\int_{\mathbb{R}^n} a(\xi)\overline{\widehat{u}(\xi)}\widehat{v}(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^n} \overline{a(D)u(x)}v(x)\,\mathrm{d}x.$$

It follows that

$$B(u,v) = (a(D)u,v)_0\,, u,v \in C_0^\infty(\mathbb{R}^n)\,.$$

THEOREM 2.4. The Hilbert space $H_a^{1/2}(\mathbb{R}^n)$ with the bilinear form $(.,.)_{1/2,a}$ is a Dirichlet space.

Proof. Using the fact that

$$(u,v)_{1/2,a} = \int_{\mathbb{R}^n} (1+a(\xi))\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\mathrm{d}\xi$$

and Theorem 2.3, the assertion holds immediatly.

3. A class of Dirichlet forms

Let $m \in \mathbf{N}$. For $1 \le j \le m$, $n_j \in \mathbf{N}$, let $a_j : \mathbb{R}^{n_j} \to \mathbb{R}$ be a continuous negative definite function

$$a_j(\xi_j) = \int_{\mathbb{R}^{n_j}\setminus\{0\}} (1-\cos(\xi_j \cdot x_j))\,\mathrm{d}\widetilde{\mu}_j(x_j)\,, \quad \xi_j := \xi_{n_j} \in \mathbb{R}^{n_j},$$

where $\widetilde{\mu}_j$ is a positive finite symmetric measure on $\mathbb{R}^{n_j} \setminus \{0\}$. We denote by

$$n := n_1 + n_2 + ... + n_m.$$

The image of $\widetilde{\mu}_j$ under the mapping

$$T_j : \mathbb{R}^{n_j} \to \mathbb{R}^n, \quad x_j \to (0,...,0,x_j,0,...,0)$$

is denoted by μ_j. Let $b_j \in L^\infty(\mathbb{R}^n)$ be independent of x_j. We denote by

$$x_j' = (x_1,...,x_{j-1},x_{j+1},...,x_m)\,,\ x_k := x_{n_k},\ 1 \le k \le m,$$

and we identify

$$\mathbb{R}_j' := \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_j-1} \times \mathbb{R}^{n_j+1} \times ... \times \mathbb{R}^{n_m}$$

with a subspace of $\mathbb{R}^n$; $b_j = b_j\left(x'_j\right)$.

Let $a : \mathbb{R}^n \to \mathbb{R}$ be the continuous negative definite function

$$a(\xi) = \sum_{j=1}^{m} a_j(\xi_j).$$

In this case

$$H_a^{1/2}(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \mid \int\limits_{\mathbb{R}^n} (1 + a(\gamma)) \mid \widehat{u}(\gamma) \mid^2 \mathrm{d}\xi < \infty \right\}$$

and $\left(H_a^{1/2}(\mathbb{R}^n), (.,.)_{1/2,a}\right)$ is a Dirichlet space. We retain that for $u \in H_a^{1/2}(\mathbb{R}^n)$ the function $(u^+ \wedge 1) \in H_a^{1/2}(\mathbb{R}^n)$.

For $\varphi \in C_0^\infty(\mathbb{R}^n)$ we define

$$L\varphi(x) = \sum_{j=1}^{m} b_j\left(x'_j\right) A_j \varphi(x),$$

where A_j, $1 \le j \le m$,

$$A_j \varphi(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} a_j(\xi_j)\, \widehat{\varphi}(\xi)\, \mathrm{d}\xi.$$

We observe

$$L\varphi(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} \left[\sum_{j=1}^{m} b_j\left(x'_j\right) a_j(\xi_j) \right] \widehat{\varphi}(\xi)\, \mathrm{d}\xi.$$

Since b_j is independent of x_j we can associate with L a symmetric bilinear form defined on $C_0^\infty(\mathbf{R}^n)$ by

$$B(\varphi, \psi) = (L\varphi, \psi)_0 = \sum_{j=1}^{m} \int\limits_{\mathbb{R}^n} b_j\left(x'_j\right) A_j^{1/2} \varphi(x)\, \overline{A_j^{1/2} \psi(x)} dx,$$

where $A_j^{1/2}$ has the symbol $a_j^{1/2}$, $1 \le j \le m$.

Proposition 3.1. For all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ we have

$$|B(\varphi, \psi)| \le C \, \|\varphi\|_{1/2,a} \cdot \|\psi\|_{1/2,a}.$$

Therefore, B has a continuous extension to $H_a^{1/2}(\mathbb{R}^n)$.

