



Harmonic Maps on Kenmotsu Manifolds

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Abstract

We study in this paper harmonic maps and harmonic morphisms on Kenmotsu manifolds. We also give some results on the spectral theory of a harmonic map for which the target manifold is a Kenmotsu manifold.

1 Introduction

Harmonic maps on Riemannian manifolds have been studied for many years, starting with the paper of J. Eells and J.H. Sampson [2]. Due to their analytic and geometric properties, harmonic maps have become an important and attractive field of research.

The study of harmonic maps on Riemannian manifolds endowed with some structures has its origin in a paper of Lichnerowicz [11], in which he proved that a holomorphic map between Kähler manifolds is not only a harmonic map but also attains the minimum of energy in its homotopy class. After that, Rawnsley [12] studied structure preserving harmonic maps between f -manifolds. Later on Ianoș, Pastore, Gherghe, Chineza and some others (see [7], [6], [1]) studied harmonic maps defined on some almost contact manifolds (i.e. Sasakian, cosymplectic etc.).

The purpose of this paper is to obtain some results concerning harmonic maps and harmonic morphism on Kenmotsu manifolds. After we recall some wellknown facts about harmonic maps, harmonic morphisms and Kenmotsu manifolds (section 2), we prove that any structure preserving map from a

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Kenmotsu manifold to a Kähler manifold is harmonic and that there are no nonconstant harmonic holomorphic maps from a Kähler manifold to a Kenmotsu manifold (section 3). In the same section we give some conditions for a map from a Kenmotsu manifold to a Kähler manifold to be a harmonic morphism.

In the last section we obtain some results on spectral theory of harmonic maps for which the target manifold is a Kenmotsu space-form.

2 Preliminaries

In this section, we recall some well known facts concerning harmonic maps and Kenmotsu manifolds.

Let $F : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds of dimensions m and n respectively. The energy density of F is a smooth function $e(F) : M \rightarrow [0, \infty)$ given by

$$e(F)_p = \frac{1}{2} \text{Tr}_g(F^*h)(p) = \frac{1}{2} \sum_{i=1}^m h(F_*u_i, F_*u_i),$$

for any $p \in M$ and any orthonormal basis $\{u_1, \dots, u_m\}$ of T_pM . If M is a compact Riemannian manifold, the energy $E(F)$ of F is the integral of its energy density:

$$E(F) = \int_M e(F) v_g,$$

where v_g is the volume measure associated with the metric g on M . A map $F \in C^\infty(M, N)$ is said to be harmonic if it is a critical point of the energy functional E on the set of all maps between (M, g) and (N, h) . Now, let (M, g) be a compact Riemannian manifold. If we look at the Euler-Lagrange equations for the corresponding variational problem, a map $F : M \rightarrow N$ is harmonic if and only if $\tau(F) \equiv 0$, where $\tau(F)$ is the tension field which is defined by

$$\tau(F) = \text{Tr}_g \tilde{\nabla} dF,$$

where $\tilde{\nabla}$ is the connection induced by the Levi-Civita connection on M and the F -pullback connection of the Levi Civita connection on N .

We take now a smooth variation $F_{s,t}$ with two parameters $s, t \in (-\epsilon, \epsilon)$ such that $F_{0,0} = F$. The corresponding variation vector fields are denoted by V and W .

The second variation formula of E is:

$$H_F(V, W) = \frac{\partial^2}{\partial s \partial t} (E(F_{s,t}))|_{(s,t)=(0,0)} = \int_M h(J_F(V), W) v_g,$$

where J_F is a second order self-adjoint elliptic operator acting on the space of variation vector fields along F (which can be identified with $\Gamma(F^{-1}(TN))$) and is defined by

$$J_F(V) = - \sum_{i=1}^m (\tilde{\nabla}_{u_i} \tilde{\nabla}_{u_i} - \tilde{\nabla}_{\nabla_{u_i} u_i}) V - \sum_{i=1}^m R^N(V, dF(u_i)) dF(u_i), \quad (1)$$

for any $V \in \Gamma(F^{-1}(TN))$ and any local orthonormal frame $\{u_1, \dots, u_m\}$ on M. Here R^N is the curvature tensor of (N, h) (see [5] for more details on harmonic maps).

Tanno [13] has classified, into three classes, the connected almost contact Riemannian manifolds whose automorphisms groups have the maximum dimensions:

- (1) homogeneous normal contact Riemannian manifolds with constant φ -holomorphic sectional curvature;
- (2) global Riemannian products of a line or a circle and a Kähler space form;
- (3) warped product spaces $L \times_f N$, where L is a line and N a Kähler manifold.

