DOI: 10.2478/auom-2013-0038
An. Şt. Univ. Ovidius Constanţa **Vol. 21(3),2013, 5–15**

Pseudo-contractibility Of Weighted L^p-Algebras

Fatemeh Abtahi

Abstract

Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G. Recently, we introduced the Lebesgue weighted L^p-algebra $\mathcal{L}^{1,p}_\omega(G)$. Here, we establish necessary and sufficient conditions for $\mathcal{L}^{1,p}_\omega(G)$ to be $\phi-$contractible, pseudo-contractible or contractible. Moreover we give some similar results about $L^p(G,\omega)$.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}-$bimodule. A derivation is a linear map $D : \mathcal{A} \to X$ such that

$$D(ab) = aD(b) + D(a)b \qquad (a, b \in \mathcal{A}).$$

A derivation D from $\mathcal{A}$ into X is inner if there is $\xi \in X$ such that

$$D(a) = a\xi - \xi a \qquad (a \in \mathcal{A}).$$

A Banach algebra $\mathcal{A}$ is called contractible if every continuous derivation from $\mathcal{A}$ into any Banach $\mathcal{A}-$bimodule is inner [15]. Accordingly, $\mathcal{A}$ is contractible if and only if it has a diagonal, i.e. there is an element $m \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ for which

Key Words: Contractible, Lebesgue weighted L^p-algebra, $\phi-$contractible, pseudo-contractible, weight function.
2010 Mathematics Subject Classification: Primary 43A15, Secondary 43A20.
Received: April, 2012.
Revised: May, 2012.
Accepted: February, 2013.

$am = ma$ and $\pi(m)a = a$, for all $a \in \mathcal{A}$ [15]. Here and in the sequel, π always denotes the product morphism from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ into $\mathcal{A}$, specified by

$$\pi(a \otimes b) = ab.$$

In fact every contractible Banach algebra is unital. The structure of contractible Banach algebras has been studied by many authors; see for example [15], [24] and also [26]. Moreover, the concept of pseudo-contractibility was introduced and investigated by Ghahramani and Zhang [14], according to the existence of a central approximate diagonal; i.e., a net (m_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$\|\pi(m_\alpha)a - a\| \to_\alpha 0$$

and also $am_\alpha = m_\alpha a$, for all $a \in \mathcal{A}$ and all α . Although there are many pseudo-contractible Banach algebras which are not contractible, but it has been proved that pseudo-contractibility of the unitization algebra of any Banach algebra $\mathcal{A}$ is equivalent to contractibility of $\mathcal{A}$ [14, Theorem 2.4].

Moreover, let $\phi \in \Omega(\mathcal{A})$, the spectrum space of $\mathcal{A}$ consisting of all non-zero characters on $\mathcal{A}$. Then $\mathcal{A}$ is called $\phi-$contractible if there is a $\phi-$diagonal, i.e. an element $m \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$\phi(\pi(m)) = 1 \quad and \quad am = \phi(a)m,$$

for each $a \in \mathcal{A}$. The notion of $\phi-$contractibility of $\mathcal{A}$ was laid by Hu, Monfared, and Traynor [17]. Furthermore it has been investigated $\phi-$contractibility of some classes of Banach algebras; see for example [2].

In the present paper, we present some characterizations for contractibility, pseudo-contractibility and also $\phi-$contractibility of the Lebesgue weighted L^p-algebra $\mathcal{L}^{1,p}_\omega(G)$ and also weighted L^p-algebra $L^p(G,\omega)$, endowed with convolution product. In fact all of the obtained results about $\mathcal{L}^{1,p}_\omega(G)$, can be considered as the consequences for Segal algebra $L^1(G) \cap L^p(G)$; see [20] for basic definitions and information concerning Segal algebras.

