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On two systems of non-resonant nonlocal boundary value problems

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Abstract

In this paper we consider the following two systems of k equations

$$x'' = f(t,x), \quad x(0) = 0, \quad x(1) = \int_0^1 x(s)dg(s)$$

and

$$x'' = f(t,x,x'), \quad x(0) = 0, \quad x'(1) = \int_0^1 x(s)dg(s),$$

where f is a vector function and the integrals are meant in the sense of Riemann-Stieltjes. We give conditions on f and g to ensure the existence of at least one solution for the above problems. Our result extends some results in the references.

1 Introduction

Recently much attention has been paid to the study of nonlocal boundary value problems (BVPs) and their study in the case of linear second-order ordinary differential equations was, as far as we are aware, initiated by Bitsadze and Samarski [2] and later continued by Il'in and Moiseev [7].

Nonlocal BVPs arise in different areas of applied mathematics and physics. Such problems, inter alia, have applications in chemical engineering, thermoelasticity, underground water flow and population dynamics (see for instance [1], [3] and the references therein).

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BVPs with Riemann-Stieltjes integral boundary conditions include as special cases multi-point and integral boundary value problems. Nowadays, the problem of the existence of solutions for various types of nonlocal BVPs is the subject of many papers. For such problems and comments on their importance, we refer the reader to [10], [11], [14], [18], [19], [23] and the references therein.

There are many papers investigating nonlocal BVPs of the second order ordinary differential equation which boundary conditions in the most general form can be written down as $x(0) = 0$, $x(1) = \int_0^1 x(s)dg(s)$ (compare for instance [6], [8], [9], [13], [15], [17], [21], [22]). In the first part of this paper we will present an existence result for problems of this type.

The second problem (which is considered in this paper) is motivated by the work of Webb and Infante [20] and Webb and Zima [21] (In both papers, the Authors also studied other boundary conditions). In [20], the Authors investigated the existence of positive solutions of the following problem

$$-x''(t) = q(t)f(t, x(t)), \quad x(0) = 0, \quad x'(1) = \int_0^1 x(s)dg(s),$$

where $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$, $q : [0, 1] \to \mathbb{R}_+$ and the integral is meant in the sense of Riemann-Stieltjes.

In [21], the Authors studied the existence of positive solutions for nonlinear nonlocal boundary value problem of the form

$$-x''(t) = f(t, x(t)) \quad x(0) = 0, \quad x'(1) = \int_0^1 x(s)dg(s).$$

There was considered the case where $f(t, x)$ is not positive for all positive x but is such that $f(t, x) + \omega^2 x \geq 0$ for $x \geq 0$ for some constant $\omega > 0$.

In this paper we study two nonlocal BVPs. In the first problem we consider the following differential equation

$$x'' = f(t, x), \tag{1}$$

with the initial condition

$$x(0) = 0 \tag{2}$$

and the non-local boundary condition

$$x(1) = \int_0^1 x(s)dg(s), \tag{3}$$

where where $f = (f_1, \ldots, f_k) : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$. The second problem is as follows

$$x'' = f(t, x, x'), \tag{4}$$

with the initial condition

$$x(0) = 0 \tag{5}$$

and the non-local boundary condition

$$x'(1) = \int_0^1 x(s)dg(s), \tag{6}$$

where $f = (f_1, \ldots, f_k) : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$. Moreover, $g = (g_1, \ldots, g_k) : [0,1] \to \mathbb{R}^k$ has bounded variation and

$$\int_0^1 x(s)dg(s) = \left[\int_0^1 x_1(s)dg_1(s), \ldots, \int_0^1 x_k(s)dg_k(s) \right].$$

Speaking precisely, (1), (2), (3) and (4), (5), (6) are the systems of k BVPs

$$\begin{cases} x_i''(t) = f_i(t, x(t)), \\ x_i(0) = 0, \\ x_i(1) = \int_0^1 x_i(s)dg_i(s) \end{cases}$$

and

$$\begin{cases} x_i''(t) = f_i(t, x(t), x'(t)), \\ x_i(0) = 0, \\ x_i'(1) = \int_0^1 x_i(s)dg_i(s), \end{cases}$$

where $t \in [0,1]$, $i = 1, \ldots, k$ and the integrals $\int_0^1 x_i(s)dg_i(s)$ are meant in the sense of Riemann-Stieltjes.