Kenmotsu [9] studied the third class and characterized it by tensor equations. A $(2m+1)$ -dimensional Riemannian manifold (M, g) is said to be a Kenmotsu manifold if it admits an endomorphism φ of its tangent bundle TM, a vector field ξ and a 1-form η , which satisfy:

$$\begin{aligned} \varphi^2 &= -I + \eta \otimes \xi, & \eta(\xi) &= 1, & \varphi(\xi) &= 0, & \eta \circ \varphi &= 0, \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), & \eta(X) &= g(X, \xi), \end{aligned}$$

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X, \quad (2)$$

for any vector fields X, Y on M, where ∇ denotes the Riemannian connection with respect to g.

Example 2.1. Let N be a Kähler manifold, with the kählerian structure (J, h) and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(t) = ce^t$, where $c \in \mathbb{R}$, $c > 0$. Then the warped product $M = \mathbb{R} \times_f N$ is defined as being the manifold $\mathbb{R} \times N$ endowed with the Riemannian metric

$$g_{(t,x)} = \begin{bmatrix} 1 & 0 \\ 0 & f^2(t)h_x \end{bmatrix}.$$

If we put $\xi = \frac{d}{dt}$, $\eta(X) = g(X, \xi)$, and

$$\varphi_{(t,x)} = \begin{bmatrix} 0 & 0 \\ 0 & \exp((t\xi)_*) J_{(x)} \exp((-t\xi)_*) \end{bmatrix},$$

for any point $(t, x) \in \mathbb{R} \times N$ and any vector field X tangent to M , then M is Kenmotsu manifold [9].

3 Harmonic maps and harmonic morphisms on Kenmotsu manifolds

A smooth map $F : M \rightarrow N$ between an almost contact metric manifold $M(\varphi, \xi, \eta, g)$ and an almost hermitian manifold $N(J, h)$ is called to be a (φ, J) -holomorphic map if its differential intertwines the structures, that is $dF \circ \varphi = J \circ dF$. We may ask now if such a map is harmonic in the case in which the domain is a Kenmotsu manifold.

Theorem 3.1. *Let $M(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold, $N(J, h)$ be a Kähler manifold and $F : M \rightarrow N$ be a (φ, J) -holomorphic map. Then F is a harmonic map.*

Proof. We know that F is a harmonic map iff $\tau(F) = 0$. So it will be enough to prove that $\tau(F) = 0$. For a (φ, J) -holomorphic map we have the following formula for its tension field see ([7]),

$$J(\tau(F)) = F_*(\text{div}\varphi) - \text{tr}_g\beta, \quad (3)$$

where $\beta(X, Y) = (\tilde{\nabla}_X J)F_*Y$, $\tilde{\nabla}$ being the connection induced in the pull-back bundle $F^{-1}TN$. Let $\{e_1, \dots, e_m, \varphi e_1, \dots, \varphi e_m, \xi\}$ be a local orthonormal φ -adapted basis on TM , Then we have

$$\begin{aligned} \text{div}\varphi &= \sum_{i=1}^{2m+1} (\nabla_{e_i}\varphi)e_i \\ &= \sum_{i=1}^{2m+1} g(\varphi e_i, e_i)\xi - \eta(e_i)\varphi e_i = 0, \end{aligned}$$

and thus the first term of the formula (3) vanishes. As N is a Kähler manifold then $\nabla J = 0$ and also the second term of the same formula vanishes. Therefore $J(\tau(F)) = 0$ i.e. $\tau(F) \equiv 0$ and F is harmonic. ■

Example 3.1. *Looking at the Example 2.1, it is not difficult to see that the canonical projection $F : M \rightarrow N$ is a (φ, J) -holomorphic map from a Kenmotsu manifold to a Kähler manifold and therefore, from Theorem 3.1, F is a harmonic map.*

A smooth map $F : N \rightarrow M$ between an almost hermitian manifold $N(J, h)$ and an almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is called to be a (J, φ) -holomorphic map if $dF \circ J = dF \circ \varphi$. After the Theorem 3.1, a good question to ask is if such a map defined from a Kähler manifold to a Kenmotsu manifold is harmonic or not.