2 Preliminaries

Let G be a locally compact group with a fixed left Haar measure λ, and let ω be a weight function on G; that is, a positive Borel measurable function on G. The weight function ω is of left moderate growth on G if for each $x \in G$,

$$ess\sup_y \frac{\omega(xy)}{\omega(y)} < \infty.$$

For $1 < p < \infty$, the weighted L^p-space $L^p(G,\omega)$ with respect to λ is the set of all complex valued measurable functions f on G such that $f\omega \in L^p(G)$, the usual Lebesgue space as defined in [16]. We denote this space by $\ell^p(G,\omega)$ when G is discrete. Then $L^p(G,\omega)$ is a Banach space with the norm $\|.\|_{p,\omega}$ defined by $\|f\|_{p,\omega} = \|f\omega\|_p$, for all $f \in L^p(G,\omega)$. Recently, in [1] we introduced the Lebesgue weighted L^p−space $\mathcal{L}^{1,p}_\omega(G) := L^1(G) \cap L^p(G,\omega)$ with norm $\|f\|_{\mathcal{L}^{1,p}_\omega(G)} = \|f\|_1 + \|f\|_{p,\omega}$, and also the space $L^\infty(G) + L^q(G, 1/\omega)$ as its dual, with the action

$$< \phi + \psi, f >= \int_G \phi f d\lambda + \int_G \psi f d\lambda \qquad (f \in \mathcal{L}^{1,p}_\omega(G))$$

and the norm of $F = \phi + \psi$ being

$$\|F\|_{\mathcal{L}^{1,p}_\omega(G)^*} = \inf\{\max\{\|\phi\|_\infty, \|\psi\|_{q,1/\omega}\}\},$$

where the infimum is taken over all $\phi \in L^\infty(G)$ and $\psi \in L^q(G, 1/\omega)$ such that $F = \phi + \psi$. Let $\mathcal{A}$ be one of the spaces $L^p(G,\omega)$ or $\mathcal{L}^{1,p}_\omega(G)$ and let $\mathcal{A}^*$ be its dual space. We will denote by $\widetilde{G}_{\mathcal{A}}$ the subset of $\mathcal{A}^*$, consisting of all continuous homomorphisms $\rho : G \to \mathbb{C}\backslash\{0\}$.

It should be noted that the spaces $L^p(G,\omega)$ and $\mathcal{L}^{1,p}_\omega(G)$ and also their algebraic properties under the convolution product, have been studied very more completely in the decade of 1970. We found a lot of interesting results related to these algebras in many earlier publications. We just refer to some of them such as [7], [8], [9], [11], [12], [13] and also [25].

Note that $\mathcal{L}^{1,p}_\omega(G)$ is a Banach algebra whenever $L^p(G,\omega)$ is a Banach algebra. We provide two examples in this field. The following examples have been essentially introduced in [9, page 454].

Example 2.1. *Take G to be either of $\mathbb{R}^m$ or $\mathbb{Z}^m$.*

1. *Define the weight function ω_a on G by*

$$\omega_a(x) = (1 + |x|)^a \qquad (x \in G).$$

   Then $L^2(G,\omega_a)$ is a Banach algebra whenever $a > m/2$. It follows that $\mathcal{L}^{1,2}_\omega(G)$ is a Banach algebra, as well.

2. *Define the weight function ω on G by*

$$\omega(x) = 1 + |x_1|^{a_1} + \cdots + |x_m|^{a_m} \qquad (x = (x_1, \cdots, x_m) \in G),$$

   whenever $0 \le a_i < \infty$. Then $\mathcal{L}^{1,2}_\omega(G)$ is a Banach algebra.

It is noticeable to know that the space $\mathcal{L}^{1,p}_w(G)$ is in fact a weighted case of Segal algebra $L^1(G) \cap L^p(G)$. However there are important differences in their structures as Banach algebras. For instance, consider $\mathbb{Z}$, the additive group of the integer numbers, and take a weight function ω on $\mathbb{Z}$ such that it is not submultiplicative and also $1/\omega \in \ell^q(\mathbb{Z})$. Then $\ell^1(\mathbb{Z}) \cap \ell^p(\mathbb{Z}) = \ell^1(\mathbb{Z})$ is always a Banach algebra whereas $\mathcal{L}^{1,p}_\omega(\mathbb{Z}) = \ell^p(\mathbb{Z}, \omega)$ is not necessarily a Banach algebra. In fact on a discrete group, the weight of any Banach algebra for all $p \geq 1$ is submultiplicative; see [19, page 573].

From now on, we assume that G is a locally compact group and ω is a weight function on G such that $L^p(G, \omega)$ or $\mathcal{L}^{1,p}_\omega(G)$ are Banach algebras under convolution, in their own situations. Moreover, we take as assumption that there are some $\rho \in \widetilde{G}_{\mathcal{A}}$. Note that the trivial character $\rho = 1$ always belongs to $\widetilde{G}_{\mathcal{A}}$, whenever $\mathcal{A} = \mathcal{L}^{1,p}_\omega(G)$. If $\mathcal{A} = L^p(G, \omega)$, then $1 \in \widetilde{G}_{\mathcal{A}}$ if and only if $1/\omega \in L^q(G)$. For example this condition is satisfied whenever G is abelian [18].

