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THE LIE ALGEBRA sl(2) AND ITS REPRESENTATIONS

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To Professor Silviu Sburlan, at his 60's anniversary

Abstract

In this paper we present same properties of Lie algebra $sl(2)$, then we prove some relations in $\mathcal{U} = \mathcal{U}(sl(2))$ - the enveloping algebra of $sl(2)$ and determine all finite-dimensional $\mathcal{U}$ -modules.

1. The Lie Algebra sl(2)

To simplify matters, we assume for the rest of this section that the ground field k is the field of complex numbers. The Lie algebra $gl(2) = \mathcal{L}(M_2(k))$ of 2×2-matrices with complex entries is four-dimensional. The four matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis of $gl(2)$. Their commutators are easily computed. We get

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y,$$

$$[I, X] = [I, Y] = [I, H] = 0. \tag{1.1}$$

The matrices of trace zero in $gl(2)$ form the subspace $sl(2)$ spanned by the basis $\{X, Y, H\}$. Relations (1.1) show that $sl(2)$ is an ideal of $gl(2)$ and that there is an isomorphism of Lie algebras

$$gl(2) \cong sl(2) \oplus kI,$$

which reduces the investigation of the Lie algebra $gl(2)$ to that of $sl(2)$.

The enveloping algebra $\mathcal{U} = \mathcal{U}(sl(2))$ of $sl(2)$ is isomorphic to the algebra generated by the three elements X, Y, H with the three relations

$$[X,Y] = H, \quad [H,X] = 2X, \quad [H,Y] = -2Y. \tag{1.2}$$

We prove some relations in $\mathcal{U}$.

Lemma 1.1. *The following relations hold in $\mathcal{U}$ for any $p, q \geq 0$:*

$$X^p H^q = (H - 2pI)^q X^p, \quad Y^p H^q = (H + 2pI) Y^p,$$

$$[X, Y^p] = pY^{p-1}(H - (p-1)I) = p(H + (p-1)I)Y^{p-1},$$

$$[X^p, Y] = pX^{p-1}(H + (p-1)I) = p(H - (p-1)I)X^{p-1}.$$

Proof. One proves the first two relations by an easy double induction on p and q using the relations $XH = (H - 2I)X$ and $YH = (H - 2I)Y$, which is another way of expressing the commutation relation (1.2).

We prove the third relation by induction on p. It trivially holds for $p = 1$. When $p > 1$, we have

$$[X, Y^p] = \left[X, Y^{p-1}\right] Y + Y^{p-1}[X,Y] =$$

$$= (p - 10Y^{p-2}(H - (p-2)I)Y + Y^{p-1}H =$$

$$Y^{p-1}\left((p-1)(H - pI) + H\right) = pY^{p-1}(H - pI + I).$$

We conclude by letting Y^{p-1} jump over H according to the second relation.

As for the last relation, it can be obtained from the third one by applying the automorphism σ of $sl(2)$ defined by

$$\sigma(X) = Y, \quad \sigma(Y) = X, \quad \sigma(H) = -H. \tag{1.3}$$

Proposition 1.2. *The set $\{X^i Y^j H^k\}_{i,j,k \in \mathbb{N}}$ is a basis of $\mathcal{U}(sl(2))$.*

Proof. It is a consequence of the Poincaré-Birkhoff-Witt Theorem. $\square$

We close this section by a few remarks on the centre of $\mathcal{U}$. Let us consider the Casimir element defined as the element

$$C = XY + YX + \frac{H^2}{2} \tag{1.4}$$

of the enveloping algebra $\mathcal{U}$.

Lemma 1.3.3. *The Casimir element C belongs to the centre of $\mathcal{U}$.*

Proof. It is enough to show that the Lie brackets of C with H, X, Y vanish. Now, $[H,C] = [H,X]Y + X[H,Y] + [H,Y]X + Y[H,X] + \frac{1}{2}[H,H^2] =$

$$= 2XY - 2XY - 2YX + 2YX = 0.$$

We also have

$$[X, C] = X[X, Y] + [X, Y]X + \frac{1}{2}[X, H]H + \frac{1}{2}H[X, H] =$$

$$= XH + HX - XH - HX = 0.$$

One shows $[Y, C] = 0$ in a similar fashion. □

Harish-Chandra constructed an isomorphism of algebras from the centre of $\mathcal{U}$ to the polynomial algebra $k[t]$. This isomorphism sends C to the generator t. As a consequence, the Casimir element generates the centre of the enveloping algebra.

