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A SEMILINEAR PERTURBATION OF THE IDENTITY IN HILBERT SPACES

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

In this note we establish an existence and uniqueness result for the equation $u - Au + F(u) = f$ where A is linear, quasi-positive and the nonlinear function F is a Lipschitz monotone operator.

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and $f \in H$.

We consider the equation

$$u - Au + F(u) = f, \tag{1}$$

where $A : H \longrightarrow H$ is a linear operator and $F : H \longrightarrow H$ is a nonlinear Lipschitz monotone operator, i.e.

$$\langle F(x) - F(y), x - y \rangle \geq 0 \tag{2}$$

for all $x, y \in H$ and there exists $M > 0$ such that

$$\|F(x) - F(y)\| \leq M \|x - y\| \tag{3}$$

for all $x, y \in H$. We suppose moreover that A is quasi-positive i.e. there exists $c > 0$ such that

$$\langle Ax, x \rangle \geq c \|Ax\|^2 , \tag{4}$$

for all $x \in H$. It$'$s clear that

Proposition 1. *The linear operator A is bounded and*

$$\|A\|_{L(H)} \leq \frac{1}{c}.$$

Proof. We have $c\|Ax\|^2 \leq \langle Ax, x\rangle = |\langle Ax, x\rangle| \leq \|Ax\| \cdot \|x\|$ for all $x \in H$. It results that

$$\|Ax\| \leq \frac{1}{c}\|x\|$$

for all $x \in H$.

An operator $B : H \longrightarrow H$ is called strongly monotone if there exists $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|x - y\|^2$$

for all $x, y \in H$.

The equation (1) can be written equivalently as

$$Vu = f \tag{5}$$

where $V = I - A + F$ and I is the identity of H.

Proposition 2 *If $c > 1$, then V is a strongly monotone operator.*

Proof. We have

$$\langle (I - A)v, v\rangle = \|v\|^2 - \langle Av, v\rangle \geq \|v\|^2 - \|Av\| \cdot \|v\| \geq$$

$$\|v\|^2 - \frac{1}{c}\|v\|^2 = \frac{c-1}{c}\|v\|^2$$

for all $v \in H$. Consequently we obtain

$$\langle Vx - Vy, x - y\rangle = \langle (I - A)(x - y), x - y\rangle + \langle F(x) - F(y), x - y\rangle \geq$$

$$\langle (I - A)(x - y), x - y\rangle \geq \frac{c-1}{c}\|x - y\|^2$$

for all $x, y \in H$.

It is clear that the operator V is continuous on H. Also, if $c > 1$, then V is coercive(i.e. $\frac{\langle Vx, x\rangle}{\|x\|} \longrightarrow \infty$, when $\|x\| \longrightarrow \infty$) and $\langle Vx - Vy, x - y\rangle > 0$ for all $x, y \in H$ with $x \neq y$, because V is strongly monotone. By the Minty-Browder theorem, we obtain that the equation (5) has a unique solution in H. (see [1] , p. 88).

So we obtained the following

Theorem. *Let $F : H \longrightarrow H$ be a nonlinear Lipschitz monotone operator and $A : H \longrightarrow H$ a linear operator such that*

$$\langle Ax, x\rangle \geq c\|Ax\|^2,$$

for all $x \in H$, with $c \in R$, $c > 1$. Then the equation $u - Au + F(u) = f$ has a unique solution in H for all $f \in H$.

References

[1] H. Brezis, *Analyse fonctionelle-Theorie et applications,* Masson Editeur, Paris 1992.

[2] D. Pascali-S. Sburlan, *Nonlinear mappings of monotone type,* Sijthoff & Noordhoff, Int. Publishers,Alphen aan den Rijn, 1978.

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