If $\mathcal{L}^{1,p}_\omega(G)$ or $L^p(G, \omega)$ is a Banach algebra, then [19, Lemma 2.1] implies that ω^p is locally summable; that is $\omega \in L^p(K)$, for each compact subset K of G.

For completeness, we also turn the attention to the spectrum space of $\mathcal{A}$, where $\mathcal{A}$ is one of the Banach algebras $\mathcal{L}^{1,p}_\omega(G)$ or $L^p(G, \omega)$. It is not hard to see that each of them contains characters ϕ_ρ, defined by

$$\phi_\rho(f) = \int_G f(x)\rho(x) d\lambda(x),$$

where $\rho \in \widetilde{G}_{\mathcal{A}}$. See [18] for the proof of this statement, whenever $\mathcal{A} = L^p(G, \omega)$ and G is abelian.

3 Main Results

We commence this section with the following proposition.

Proposition 3.1. *Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G and $\rho \in \widetilde{G}_{\mathcal{A}}$. Then the following assertions are equivalent.*

(i) $\mathcal{L}^{1,p}_\omega(G)$ is $\phi_\rho-$contractible.

(ii) $L^p(G, \omega)$ is $\phi_\rho-$contractible.

(iii) G is compact.

Proof. $(i) \Rightarrow (iii)$. Let $\mathcal{L}^{1,p}_\omega(G)$ be $\phi_\rho-$contractible. By assumption, there exists $m \in \mathcal{L}^{1,p}_\omega(G) \widehat{\otimes} \mathcal{L}^{1,p}_\omega(G)$ such that for all $h \in \mathcal{L}^{1,p}_\omega(G)$

$$\phi_\rho(\pi(m)) = 1 \quad \text{and} \quad h * \pi(m) = \phi_\rho(h)\pi(m).$$

Set $g=\rho\pi(m)$. So $g\in L^1(G)$ and also $\phi_1(g)=\phi_\rho(\pi(m))=1$. Thus $f*g=\phi_1(f)g$, for all $f\in C_{00}(G)$; indeed,

$$\begin{aligned} f*g &= f*(\rho\pi(m))\\ &= \rho((\widetilde{\rho}f)*\pi(m))\\ &= \rho\phi_1(f)\pi(m)\\ &= \phi_1(f)g. \end{aligned}$$

Since $C_{00}(G)$ is dense in $L^1(G)$, then for each $f\in L^1(G)$

$$f*g=\left(\int_G f(x)d\lambda(x)\right)g,$$

almost every where on G. It follows that G is compact; see for example [21, Exercise 1.1.7].

$(iii)\Rightarrow(i)$. Let G be compact. Since ω^p is locally summable, it follows that $\rho\in\mathcal{L}^{1,p}_\omega(G)$. Set $m=\widetilde{\rho}\otimes\rho$, where $\widetilde{\rho}(x)=\rho(x^{-1})$, for each $x\in G$. Then

$$\phi_\rho(\pi(m))=\phi_\rho(\widetilde{\rho})\phi_\rho(\rho)=1.$$

For $f\in\mathcal{L}^{1,p}_\omega(G)$ and all $g,h\in\mathcal{L}^{1,p}_\omega(G)^*$, we have

$$\begin{aligned} \left((f*\widetilde{\rho})\otimes\rho\right)(g,h) &= g(f*\widetilde{\rho})h(\rho)\\ &= h(\rho)\int_G\int_G g(x)f(y)\widetilde{\rho}(y^{-1}x)d\lambda(y)d\lambda(x)\\ &= h(\rho)\int_G g(x)\widetilde{\rho}(x)d\lambda(x)\int_G f(y)\rho(y)d\lambda(y)\\ &= h(\rho)\langle g,\widetilde{\rho}\rangle\phi_\rho(f)\\ &= \phi_\rho(f)(\widetilde{\rho}\otimes\rho)(g,h). \end{aligned}$$

Thus for each $f\in\mathcal{L}^{1,p}_\omega(G)$,

$$(f*\widetilde{\rho})\otimes\rho=\phi_\rho(f)(\widetilde{\rho}\otimes\rho).$$

It follows that $\mathcal{L}^{1,p}_\omega(G)$ is $\phi_\rho-$contractible and so (i) is equivalent to (iii). One can easily follow the same steps to prove the equivalence between (ii) and (iii). □

We state here the following theorem, as an immediate result of Proposition 3.1.