1.2. Representations of sl(2)

We now determine all finite-dimensional $\mathcal{U}$-modules. We start with the concept of a highest weight vector.

Definition 2.1. Let V be a $\mathcal{U}$-module and λ be a scalar. A vector $v \in V$, $v \neq 0$ is a said to be of weight $\lambda \in K$ if $Hv = \lambda v$. If, in addition, we have $Xv = 0$, then we say that v is a highest weight vector of weight λ.

Definition 2.2. Any non-zero finite-dimensional $\mathcal{U}$-module V has a highest weight vector.

Proof. Since k is algebraically closed and V is finite-dimensional, the operator H has an eigenvector $w \neq 0$ with eingenvalue $\alpha : Hw = \alpha w$. If $Xw = 0$, then w is a highest weight vector and we are done. If not, let us consider the sequence of vector $X^n w$. By Lemma 1.1 we have

$$H(X^n w) = (\alpha + 2n)(X^n w).$$

Consequently, $(X^n w)_{n \geq 0}$ is a sequence of eingenvectors for H with distinct eingenvalues. As V is finite-dimensional, H can have but a finite number of eingenvalues; consequently, there exists an integer n such that $X^n w \neq 0$ and $X^{n+1} w = 0$. The vector $X^n w$ is a highest weight vector. □

Lemma 2.3. *Let v be a highest weight vector of weight λ. For $p \in \mathbb{N}$, set $v_p = \frac{1}{p!} Y^p v$. Then*

$$Hv_p = (\lambda - 2p)v_p, \quad Xv_p = (\lambda - p + 1)v_{p-1}, \quad Yv_p = (p+1)v_{p+1}.$$

Proof. The first two result from Lemma 1.1 and the third relation is trivial. □

We now state the theorem describing simple finite-dimensional $\mathcal{U}$-modules.

Theorem 2.4. *(a) Let V be a finite-dimensional $\mathcal{U}$-module generated by a highest vector v of weight λ. Then*

(i) The scalar λ is an integer equal to $\dim(V) - 1$.

(ii) Setting $v_p = \frac{1}{p!}Y^p v$, we have $v_p = 0$ for $p > \lambda$ and in addition, $\{v = v_0, v_1, ..., v_\lambda\}$ is a basis for V.

(iii) The operator H acting on V is diagonalizable with the $(\lambda + 1)$ distinct eingenvalues $\{\lambda, \lambda - 2, ..., \lambda - 2\lambda = -\lambda\}$.

(iv) Any other highest weight vector in V is a scalar multiple of v and is of weight λ.

(v) The module V is simple.

(b) Any simple finite-dimensional $\mathcal{U}$-module is generated by a highest weight vector. Two finite-dimensional $\mathcal{U}$-modules generated by highest weight vectors of the same weight are isomorphic.

Proof. *(a)* According to Lemma 2.3, the sequence $\{v_p\}_{p\geq 0}$ is a sequence of eingenvectors for H with distinct eingenvalues. Since V is finite-dimensional, there has to exist an integer n such that $v_n \neq 0$ and $v_{n+1} = 0$. The formulas of Lemma 2.3 then show that $v_m = 0$ for all $m > n$ and $v_m \neq 0$ for all $m \leq n$. We get $n = \lambda$ since we have $0 = Xv_{n+1} = (\lambda - n)v_n$ by Lemma 2.3. The family $\{v = v_0, ..., v\lambda\}$ is free, for it is composed of non-zero eingenvectors for H with distinct eingenvalues. It also generates V; indeed the formulas of Lemma 2.3 show that any element of V, which is generated by v as a module, is a linear combination of the set $\{v_i\}_i$. It results that $\dim V = \lambda + 1$. We have thus proved *(i)* and *(ii)*. The assertion *(iii)* is also a consequence of Lemma 2.3.

(iv) Let v' be another highest weight vector. It is an eigenvector for the action of H; hence, it is a scalar multiple of some vector v_i. But, again by Lemma 2.3 the vector v_i is killed by X if and only if $i = 0$.