Imposing an a priori bound condition on f and applying Leray-Schauder fixed point theorem, we have proved the existence of at least one solution to the problem (1), (2), (3) and (4), (5), (6). Similar a priori bound conditions one can find for instance in the following papers [4], [5], [12], [16].

2 The existence of solutions for the first BVP

First, let us consider BVP (1), (2), (3). Denote by $C([0,1], \mathbb{R}^k)$ the Banach space of all continuous functions $x : [0,1] \to \mathbb{R}^k$ with the supremum norm.

The following assumptions will be needed:

(i) $f = (f_1, \dots, f_k) : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function;

(ii) there exists $M > 0$ such that $\langle x, f(t,x)\rangle > 0$ for $t \in [0,1]$, $|x| \geq M$, where $|\cdot|$ means the Euclidean norm in $\mathbb{R}^k$ and $\langle \cdot\,, \cdot\rangle$ means the scalar product in $\mathbb{R}^k$ corresponding to the Euclidean norm;

(iii) $g = (g_1, \dots, g_k) : [0,1] \to \mathbb{R}^k$ and $\mathrm{Var}(g) < 1$, where $\mathrm{Var}(g)$ means the variation of g on the interval $[0,1]$;

(iv) $\int_0^1 s\, dg_i(s) \neq 1$, $i = 1, \dots, k$.

By assumption (iv), the considering problem is non-resonant. Hence, there exists an equivalent integral equation. Let us consider the equation (1) and integrate it from 0 to t, we get

$$x'(t) = \int_0^t f(s, x(s))ds + E \tag{7}$$

Now, integrating (7) from 0 to t, we have

$$\begin{aligned} x(t) &= \int_0^t \int_0^s f(u, x(u))duds + Et + F \\ &= \int_0^t (t-u) f(u, x(u))du + Et + F. \end{aligned} \tag{8}$$

By (2), $F = 0 \in \mathbb{R}^k$. Moreover, by (3) and (8), we obtain

$$\int_0^1 (1-u) f(u, x(u))du + E =$$

$$= \int_0^1 \int_0^s (s-u) f(u, x(u))dudg(s) + E \int_0^1 sdg(s).$$

Set

$$\alpha_i := \left(1 - \int_0^1 sdg_i(s)\right)^{-1},$$

$i = 1, \dots, k$. Then, we have

$$E_i = \alpha_i \left[\int_0^1 \int_0^s (s-u) f_i(u, x(u))dudg_i(s) - \int_0^1 (1-u) f_i(u, x(u))du\right].$$

Now, is easy to see that the following Lemma holds:

Lemma 1. *A function $x \in C([0,1],\mathbb{R}^k)$ is a solution of the problem* (1), (2), (3) *only, and only if $x = (x_1, \dots, x_k)$ satisfies the following integral equation*

$$x_i(t) \quad = \quad \int_0^t (t-u) f_i(u, x(u)) du + E_i t,$$

for every $i = 1, \dots, k$.

Given $x \in C([0,1],\mathbb{R}^k)$ let

$$\begin{aligned} (Ax)(t) \quad &= \quad \int_0^t (t-u) f(u, x(u)) du + \\ &+ \quad \alpha t \left[\int_0^1 \int_0^s (s-u) f(u, x(u)) du dg(s) - \int_0^1 (1-u) f(u, x(u)) du \right]. \end{aligned}$$

By assumptions (i), (iii) and (iv) and the classical Arzelà$-$Ascoli theorem, for $A : C([0,1],\mathbb{R}^k) \to C([0,1],\mathbb{R}^k)$, we get

Lemma 2. *The operator A is completely continuous.*

Now we are in the position to establish the main result.