Theorem 3.2. *Let G be a locally compact group, $1<p<\infty$ and let ω be a weight function on G. Then the following assertions are equivalent.*

(i) $\mathcal{L}^{1,p}_{\omega}(G)$ *is* $\phi_\rho-$*contractible, for all* $\rho \in \widetilde{G}_{\mathcal{A}}$.

(ii) $\mathcal{L}^{1,p}_{\omega}(G)$ *is* $\phi_\rho-$*contractible, for some* $\rho \in \widetilde{G}_{\mathcal{A}}$.

(iii) $L^p(G,\omega)$ *is* $\phi_\rho-$*contractible, for all* $\rho \in \widetilde{G}_{\mathcal{A}}$.

(iv) $L^p(G,\omega)$ *is* $\phi_\rho-$*contractible, for some* $\rho \in \widetilde{G}_{\mathcal{A}}$.

(v) G *is compact.*

By [2], every pseudo-contractible Banach algebra $\mathcal{A}$ is $\phi-$contractible, for each $\phi \in \Omega(\mathcal{A})$. Thus the following result is obtained from Proposition 3.1.

Corollary 3.3. *Let* G *be a locally compact group,* $1 < p < \infty$ *and let* ω *be a weight function on* G. *If* $\mathcal{L}^{1,p}_{\omega}(G)$ *or* $L^p(G,\omega)$ *is pseudo-contractible then* G *is compact.*

It would be interesting to know whether if the converse of Corollary 3.3 also holds. An affirmative answer for a particular case is given bellow. As a basic property of Segal algebras, $L^1(G) \cap L^p(G)$ is always a Banach left $L^1(G)-$module. Also [22, Theorem 3.5] implies that $L^1(G) \cap L^p(G)$ is pseudo-contractible if and only if G is compact. Since $\mathcal{L}^{1,p}_{\omega}(G) \subseteq L^1(G)$, it also comes to mind to verify this result for $\mathcal{L}^{1,p}_{\omega}(G)$, in the case where it is a Banach left $L^1(G)-$module. In the next result, we take this property as an assumption for $\mathcal{L}^{1,p}_{\omega}(G)$ and $L^p(G,\omega)$.

Theorem 3.4. *Let* G *be a locally compact group,* $1 < p < \infty$ *and let* ω *be a weight function on* G. *Then the following assertions are equivalent.*

(i) $\mathcal{L}^{1,p}_{\omega}(G)$ *is pseudo-contractible and it is a Banach left* $L^1(G)-$*module.*

(ii) $L^p(G,\omega)$ *is pseudo-contractible and it is a Banach left* $L^1(G)-$*module.*

(iii) G *is compact and* ω *is of left moderate growth.*

Proof. $(i) \Rightarrow (iii)$. Corollary 3.3 implies that G is compact. Now we show that ω is of left moderate growth. We follow an argument similar to [19, Theorem 3.1]. Since $\mathcal{L}^{1,p}_{\omega}(G)$ is a Banach left $L^1(G)-$module, it follows that with some constant $K > 0$

$$\|f * g\|_{\mathcal{L}^{1,p}_{\omega}(G)} \leq K\|f\|_1\|g\|_{\mathcal{L}^{1,p}_{\omega}(G)},$$

for all $f \in L^1(G)$ and $g \in \mathcal{L}^{1,p}_{\omega}(G)$. Since ω^p is locally summable, then $\mathcal{L}^{1,p}_{\omega}(G)$ contains characteristic functions χ_A, for each subset A of G of finite measure. We also have the following inequality, pointwise

$$\lambda(A)\chi_{xyB} \leq \chi_{xA} * \chi_{A^{-1}yB}, \tag{1}$$

for all $x, y \in G$ and relatively compact subsets A and B of G with positive measure. Hence inequality (1) implies that

$$\lambda(A)\|\chi_{xyB}\|_{\mathcal{L}^{1,p}_\omega(G)} \le \|\chi_{xA} * \chi_{A^{-1}yB}\|_{\mathcal{L}^{1,p}_\omega(G)} \le K\|\chi_{xA}\|_1\|\chi_{A^{-1}yB}\|_{\mathcal{L}^{1,p}_\omega(G)}.$$