Theorem 3.2. *Let $N(J, h)$ be a Kähler manifold, $M(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold and $F : N \rightarrow M$ be a (J, φ) -holomorphic map. Then F is harmonic map if and only if F is a constant map.*

Proof. For a (J, φ) -holomorphic map we have a similar formula as (3)

$$\varphi(\tau(F)) = dF(\operatorname{div}J) - \operatorname{tr}_h\beta,$$

where $\beta(X, Y) = (\tilde{\nabla}_X\varphi)(dFY)$.

As N is a Kähler manifold we have

$$\operatorname{div}J = \sum_{i=1}^{2n} (\nabla_{e_i}J)e_i = 0,$$

where $\{e_i\}_{i=1\dots 2n}$ is an orthonormal local basis on TN . Now, using the formula (2) we obtain

$$\operatorname{Tr}_h\beta = \sum_{i=1}^{2n} (\tilde{\nabla}_{e_i}\varphi)(dFe_i) = - \sum_{i=1}^{2n} \eta(F_*e_i)\varphi F_*e_i,$$

and thus

$$\varphi(\tau(F)) = - \sum_{i=1}^{2n} \eta(F_*e_i)\varphi F_*e_i.$$

As F is a (J, φ) -holomorphic map, we have

$$\eta(F_*e_i) = -\eta(F_*J^2e_i) = -\eta(\varphi F_*Je_i) = 0,$$

and thus $\varphi(\tau(F)) = 0$, that is $\tau(F) = \eta(\tau(F))\xi$. We have just obtained that F is harmonic if and only if $g(\tau(F), \xi) = 0$.

On the other hand

$$\begin{aligned}
 g(\tau(F), \xi) &= \sum_{i=1}^{2n} g(\tilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i, \xi) \\
 &= \sum_{i=1}^{2n} g(\tilde{\nabla}_{e_i} F_* e_i, \xi) - \sum_{i=1}^{2n} g(F_* \nabla_{e_i} e_i, \xi) \\
 &= - \sum_{i=1}^{2n} \{g(\tilde{\nabla}_{e_i} \varphi(F_*(Je_i)), \xi) - g(\varphi \circ F_*(J\nabla_{e_i} e_i), \xi)\}.
 \end{aligned}$$

In the last equality we have used that N is Kähler and F is a (J, φ) -holomorphic map. Now the second term vanishes and we get

$$\begin{aligned}
 g(\tau(F), \xi) &= \sum_{i=1}^{2n} g(\nabla_{F_* e_i}^M \varphi F_*(Je_i), \xi) - \sum_{i=1}^{2n} g(\varphi(\nabla_{F_* e_i}^M F_* Je_i), \xi) \\
 &= - \sum_{i=1}^{2n} g(\nabla_{F_* e_i}^M \varphi F_*(Je_i), \xi).
 \end{aligned}$$

Using the formula (2) and the fact that $\eta(F_* e_i) = 0$ we get

$$g(\tau(F), \xi) = \sum_{i=1}^{2n} g(F_* e_i, F_* e_i).$$

Therefore F is a harmonic map iff $g(F_* e_i, F_* e_i) = 0$ for any $i = 1, \dots, 2n$ and thus F is a constant map. ■

Harmonic morphism are maps which pull back germs of real valued harmonic functions on the target manifold to germs of harmonic functions on the domain, that is, a smooth map $F : (M, g) \rightarrow (N, h)$ is a harmonic morphism if for any harmonic function $f : U \rightarrow \mathbb{R}$, defined on an open subset U of N such that $\pi^{-1}(U)$ is non-empty, the composition $f \circ F : \pi^{-1}(U) \rightarrow \mathbb{R}$ is a harmonic function. The following characterization of harmonic morphisms is due to Fuglede and Ishihara: A smooth map F is a harmonic morphism if and only if F is horizontally conformal harmonic map (see [3] and [4]). Now we look for harmonic morphisms defined on Kenmotsu manifolds.

Theorem 3.3. *Let $F : M \rightarrow N$ be a horizontally conformal (φ, J) -holomorphic map from a Kenmotsu manifold $M(\varphi, \xi, \eta, g)$ into an almost Hermitian manifold $N(J, h)$. Then F is a harmonic morphism if and only if N is a semi Kähler manifold.*