Theorem 1. *Under assumptions* (i)-(iv) *problem* (1), (2) *and* (3) *has at least one solution.*

Proof. Consider the continuous family of BVPs:

$$\varphi''(t) = \lambda f(t, \varphi(t)), \quad \varphi(0) = 0, \quad \varphi(1) = \int_0^1 \varphi(s) dg(s), \tag{9}$$

depending on a parameter $\lambda \in [0,1]$. Then problem (9) is equivalent to an integral equation $\varphi(t) = \lambda A \varphi(t)$. By Lemma 2 we get that operator λA is completely continuous.

Let us consider the homotopy $H : [0,1] \times C([0,1],\mathbb{R}^k) \to C([0,1],\mathbb{R}^k)$ given by

$$H(\lambda, \varphi) = \varphi - \lambda A \varphi$$

in $\Omega = \left\{ x \in C([0,1],\mathbb{R}^k) \mid \|x\| \leq M \right\}$, where M is the positive constant from the assumption (ii).

Now, we shall show that $H(\lambda, \varphi) = 0$ has no solution for $\lambda \in [0,1]$ and φ belonging to the boundary of the ball Ω.

Indeed, $H(0, \varphi) = 0$ has only a trivial solution, which does not lay on the boundary of Ω, so $\lambda \neq 0$.

Suppose that there exists a solution of the equation $H(\lambda, \varphi) = 0$ with $\lambda \in (0,1]$ and $\varphi \in \partial\Omega$. Notice that $\varphi(0) = 0$. Hence $|\varphi(t)| = M$ for some $t_0 \in (0,1]$.

Assume that $|\varphi(1)| = M$. Then, by (3) and (*iii*), we get a contradiction. Indeed, we have

$$M = |\varphi(1)| = \left| \int_0^1 \varphi(s) dg(s) \right| \leq M \mathrm{Var}(g) < M.$$

Hence $|\varphi(t)| = M$ for some $t_0 \in (0,1)$. Let us consider a function $\psi(t) = |\varphi(t)|^2$ and observe that ψ has a maximum equal to M^2 for t_0. Then, by assumption (*ii*), since φ is a solution of (9) and $|\varphi(t_0)| = M$, we get a contradiction. Indeed, we get

$$\begin{aligned} 0 \geq \psi''(t_0) &= 2|\varphi'(t_0)|^2 + 2\lambda \langle \varphi(t_0), \varphi''(t_0) \rangle = \\ &= 2|\varphi'(t_0)|^2 + 2\lambda \langle \varphi(t_0), f(t_0, \varphi(t_0)) \rangle > 0. \end{aligned}$$

Hence homotopy H does not vanish on the boundary of Ω for $\lambda > 0$. Finally, $H(\lambda, \varphi) \neq 0$ for $\lambda \in [0,1]$ and $\varphi \in \partial\Omega$.

Therefore, by the properties of the Leray-Schauder topological degree, we have

$$\deg(I - A, \Omega) = \deg(H(1, \cdot), \Omega) = \deg(H(0, \cdot), \Omega) = \deg(I, \Omega) = 1 \neq 0.$$

Hence A has a fixed point in Ω, i.e. BVP (1), (2) and (3) has a solution in Ω. $\square$

3 The existence of solutions for the second BVP

Now, we shall prove an existence result for BVP (4), (5) and (6).

Denote by $C^1([0,1], \mathbb{R}^k)$ the Banach space of all continuous functions $x : [0,1] \to \mathbb{R}^k$ which have continuous first derivatives x' with the norm

$$\|x\| = \max \left\{ |x(0)|, \sup_{t \in [0,1]} |x'(t)| \right\}. \tag{10}$$

Let $x \in C^1([0,1], \mathbb{R}^k)$ and $\|x\| = M$. Observe that

$$|x(t)| \leq t \sup_{t \in [0,1]} |x'(t)| + |x(0)| \leq M + |x(0)|. \tag{11}$$

The Lemma below, which is a straightforward consequence of the classical Arzelà−Ascoli theorem, gives a compactness criterion in $C^1([0,1], \mathbb{R}^k)$.