Thus

$$\|\chi_{xyB}\|_{\mathcal{L}^{1,p}_\omega(G)} \le K\|\chi_{A^{-1}yB}\|_{\mathcal{L}^{1,p}_\omega(G)}.$$

It follows that

$$\|\chi_{xyB}\|_1 + \|\chi_{xyB}\|_{p,\omega} \le K\left(\|\chi_{A^{-1}yB}\|_1 + \|\chi_{A^{-1}yB}\|_{p,\omega}\right). \tag{2}$$

Let $x \in G$ be fixed. Since ω^p is locally summable, it follows that there exists a family $\mathcal{V}$ of sets of positive measure such that every $V \in \mathcal{V}$ contains the identity and every neighborhood of identity contains eventually all $V \in \mathcal{V}$ and also the following equations hold:

$$\lim_{V\in\mathcal{V}} \frac{1}{\lambda(V)} \int_{yV} \omega^p(r)dr = \omega^p(y)$$

and

$$\lim_{V\in\mathcal{V}} \frac{1}{\lambda(V)} \int_{yV} \omega^p(xr)dr = \omega^p(xy),$$

for locally almost all $y \in G$; see [3]. For such y and any $\varepsilon > 0$ for sufficiently small $V \in \mathcal{V}$ we have

$$\|\chi_{yV}\|^p_{p,\omega} = \int_{yV} \omega^p(r)dr < \lambda(V)\omega^p(y)(\varepsilon + 1) \tag{3}$$

and

$$\|\chi_{xyV}\|^p_{p,\omega} = \int_{yV} \omega^p(xr)dr > \frac{\lambda(V)\omega^p(xy)}{\varepsilon + 1}. \tag{4}$$

Moreover, there exists a relatively compact neighborhood U of identity such that

$$\|\chi_{U^{-1}yV}\|_{p,\omega} < (1+\varepsilon)\|\chi_{yV}\|_{p,\omega} \tag{5}$$

and also

$$\|\chi_{U^{-1}yV}\|_1 < \frac{(1+\varepsilon)}{K}\|\chi_{yV}\|_1. \tag{6}$$

By (2) we have

$$\|\chi_{xyV}\|_1 + \|\chi_{xyV}\|_{p,\omega} \le K(\|\chi_{U^{-1}yV}\|_1 + \|\chi_{U^{-1}yV}\|_{p,\omega}). \tag{7}$$

As a consequence of the observations (3), (4), (5) and (6) together with (7), we have the following inequality,

$$\lambda(V) + \frac{\lambda(V)^{1/p}\omega(xy)}{(\varepsilon+1)^{1/p}} \leq K\left(\frac{1+\varepsilon}{K}\lambda(V) + (1+\varepsilon)^{1+1/p}\omega(y)\lambda(V)^{1/p}\right)$$

and hence

$$(1+\varepsilon)^{1/p}\lambda(V)^{1-1/p}+\omega(xy) \leq \left((1+\varepsilon)^{1+1/p}\lambda(V)^{1-1/p} + K(1+\varepsilon)^{1+2/p}\omega(y)\right).$$

Since G is compact, it follows that the net $(\lambda(V))_{V\in\mathcal{V}}$ is bounded and in the limit as $\varepsilon \to 0$, we conclude that

$$\frac{\omega(xy)}{\omega(y)} < K,$$

locally almost every where $y \in G$. Consequently ω is of left moderate growth.

$(iii) \Rightarrow (i)$. Let G be compact and let ω be of left moderate growth. Since ω is locally summable, it follows that ω is equivalent to a continuous function [10, Theorem 2.7]. So the compactness of G implies that $L^p(G,\omega) = L^p(G)$. Consequently $\mathcal{L}^{1,p}_\omega(G) = L^p(G)$. Now the identity map is a topological isomorphism of $L^p(G)$ onto $\mathcal{L}^{1,p}_\omega(G)$. Since $L^p(G)$ is pseudo-contractible [14, Theorem 4.5], it follows that $\mathcal{L}^{1,p}_\omega(G)$ is also pseudo-contractible.

A similar argument shows that $(i) \Leftrightarrow (ii)$. □

The following known result is immediately obtained from Theorem 3.4, which is in fact [22, Theorem 3.5].