Proof. We know that for a horizontally conformal (φ, J) -holomorphic map F from an almost contact metric manifold to an almost hermitian manifold, any two of the following conditions imply the third: (i) $divJ = 0$ (ii) $dF(div\varphi) = 0$ (iii) F is harmonic and so is harmonic morphism see ([7]). Let $\{e_1, \dots, e_m, \varphi e_1, \dots, \varphi e_m, \xi\}$ be a φ - adapted local frame on TM then taking $e_{2m+1} = \xi$

$$div\varphi = \sum_{i=1}^{2m+1} (\nabla_{e_i}\varphi)e_i = \sum_{i=1}^{2m+1} g(\varphi e_i, e_i)\xi - \eta(e_i)\varphi e_i = 0.$$

As F is a horizontally conformal (φ, J) -holomorphic map, it follows that F is a harmonic morphism if and only if $divJ=0$, i.e. N is semi-Kähler. ■

4 Spectral geometry on Kenmotsu manifolds

Let $f : (M, g) \rightarrow (N, h)$ be a harmonic map defined on a compact manifold M . The corresponding Jacobi operator is an elliptic self-adjoint operator which has discrete spectrum of eigenvalues with finite multiplicities, denoted by

$$Spec(J) = \{\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots \uparrow \infty\}.$$

Then the trace $Z(t) = \sum_{j=1}^{\infty} exp(-t\lambda_j)$ of the heat kernel for the operator J has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-m/2} \{a_0(J) + a_1(J)t + a_2(J)t^2 + \dots\} \quad as \quad t \rightarrow \infty. \quad (4)$$

Using the results of Gilkey (see [8]) Urakawa obtained the expressions for the first three coefficients (see [14]):

Theorem 4.1. *For a harmonic map $f : (M^m, g) \rightarrow (N^n, h)$, the first three coefficient of the expansion are given by*

$$a_0(D) = nVol(M, g), \tag{5}$$

$$a_1(D) = \frac{n}{6} \int_M \tau_g v_g + \int_M Tr(R_f)v_g, \tag{6}$$

$$a_2(D) = \frac{n}{360} \int_M (5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2)dv_g + \frac{1}{360} \int_M [-30\|R^{\tilde{\nabla}}\|^2 + 60\tau_g Tr(R_f) + 180Tr(R_f^2)]dv_g, \tag{7}$$

where $R^{\tilde{\nabla}}$ is the curvature tensor of the connection $\tilde{\nabla}$ on the induced bundle, which is defined by $R^{\tilde{\nabla}} = f^*R_h$ (R_h is the Riemann curvature tensor of

(N, h) , R_g, ρ_g, τ_g are the curvature tensor, Ricci tensor, scalar curvature on M respectively, and R_f is the endomorphism of the induced bundle defined by $R_f(V) = \text{Tr}_g f^* R(V, -)$.

The spectral geometry for the Jacobi operators of harmonic maps into a Sasakian or cosymplectic space form was studied by Kang and Kim (see [10]). We study now the spectral geometry for the case when the target manifold is a Kenmotsu space form.

A Kenmotsu manifold with constant φ -sectional curvature c is called a Kenmotsu space form and its curvature tensor R is expressed by

$$\begin{aligned} R(X, Y)Z &= \frac{(c-3)}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{(c+1)}{4} \{g(X, \phi Z)\phi Y - \\ &\quad - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z + \eta(X)\eta(Z)Y - \\ &\quad - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}. \end{aligned}$$

Let $N(c)$ be a $(2n+1)$ -dimensional Kenmotsu space form with constant φ -sectional curvature c . Let $f : (M^m, g) \rightarrow N(c)$ be a harmonic map from a compact Riemannian manifold into Kenmotsu space form. If we make the notations $c_1 = \frac{c-3}{4}$ and $c_2 = \frac{c+1}{4}$, after some long but straightforward computations we get:

$$\begin{aligned} \text{Tr}(R_f) &= \sum_{i=1}^m \sum_{a=1}^{2n+1} h(R(v_a, f_*e_i)f_*e_i, v_a) \\ &= 4(nc_1 + c_2)e(f) - 2c_2(n+1)\|f^*\eta\|^2, \end{aligned} \quad (8)$$

$$\begin{aligned} \text{Tr}(R_f^2) &= \sum_{i,j=1}^m \sum_{a=1}^{2n+1} h(R(v_a, f_*e_i)f_*e_i, R(v_a, f_*e_j)f_*e_j) \\ &= 4[(2n-1)c_1^2 + 4c_1c_2 + c_2^2]e(f)^2 + (c_1^2 + 9c_2^2)\|f^*h\|^2 - \\ &\quad - 6c_1c_2\|f^*\phi\|^2 - 4(c_1c_2 + 4c_2^2)f^*(\eta \times \eta \times g) + \\ &\quad + 2(n+7)c_2^2\|f^*\eta\|^4 - 8(nc_1c_2 + 2c_2^2)e(f)\|f^*\eta\|^2, \end{aligned} \quad (9)$$