Lemma 3. *For a set $Z \subset C^1\left([0,1], \mathbb{R}^k\right)$ to be relatively compact, it is necessary and sufficient that:*
(1) there exists $M > 0$ such that for any $x \in Z$ and $t \in [0,1]$ we have $|x(0)| \leq M$ and $|x'(t)| \leq M$;
(2) for every $t_0 \in [0,1]$ the family $Z' := \{x' \mid x \in Z\}$ is equicontinuous at t_0.

Let us introduce the following assumptions:

(i) $f = (f_1, \ldots, f_k) : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function;

(ii) there exists $M > 0$ such that $\langle y, f(t,x,y)\rangle > 0$ for $t \in [0,1]$, $x \in \mathbb{R}^k$ and $|y| \geq M$;

(iii) $g = (g_1, \ldots, g_k) : [0,1] \to \mathbb{R}^k$ and $\mathrm{Var}(g) < 1$;

(iv) $\int_0^1 s\, dg_i(s) \neq 1$, $i = 1, \ldots, k$.

Proceeding similarly as in the case of the first problem, by assumption (iv), we get

Lemma 4. *A function $x \in C^1([0,1], \mathbb{R}^k)$ is a solution of the problem* (4), (5), (6) *only, and only if x satisfies the following integral equation*

$$\begin{aligned} x_i(t) \quad &= \quad \int_0^t (t-u) f_i(u, x(u), x'(u)) du + \\ &+ \alpha_i t \left[\int_0^1 \int_0^s (s-u) f_i(u, x(u), x'(u)) du dg_i(s) - \int_0^1 f_i(u, x(u), x'(u)) du \right], \end{aligned}$$

for every $i = 1, \ldots, k$.

Let $B : C^1\left([0,1], \mathbb{R}^k\right) \to C^1\left([0,1], \mathbb{R}^k\right)$ is given by

$$\begin{aligned} (Bx)(t) \quad &= \quad \int_0^t (t-u) f(u, x(u), x'(u)) du + \\ &+ \alpha t \left[\int_0^1 \int_0^s (s-u) f(u, x(u), x'(u)) du dg(s) - \int_0^1 f(u, x(u), x'(u)) du \right]. \end{aligned}$$

It is clear that $Bx, (Bx)' : [0,1] \to \mathbb{R}^k$ are continuous. It follows that B is well-defined. Moreover, by assumptions (i) and (iv), (11) and Lemma 3, we get the following

Lemma 5. *The operator B is completely continuous.*

Our main result is given in the following theorem

Theorem 2. *Under assumptions* (i)-(iv) *problem* (4), (5), (6) *has at least one solution.*

Proof. Let us consider the continuous family of BVPs

$$\varphi''(t) = \lambda f(t, \varphi(t), \varphi'(t)), \quad \varphi(0) = 0, \quad \varphi'(1) = \int_0^1 \varphi(s) dg(s), \tag{12}$$

where $\lambda \in [0, 1]$, which is equivalent to an integral equation $\varphi(t) = \lambda B\varphi(t)$. By Lemma 5, λB is completely continuous. Now, let us consider the homotopy $H : [0, 1] \times C^1\left([0, 1], \mathbb{R}^k\right) \to C^1\left([0, 1], \mathbb{R}^k\right)$ given by

$$H(\lambda, \varphi) = \varphi - \lambda B\varphi$$

in $\Omega = \left\{x \in C^1([0, 1], \mathbb{R}^k) \mid \|x\| \le M\right\}$, where M is the positive constant from the assumption (ii).

Now, we show that H does not vanish on the boundary of Ω for $\lambda \in [0, 1]$.

If $H(\lambda, \varphi) = 0$ for $\lambda = 0$ and $\varphi \in \partial\Omega$, then BVP (12) has only a trivial solution.