Corollary 3.5. *Let G be a locally compact group and $1 < p < \infty$. Then $L^1(G) \cap L^p(G)$ is pseudo-contractible if and only if G is compact.*

Remark 3.6. Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G. Also let $\mathcal{A}$ denote one of the Banach space $\mathcal{L}^{1,p}_\omega(G)$ or $L^p(G,\omega)$ (not necessarily an algebra), such that $\mathcal{A}$ is a Banach left $L^1(G)-$module.

(i) Whenever $\mathcal{A}$ is an essential Banach left $L^1(G)-$module, a stronger result may be obtained. Indeed, we show that if $\mathcal{A}$ is an essential Banach left $L^1(G)-$module then $\mathcal{A}$ is left translation invariant. Since $L^1(G)$ has a bounded approximate identity, one can conclude that $L^1(G) * \mathcal{A} = \mathcal{A}$, by the factorization theorem [16]. Then for every $f \in \mathcal{A}$ one can choose such $g \in L^1(G)$, $h \in \mathcal{A}$ that $f = g * h$. Now take an approximate identity ξ_ν in $L^1(G)$ and any $x \in G$. Denote by ϕ^x the left translation of a function ϕ by x. We have then $f^x = g^x * h$ and $\xi^x_\nu * g \to g^x$ in $L^1(G)$. So $\xi^x_\nu * f = (\xi^x_\nu * g) * h$ is a Cauchy net in $\mathcal{A}$. Its limit is obviously f^x. Thus $f^x \in \mathcal{A}$ and consequently $\mathcal{A}$ is translation-invariant.

(ii) If ω is locally summable then $C_{00}(G)$, the set of all compactly supported continuous functions on G, is contained in $\mathcal{A}$. It follows that $L^1(G) * \mathcal{A}$ is dense in $\mathcal{A}$, whenever $\mathcal{A} = L^p(G, \omega)$ and G is compact. By a similar proof to part (i) we obtain that $\mathcal{A}$ is translation invariant.

(iii) By [19, Lemma 2.1], if $L^p(G, \omega)$ is a Banach algebra then ω is locally summable. Thus part (ii) implies that if $\mathcal{A} = L^p(G, \omega)$ is a Banach algebra and G is compact then $\mathcal{A}$ is essential and again left translation invariance of $\mathcal{A}$ is obtained by part (i).

(iv) Let G be compact and let $\mathcal{A} = L^p(G, \omega)$ be a Banach algebra. Then one gets not only $\mathcal{A}$ is translation invariant, but also the translation is a bounded operator on $\mathcal{A}$; see for example [6]. Now one can concludes that ω is of left moderate growth by [7]. Consequently the implication $(ii) \Rightarrow (iii)$ of Theorem 3.4 can be proved more easier.

We finish the paper with the presentation of an equivalent condition to contractibility of $\mathcal{L}^{1,p}_{\omega}(G)$ and also $L^p(G, \omega)$. Let us first recall the concept of approximation property (AP).

A Banach space $\mathcal{X}$ is said to have the approximation property (AP), if every compact operator on $\mathcal{X}$ is a limit of finite-rank operators.

Proposition 3.7. *Let G be a locally compact group, $1 < p < \infty$ and let ω be a weight function on G. Then the following assertions are equivalent.*

(i) $L^p(G, \omega)$ *is contractible.*

(ii) $\mathcal{L}^{1,p}_{\omega}(G)$ *is contractible.*

(iii) G *is finite.*

Proof. $(i) \Rightarrow (iii)$. It follows from previously known results. First, one can show that $L^p(G, \omega)$ has the approximation property (AP); see for example [5, Example 11, p. 245]. From a result of Selivanov [23], it follows then that if $L^p(G, \omega)$ is contractible then G is finite.

$(iii) \Rightarrow (i)$ is obvious.

$(ii) \Rightarrow (iii)$. Let $\mathcal{L}^{1,p}_{\omega}(G)$ be contractible. Then it is unital and since $\mathcal{L}^{1,p}_{\omega}(G)$ is dense in $L^1(G)$, it follows that $L^1(G)$ is unital. Consequently G is discrete [4]. Since contractibility implies pseudo contractibility, it follows by Corollary 3.3 that G is compact. Therefore G is finite.

The statement $(iii) \Rightarrow (ii)$ is clear. □

Acknowledgments. The author expresses its sincere gratitude to Professor Hans Georg Feichtinger for his invaluable advice in continuously pointing out a number of earlier publications in the field. This research was partially supported by the Banach algebra Center of Excellence for Mathematics, University of Isfahan

References

[1] F. Abtahi, *Lebesgue weighted L^p-algebra on locally compact groups*, Acta. Math. Hungar., **133/4**, (2011), 324-331.