(v) Let V' be a non-zero $\mathcal{U}$-submodule of V and let v' be a highest weight vector of V'. Then v' also is a highest weight vector for V. By *(iv)*, v' is a non-zero scalar multiple on V. Therefore v is in V'. Since v generates V, we must have $V \subset V'$, which proves that V is simple.

(b) Let v be a highest weight vector of V; if V is simple, then the submodule generated by v is necessarily equal to V. Consequently, V is generated by a highest weight vector.

If V and V' are generated by highest weight vectors v and v' with the same weight λ, then the linear map sending v_i to v'_i for all i is an isomorphism of $\mathcal{U}$-modules. □

Up to isomorphisms, the simple $\mathcal{U}$-modules are classified by the nonnegative integers: given such an integer n, there exists a unique (up to isomorphism) simple $\mathcal{U}$-module of dimension $n+1$, generated by a highest weight vector of weight n. We denote this module by $V(n)$ and the corresponding morphism of Lie algebras by $\rho(n) : sl(2) \to gl(n+1)$.

For instance, we have $V(0) = k$ and $\rho(0) = 0$, which means that the module $V(0)$ is trivial, as is also the case for all modules of dimension 1.

More generally, any trivial $\mathcal{U}$-module is isomorphic to a direct sum of copies of $V(0)$.

Observe that the morphism $\rho(1) : sl(2) \to gl(2)$ is the natural embedding of $sl(2)$ into $gl(2)$ and that the module $V(2)$ is isomorphic to the adjoint representation of $sl(2)$ via the map sending the highest weight vector v_0 onto X, v_1 onto H and v_2 onto Y.

As for the higher-dimensional module $V(n)$, the generators X, Y and H acts by operators represented by the following matrices in the basis $\{v_0, v_1, ..., v_n\}$:

$$\rho(n)(X) = \begin{pmatrix} 0 & n & 0 & \dots & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

$$\rho(n)(Y) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 2 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & n & 0 \end{pmatrix}$$

$$\text{and} \quad \rho(n)(H) = \begin{pmatrix} n & 0 & \cdots & \cdots & 0 & 0 \\ 0 & n-2 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & -n \end{pmatrix}$$

Let us determine the action of the Casimir element on the simple module $V(n)$.

Lemma 2.5. *Any central element of $\mathcal{U}$ acts by a scalar on the simple module $V(n)$. In particular, the Casimir element C acts on $V(n)$ by multiplication by the scalar $\frac{n(n+2)}{2}$, which is non-zero when $n > 0$.*

Proof. Let Z be a central element in $\mathcal{U}$. It commutes with H which decomposes $V(n)$ into a direct sum of one-dimensional eingenspaces.

Consequently, the operator Z is diagonal with the same eingenvectors $\{v = v_0, ..., v_n\}$ as H. In particular, there exists scalars $\alpha_0, ..., \alpha_n$ such that $Zv_p = \alpha_p v_p$ for all p. Now

$$\alpha_{p+1} Y v_p = \alpha_{p+1}(p+1)v_{p+1} = (p+1)Zv_{p+1} = ZYv_p = YZv_p = \alpha_p Y v_p.$$

Consequently, all scalars α_p are equal, which shows that Z acts as a scalar.

In order to determine the action of the Casimir element on $V(n)$, we have only to compute Cv for the highest weight vector v. By (1.4) and by Lemma 2.3 we get

$$Cv = XYv + YXv + \frac{H^2}{2}v = nv + \frac{n^2}{2}v = \frac{n(n+2)}{2}v. \ \square$$

We finally show that any finite-dimensional $\mathcal{U}$-module is a direct sum of simple $\mathcal{U}$-modules.

Theorem 2.6. *Any finite-dimensional $\mathcal{U}$-module is semisimple.*

Proof. We know that is suffices to show that for any finite-dimensional $\mathcal{U}$-module V and any submodule V' of V, there exists another submodule V'' such that V is isomorphic to the direct sum $V' \oplus V''$. Set $\mathcal{L} = sl(2)$.

1. We shall first prove the existence of such a submodule V'' in the case when V' is of codimension 1 in V. We proceed by induction on the dimension of V'.