Proof. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H_a^{1/2}(\mathbb{R}^n)$, we consider $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned} |B(\varphi,\psi)| &= \left| \sum_{j=1}^{m} \left(b_j(x_j') A_j^{1/2}\varphi(x), A_j^{1/2}\psi(x) \right)_0 \right| \leq \\ &\leq C \sum_{j=1}^{m} \left\| A_j^{1/2}\varphi \right\|_0 \cdot \left\| A_j^{1/2}\psi \right\|_0 . \end{aligned}$$

and

$$\left\| A_j^{1/2}\varphi \right\|_0 \leq C \, \| \varphi \|_{1/2,a}, \quad \left\| A_j^{1/2}\psi \right\|_0 \leq C \, \| \psi \|_{1/2,a} .$$

We assume that there exists a constant $c_0 > 0$ such that

$$b_j(x_j') \geq c_0, (\forall)\, j = 1, ..., n.$$

Proposition 3.2. For all $\varphi \in H_a^{1/2}(\mathbb{R}^n)$ we have

$$B(\varphi,\varphi) \geq 0, \quad B(\varphi,\varphi) \geq c_0 \| \varphi \|_{1/2,a}^2 - c_0 \| \varphi \|_0^2 .$$

Proof. For $\varphi \in H_a^{1/2}(\mathbb{R}^n)$, we have

$$\begin{aligned} B(\varphi,\varphi) &= \sum_{j=1}^{m} \left(b_j(x_j') A_j^{1/2}\varphi, A_j^{1/2}\varphi \right)_0 = \sum_{j=1}^{m} \int_{\mathbb{R}^n} b_j(x_j') \left| A_j^{1/2}\varphi(x) \right|^2 \mathrm{d}x \geq \\ &\geq C_0 \sum_{j=1}^{m} \int_{\mathbb{R}^n} \left| A_j^{1/2}\varphi(x) \right|^2 \mathrm{d}x \geq 0. \end{aligned}$$

Moreover,

$$B(\varphi,\varphi) \geq C_0 \int_{\mathbb{R}^n} \sum_{j=1}^{m} a_j(\xi_j) \mid \widehat{\varphi}(\xi) \mid^2 \mathrm{d}\xi = c_0 \| \varphi \|_{1/2,a}^2 - c_0 \| \varphi \|_0^2 .$$

Thus, B with the domain $D(B) = H_a^{1/2}(\mathbb{R}^n)$ is a closed symmetric bilinear form on $L^2(\mathbb{R}^n)$.

Proposition 3.3. For all $\varphi, \psi \in H_a^{1/2}(\mathbb{R}^n)$ we have

$$B(\varphi,\psi) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \left[\varphi(x+y) - \varphi(x) \right] \left[\psi(x+y) - \psi(x) \right] J(x, \mathrm{d}y) \, \mathrm{d}x,$$

where

$$J\left(x,\mathrm{d}y\right)=\frac{1}{2}\sum_{j=1}^{n}b_j\left(x_j'\right)\mu_j\left(\mathrm{d}y\right)$$

and μ_j are measures on $\mathbb{R}^n\backslash\{0\}$.

Proof. The idea of the proof is the formula

$$B\left(\varphi,\psi\right)=\sum_{j=1}^{n}\int\limits_{\mathbb{R}_j'}b_j\left(x_j'\right)\int\limits_{\mathbb{R}^{n_j}}A_j\varphi\left(x\right)\overline{A_j\psi\left(x\right)}\mathrm{d}x_j\mathrm{d}\widetilde{x}_j$$

and to apply the partial Fourier transform

$$F_j\varphi(\gamma_j,x_j')=\int\limits_{G_j}e^{-i\xi_j\cdot x_j}\varphi(x_j,x_j')\mathrm{d}x_j.$$

THEOREM 3.4. B is a Dirichlet form with the domain $D\left(B\right)=H_a^{1/2}\left(\mathbb{R}^n\right)$ on $L^2(\mathbb{R}^n)$.

Proof. If $\varphi\in H_a^{1/2}\left(\mathbf{R}^n\right)$ then, from Theorem 2.4,

$$\varphi^+\wedge 1\in H_a^{1/2}\left(\mathbb{R}^n\right).$$

The fact that $T_1\left(\varphi\right)=\inf\left(\varphi^+,1\right)$ is a normal contraction, i.e.

$$T_1\left(0\right)=0,\ \ \left|T_1x-T_1y\right|\leq\left|x-y\right|,$$

implies that

$$\left|\left(\varphi^+\wedge 1\right)\left(x+y\right)-\left(\varphi^+\wedge 1\right)\left(x\right)\right|\leq\left|\varphi\left(x+y\right)-\varphi\left(x\right)\right|$$

holds for all $x,y\in\mathbb{R}^n$. It follows that

$$B\left(\varphi^+\wedge 1,\varphi^+\wedge 1\right)\leq B\left(\varphi,\varphi\right).$$

REMARK 3.5. If $n_1=n_2=...=n_m=1$ then we obtain the results from [4].

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