$$\begin{aligned} \|R^{\tilde{\nabla}}\|^2 &= \sum_{i,j=1}^m \sum_{a,b=1}^{2n+1} h(R_h(f_*e_i, f_*e_j)v_a, v_b)h(R_h(f_*e_i, f_*e_j)v_a, v_b) \\ &= -2(c_1^2 + c_2^2)\|f^*h\|^2 + 8c_1c_2f^*(\eta \times \eta \times g) + \\ &\quad + 8(c_1^2 + c_2^2)e(f)^2 - 16c_1c_2e(f)\|f^*\eta\|^2 + \\ &\quad + [12c_1c_2 + 8(n+1)\beta^2]\|f^*\phi\|^2, \end{aligned} \quad (10)$$

for any local orthonormal basis $\{e_i : i = 1 \dots m\}$ and $\{v_a : a = 1 \dots 2n+1\}$ on M and N respectively. In the above formulas we have used the following

notations:

$$\begin{aligned} \|f^*\eta\|^2 &= \sum_{i=1}^m \eta(f_*e_i)\eta(f_*e_i), \\ f^*(\eta \times \eta \times g) &= \sum_{i,j=1}^m h(f_*e_i, f_*e_j)\eta(f_*e_i)\eta(f_*e_j), \\ \|f^*\phi\|^2 &= \sum_{i=1}^m h(f_*e_i, \varphi f_*e_j), \\ \|f^*h\|^2 &= \sum_{i=1}^m h(f_*e_i, f_*e_j). \end{aligned}$$

Finally, substituting (8) ~ (10) into (5) ~ (7), we get

Theorem 4.2. *Let $f : (M, g) \rightarrow N(c)$ be a harmonic map from a m -dimensional compact Riemannian manifold (M, g) into a $(2n+1)$ -dimensional Kenmotsu space form $N(c)$. Then the coefficients $a_0(J_f)$, $a_1(J_f)$ and $a_2(J_f)$ of the asymptotic expansion for the Jacobi operator J_f are respectively given by*

$$a_0(J_f) = (2n + 1)Vol(M, g), \tag{11}$$

$$\begin{aligned} a_1(J_f) &= \frac{2n + 1}{6} \int_M \tau_g v_g + 4(nc_1 + c_2)E(f) - \\ &\quad - 2c_2(n + 1) \int_M \|f^*\eta\|^2 v_g, \end{aligned} \tag{12}$$

$$\begin{aligned} a_2(J_f) &= \frac{2n + 1}{360} \int_M (5\tau_g^2 - 2\|\rho_g\|^2 + \|R_g\|^2)dv_g + \\ &\quad + \frac{2}{3} \int_M (c_1^2 + 7c_2^2)\|f^*h\|^2 v_g - \\ &\quad - \frac{8}{3} \int_M (c_1c_2 + 3c_2^2)f^*(\eta \times \eta \times g)v_g + \\ &\quad + \frac{4}{3} \int_M [(3n - 2)c_1^2 + c_2^2 + 6c_1c_2]e(f)^2 v_g - \\ &\quad - \frac{2}{3} \int_M [6c_1c_2 + (n + 1)c_2^2]\|f^*\phi\|^2 v_g + \int_M (n + 7)c_2^2\|f^*\|^4 v_g + \\ &\quad + \frac{2}{3} \int_M (c_1n + c_2)\tau_g e(f)v_g - \frac{1}{3} \int_M c_2(n + 1)\|f^*\eta\|^2 \tau_g v_g + \\ &\quad + \frac{4}{3} \int_M [(1 - 3n)c_1c_2 - 6c_2^2]\|f^*\eta\|^2 e(f)v_g. \end{aligned} \tag{13}$$

A first application of the above theorem is the following

Corollary 4.1. *Let f, \tilde{f} be two harmonic maps from a compact Riemannian manifold M into a Kenmotsu space form $N(c)$. If $Spec(J_f) = Spec(J_{\tilde{f}})$ and the structure vector field ξ is normal to $f(M)$ and $\tilde{f}(M)$, then $E(f) = E(\tilde{f})$.*