If $\dim(V') = 0$, we may take $V'' = V$. If $\dim(V') = 1$, then necessarily V' and V/V' are trivial one-dimensional representations. Therefore there exist a basis $\{v_1 \in V', v_2\}$ of V such that $\mathcal{L}v_1 = 0$ and $\mathcal{L}v_2 \subset V' = kv_1$.

Consequently, we have $[\mathcal{L}, \mathcal{L}]v_i = 0$ for $i = 1, 2$. Formulas (1.2) show that the action of $\mathcal{L}$ on V is trivial. We thus may take for V'' any supplementary subspace of V' in V.

We now assume that $\dim(V') = p > 1$ and that the assertion to be proved holds in all dimensions $< p$. We have the following alternative: either V' is simple, or it is not.

(i) Let us first suppose that V' is not simple; then there exists a submodule V_1 of V' such that $0 < \dim(V_1) < \dim(V') = p$. Let π be the canonical projection of V onto $\overline{V} = V/V_1$. The module $\overline{V'} = \pi(V')$ is a submodule of $\overline{V}$ of codimension one and its dimension is $< p$. This allows us to apply the induction hypothesis and to find a submodule $\overline{V''}$of $\overline{V}$ such that $\overline{V} \cong \overline{V'} \oplus \overline{V''}$. Lifting this isomorphism to V, we get

$$V = V' + \pi^{-1}(\overline{V''}).$$

Now, since $\dim(\overline{V''}) = 1$, the vector space V_1 is a submodule of codimension one of $\pi^{-1}(\overline{V''})$. We again apply the induction hypothesis in order to find a submodule V'' of $\pi^{-1}(\overline{V''})$ such that $\pi^{-1}(\overline{V''}) \cong V_1 \oplus V''$. Let us prove that the one-dimensional submodule V'' has the expected properties, namely $V \cong V' \oplus V''$. Indeed, the above argument implies that $V = V' + V_1 + V''$; now $V_1$is contained in V', which shows that V is the sum of V' and of V''. The formula $\dim(V) = \dim(V') + \dim(V'')$ implies that this is a direct sum.

(ii) If the submodule V' is simple of dimension > 1, then Lemma 2.5 implies that the Casimir element C acts on V' as a scalar $\alpha \neq 0$. Consequently, the operator C/α is the identity on V'. Now V/V' is one-dimensional, hence a trivial module. Therefore C sends V into the submodule V', which means that the map C/α is a projector of V onto V'. As C/α commutes with any element of $\mathcal{U}$,the map C/α is a morphism of $\mathcal{U}$-modules, then the submodule $V'' = Ker(C/\alpha)$ is a supplementary submodule to V'.

2. General case. We are now given two finite-dimensional modules $V' \subset V$ without any restriction on the codimension. We shall reduce the situation to the codimension-one case by considering vector spaces $W' \subset W$ defined as follows: W(resp W') is the subspace of all linear maps from V to V' whose restriction to V' is a homothety (respectiv is zero). It is clear that W' is of codimension one in W. In order to reduce to Part 1, we have to equip W and W' with $\mathcal{U}$-modules structures. We give $Hom(V, V')$ the $\mathcal{U}$-module structure defined by relation $(xf)(v) = xf(v) - f(xv)$ for all $x \in \mathcal{L}$, $v \in V$. Let us check that W and W' are $\mathcal{U}$-submodules. For $f \in W$, let α be the scalar such that $f(v) = \alpha v$ for all $v \in V'$; then for any $x \in \mathcal{L}$, we have

$$(xf)(v) = xf(v) - f(xv) = x(\alpha v) - \alpha(xv) = 0.$$

A similar argument proves that W' is a submodules. Appling Part 1, we get a one-dimensional submodule W'' such that $W \cong W' \oplus W''$. Let f be a generator of W''. By definition, it acts on V' as a scalar $\alpha \neq 0$. It follows that f/α is a projection of V onto V'. To conclude, it suffices to check that f (hence f/α) is a morphism o modules. Now, since W'' is a one-dimensional submodule, it is trivial.

Therefore, we have $xf = 0$ for all $x \in \mathcal{L}$, which by relation $(xf)(v) =$ $= xf(v) - f(xv)$, $x \in \mathcal{L}, v \in V$, translates into $xf(v) - f(xv)$ for all $v \in V$. □

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