Suppose that there exists a solution of $H(\lambda, \varphi) = 0$ with $\lambda \in (0, 1]$ and $\varphi \in \partial\Omega$. Notice that $\varphi(0) = 0$. Hence $M = \|\varphi\| = \sup_{t\in[0,1]} |\varphi'(t)|$.

Assume that $|\varphi'(1)| = M$. Then, by (3), (11) and (iv), we have

$$M = |\varphi'(1)| = \left|\int_0^1 \varphi(s) dg(s)\right| \le M\mathrm{Var}(g) < M.$$

Hence $|\varphi'(t)| = M$ for some $t \in [0, 1)$. Let us consider a function $\psi(t) = |\varphi'(t)|^2$ and observe that ψ has a maximum equal to M^2 for certain $t_0 \in [0, 1)$.

If $t_0 = 0$, then, by assumptions (ii) and (iii), since $|\varphi'(0)| = M$, we have

$$\begin{aligned} 0 \ge \psi'(0) &= 2\lambda \langle \varphi'(0), \varphi''(0)\rangle = \\ &= 2\lambda \langle \varphi'(0) \mid f(0, \varphi(0), \varphi'(0))\rangle > 0. \end{aligned}$$

Hence, we get a contradiction.

If $t_0 \in (0, 1)$, then $|\varphi'(t_0)| = M$. Now, by assumptions (ii) and (iii), we get a contradiction

$$0 = \psi'(t_0) = 2\lambda \langle \varphi'(t_0), f(t_0, \varphi(t_0), \varphi'(t_0))\rangle > 0.$$

Finally, $H(\lambda, \varphi) \ne 0$ for $\lambda \in [0, 1]$ and $\varphi \in \partial\Omega$. Hence B has a fixed point in Ω, i.e. BVP (4), (5) and (6) has a solution in Ω. □

4 Example

Consider the following BVP

$$x'' = f(t, x, x'), \quad x(0) = 0, \quad x'(1) = \int_0^1 x(s)dg(s), \tag{13}$$

where

$$f_1(t, x, y) = \left[\sin^2 t + 1\right] \left[\exp(-(x_1^2 + x_2^2)) + 1\right] (y_1 + y_2),$$
$$f_2(t, x, y) = \left[\sin^2 t + 1\right] \left[\exp(-(x_1^2 + x_2^2)) + 1\right] (y_2 - y_1 + 1)$$

and g is arbitrary function satisfying the conditions (iii) and (iv).

Function f is continuous. Let us check if f satisfies the assumption (ii). For any $M > 1$ and $|y| \geq M$, $x \in \mathbb{R}^2$ and $t \in [0, 1]$, we have

$$\begin{aligned} \langle y, f(t, x, y)\rangle &= \left[\sin^2 t + 1\right] \left[\exp(-(x_1^2 + x_2^2)) + 1\right] y_1(y_1 + y_2) + \\ &+ \left[\sin^2 t + 1\right] \left[\exp(-(x_1^2 + x_2^2)) + 1\right] y_2(y_2 - y_1 + 1) > \\ &> \left[\sin^2 t + 1\right] y_1^2 + \left[\sin^2 t + 1\right] y_2^2 + \left[\sin^2 t + 1\right] y_2 > \\ &> \left[\sin^2 t + 1\right] \left(y_1^2 + y_2^2 + y_2\right). \end{aligned}$$

Notice that $y_1^2 + y_2^2 + y_2 > 0$, if $y_2 \in (-\infty, -1] \cup [0, \infty)$. If $y_2 \in (-1, 0)$, we obtain

$$\begin{aligned} \langle y \mid f(t, x, y)\rangle &> \left[\sin^2 t + 1\right] \left(y_1^2 + y_2^2 + y_2\right) \geq \\ &\geq \left[\sin^2 t + 1\right] (M^2 + y_2) > \\ &> \left[\sin^2 t + 1\right] (1 + y_2) > 0. \end{aligned}$$

Hence, there exists at least one nontrivial solution of (13).

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