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WHAT IS THE VALUE OF TAXICAB(6)? Extended Abstract

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To Professor Silviu Sburlan, at his 60's anniversary

Srinivasa Ramanujan was one of India's greatest, largely self-taught, mathematical geniuses. In January 1913 Ramanujan, after seeing the book *Orders of Infinity*, wrote to its author, the distinguished number theorist G. H. Hardy. Hardy and Littlewod studied the long list of unproved theorems which Ramanujan enclosed with his letter and sent him the letter beginning with the famous paragraph:

„*I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions. Your results seem to me to fall into roughly three classes:*
(1) there are a number of results that are already known, or easily deducible from known theorems;
(2) there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance;
(3) there are results which appear to be new and important ..."

In 1914 Ramanujan arrived in England and for a couple of years Hardy and Ramanjuan had a collaboration which led to important results. The following is a famous story told by Hardy, repeated word for word in various sources. Ramanjuan became ill in 1917 and at the age of 33 lay dying in Putney hospital. Hardy on a visit to his colleague, moved and at a loss for words, could only say "I *came in a taxi 1729, that's a pretty dull number.*" Ramanujan's immediate rejoinder was "*Oh no Hardy. 1729 is the smallest integer which can be expressed in two different ways as the sum of two cubes.*"*

*More accurately, "1729 is the smallest integer which can be expressed as the sum of two positive cubes in two different ways."

This episode prompted Littlewood to say that "every positive integer was one of [Ramanujans'] *personal friends*".

The number 1729 has since become known as the Hardy-Ramanujan Number, even though this feature of 1729 was known more than 300 years before Ramanujan (more precisely, by Bernard Fréenicleê de Bessy in 1657, cf. [1]).

The smallest number expressible as the sum of two cubes in n different ways is called the Taxicab Number $Taxicab(n)$. Hardy and Wright [6] (Theorem 412) have proven that the $Taxicab(n)$ exists for every positive integer n, but the proof is of little use in finding the number.

Trivially, $Taxicab(1) = 2 = 1^3 + 1^3$. The next number, $Taxicab(2) = 1729 = 1^3 + 12^3 = 9^3 + 10^3$, is the Hardy-Ramanujan Number. In 1957, Leech [7] computed

$$Taxicab(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3,$$

in 1991 Rosenstiel, Dardis, and Rosenstiel [10] (see also Butler's program [3]) showed that

$$\begin{aligned} Taxicab(4) = 6963472309248 &= 2421^3 + 19083^3 \\ &= 5436^3 + 18948^3 \\ &= 10200^3 + 18072^3 \\ &= 13322^3 + 16630^3, \end{aligned}$$

and in 1997 Wilson [12] discovered the fifth Taxicab Number,

$$\begin{aligned} Taxicab(5) = 48988659276962496 &= 38787^3 + 365757^3 \\ &= 107839^3 + 362753^3 \\ &= 205292^3 + 342952^3 \\ &= 221424^3 + 336588^3 \\ &= 231518^3 + 331954^3. \end{aligned}$$

In 1998 Bernstein (see [2]) discovered that 391909274215699968 is a six-way sum of cubes and showed that $Taxicab(6) \geq 10^{18}$. In 2002 Rathbun [9] has found a smaller six-way sum of cubes:

$$\begin{aligned} Taxicab(6) \leq 24153319581254312065344 &= 28906206^3 + 582162^3 \\ &= 28894803^3 + 3064173^3 \\ &= 28657487^3 + 8519281^3 \\ &= 27093208^3 + 16218068^3 \\ &= 26590452^3 + 17492496^3 \\ &= 26224366^3 + 18289922^3. \end{aligned}$$

The exact value of $Taxicab(6) = Taxicab(2, 3, 6)$ is not known. For more information see [13, 8, 11, 14].

The main result announced here is that *with probability greater than 99%,* $Taxicab(6) = 24153319581254312065344$. This result was obtained using a sampling approach (see [4, 5]) using $22,500$ randomly chosen numbers in the interval $[10^{18}, 24153319581254312065344]$.

The full details of the method and computation will appear elsewhere.

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