Proof. Since the vector field ξ is normal to $f(M)$ and $\tilde{f}(M)$, then

$$\|f^*\eta\|^2 = \sum_{i=1}^m \eta(f_*e_i)\eta(f_*e_i) = \sum_{i=1}^m g(f_*e_i, \xi)g(f_*e_i, \xi) = 0$$

and similar for $\|\tilde{f}^*\eta\|$. On the other hand, as $\text{Spec}(J_f) = \text{Spec}(J_{\tilde{f}})$ we have $a_1(f) = a_1(\tilde{f})$, put and we get $E(f) = E(\tilde{f})$. ■

Let $N(\varphi, \xi, \eta, h)$ be a $(2n+1)$ -dimensional Kenmotsu manifold and $f : M \rightarrow N$ be isometric immersion of a Riemannian manifold (M, g) into N . f is said to be an invariant immersion if $\varphi(f_*TM) \subset f_*TM$ and ξ is tangent to $f(M)$ everywhere on M . If f is an invariant immersion then it is minimal. Indeed, any invariant submanifold M with induced structure tensors, which will be denoted by same letters (φ, ξ, η, g) as \overline{M} , is also a Kenmotsu manifold. Using the Gauss formula, is not difficult to prove that $B(X, \xi) = 0$ and $B(X, \varphi Y) = B(\varphi X, Y) = \varphi B(X, Y)$ for any vector fields X and Y tangent to M . Here we have denoted by B the second fundamental form of M . Now, for any $x \in M$ and any φ -adapted basis of $T_x(M)$ $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi\}$ we have

$$\begin{aligned} \sum_{i=1}^n [B(e_i, e_i) + B(\varphi e_i, \varphi e_i)] + B(\xi, \xi) &= \sum_{i=1}^n [B(e_i, e_i) + \varphi^2 B(e_i, e_i)] + 0 = \\ &= \sum_{i=1}^n [B(e_i, e_i) - B(e_i, e_i)] = 0, \end{aligned}$$

that is f is minimal. On the other hand any isometric immersion is harmonic if and only if is minimal.

Using the above corollary and the asymptotic expansions we get the following

Proposition 4.1. *Let f and \tilde{f} be isometric minimal immersions of compact Riemannian manifolds (M, g) and (\tilde{M}, \tilde{g}) into a Kenmotsu space form respectively. Assume that $\text{Spec}(J_f) = \text{Spec}(J_{\tilde{f}})$ and the structure vector field ξ is normal (or tangent) to $f(M)$ and $\tilde{f}(\tilde{M})$. Then we have*

1. $\dim(M) = \dim(\tilde{M})$;
2. $\text{Vol}(M, g) = \text{Vol}(\tilde{M}, \tilde{g})$;
3. $\int_M \tau_g dv_g = \int_{\tilde{M}} \tau_{\tilde{g}} dv_{\tilde{g}}$.

Proposition 4.2. *Let f, \tilde{f} be invariant immersions of compact Riemannian manifolds (M, g) and (\tilde{M}, \tilde{g}) into a Kenmotsu space form N respectively. Assume that $\text{Spec}(J_f) = \text{Spec}(J_{\tilde{f}})$. If f is a totally geodesic immersion, then so is \tilde{f} .*

Proof. As $\text{Spec}(J_f) = \text{Spec}(J_{\tilde{f}})$, using the relation (6) we have

$$\frac{n}{6} \int_M \tau_g dv_g + \int_M \text{Tr}(R_f) dv_g = \frac{n}{6} \int_{\tilde{M}} \tau_{\tilde{g}} dv_{\tilde{g}} + \int_{\tilde{M}} \text{Tr}(R_{\tilde{f}}) dv_{\tilde{g}}$$

Using the part (3) of the previous proposition (i.e. $\int_M \tau_g dv_g = \int_{\tilde{M}} \tau_{\tilde{g}} dv_{\tilde{g}}$), we get

$$\int_M \text{Tr}(R_f) dv_g = \int_{\tilde{M}} \text{Tr}(R_{\tilde{f}}) dv_{\tilde{g}}, \tag{14}$$

where

$$\begin{aligned} \text{Tr}(R_f) &= \sum_{i=1}^m \sum_{a=1}^{2n+1} h(R_h(V_a, f_*e_i)f_*e_i, V_a) \\ \text{Tr}(R_{\tilde{f}}) &= \sum_{i=1}^{\tilde{m}} \sum_{a=1}^{2n+1} h(R_h(V_a, \tilde{f}_*\tilde{e}_i)\tilde{f}_*\tilde{e}_i, V_a) \end{aligned}$$

Now, the proposition follows by using the Gauss equation. ■

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