[2] M. Alaghmandan, R. Nasr Isfahani and M. Nemati, *On $\phi-$contractibility of the Lebesgue-Fourier algebra of a locally compact group*, Arch. Math. (Basel), **95**, (2010), 373-379.

[3] A. M. Bruckner, *Differentiation of integrals*, Amer. Math. Monthly, **78** No. 9 pt. 2, (1971).

[4] A. Deitmar and S. Echterhoff, Principles of harmonic analysis, Springer, New York, 2009.

[5] J. Diestel and J. J. Uhl, Jr., Vector measures, AMS, 1977.

[6] D. H. Dunford, *Segal algebras and left normed ideals*, *J. London Math. Soc. (2),* **8**, (1974), 514-516.

[7] R. E. Edwards, *The stability of weighted Lebesgue spaces*, Trans. Amer. Math. Soc., **93**, (1959), 369-394.

[8] H. G. Feichtinger, *On a class of convolution algebras of functions*, Ann. Inst. Fourier (Grenoble), **27**, no. 3, (1977), 135-162.

[9] H. G. Feichtinger, *Some remarks on Banach convolution algebras of functions (*)*, Istituto Nazionale di Alta Matematica Symposia Mathematica, Volume XXII, (1977), 453-455.

[10] H. G. Feichtinger, *Gewichtsfunktionen auf, lokalkompakten Gruppen*, Osterreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II, **188/8-10**, (1979), 451-471.

[11] H. G. Feichtinger, *Weighted L^p-spaces and the canonical mapping $T_H : L^1(G) \to L^1(G/H)$*, Bollettino U. M. I. **(5)16-B**, (1979), 989-999.

[12] H. G. Feichtinger and A. T. Gürkanli, *On a family of weighted convolution algebras*, Internat. J. Math. Math. Sci., **13/3**, (1990), 517-525.

[13] G. Fendler, K. Grchenig, M. Leinert, J. Ludwig and C. Molitor-Braun, *Weighted group algebras on groups of polynomial growth*, Math. Z., **245/4**, (2003), 791-821.

[14] F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Cambridge Philos. Soc., **142**, (2007), no. 1, 111-123.

[15] A. Ya. Helemskii, A. Ya. Helemskii, Banach and Locally Convex Algebras, The Clarendon Press Oxford University Press, New York, 1993.

[16] E. Hewitt and K. Ross, Abstract harmonic analysis I,II, Springer-Verlag, New York, 1970.

[17] Z. Hu, M. S. Monfared and T. Traynor, *On character amenable Banach algebras*, Studia Math., **193**, (2009), no. 1, 53-78.

[18] Yu. N. Kuznetsova, *Weighted Lp-Algebras on Groups*, Funct. Anal. Appl., **40/3**, (2006), 234-236.

[19] Yu. N. Kuznetsova, *Invariant weighted algebras $L^p(G,\omega)$*, Mat. Zametki., **84/4**, (2008), 567-576.

[20] H. Reiter, L^1-algebras and segal algebras, Lecture Notes in Mathematics, **231**, Springer-Verlag, Berlin, 1971.

[21] V. Runde, Lectures on amenability, Lecture Notes in Mathematics, **1774**, Springer-Verlag, Berlin, 2002.

[22] E. Samei, N. Spronk and R. Stokke, *Biflatness and pseudo-amenability of Segal algebras*, Canad. J. Math., **62**, (2010), no. 4, 845-869.

[23] Ju. V. Selivanov, *Banach Algebras of Small Global Dimension Zero*, Uspehi Mat. Nauk., **31**, (1976), 227-228.

[24] J. L. Taylor, *Homology and cohomology for topological algebras*, Advances in Math., **9**, (1972), 137-182.

[25] J. Wermer, *On a class of normed rings*, Ark. Mat., **2**, (1954), 537-551.

[26] Y. Zhang, *Maximal ideals and the structure of contractible and amenable Banach algebras*, Bull. Austral. Math. Soc., **62**, (2000), 221-226.

Fatemeh ABTAHI,
Department of Mathematics,
University of Isfahan,
P. O. Box 81746-73441,
Isfahan, Iran.
Email: f.abtahi@sci.ui.ac.ir