



A viscosity projection method for class \mathfrak{T} mappings

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Abstract

In this paper, we firstly introduce a viscosity projection method for the class \mathfrak{T} mappings

$$x_{n+1} = \alpha_n P_{H(x_n, S_n x_n)} f(x_n) + (1 - \alpha_n) S_n x_n,$$

where $S_n = (1 - w)I + wT_n$, $w \in (0, 1)$, $T_n \in \mathfrak{T}$ and prove strong convergence theorems of the proposed method. It is verified that the viscosity projection method converges locally faster than the viscosity method. Furthermore, we present a viscosity projection method for a quasi-nonexpansive and nonexpansive mappings in Hilbert spaces. A numerical test provided in the paper shows that the viscosity projection method converges faster than the viscosity method.

1 Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Recall that a mapping $T : H \rightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is $\text{Fix}(T) := \{x \in H : Tx = x\}$. A mapping $T : H \rightarrow H$ is said to be quasi-nonexpansive if $\text{Fix}(T)$ is nonempty and $\|Tx - p\| \leq \|x - p\|$ for all $x \in H$ and $p \in \text{Fix}(T)$. A mapping $f : H \rightarrow H$ is said to be a contraction with constant $\rho \in [0, 1)$ if

$$\|f(x) - f(y)\| \leq \rho \|x - y\| \quad \forall x, y \in H.$$

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Given $x, y \in H$, let

$$H(x, y) := \{z \in H : \langle z - y, x - y \rangle \leq 0\},$$

be the half-space generated by (x, y) . The boundary ∂H of H is

$$\partial H(x, y) = \{z \in H : \langle z - y, x - y \rangle = 0\}.$$

It is clear that $\partial H(x, y)$ is a closed and convex subset of H . A mapping $T : H \rightarrow H$ is said to be the class \mathfrak{T} (or a cutter) if $T \in \mathfrak{T} = \{T : H \rightarrow H \mid \text{dom}(T) = H \text{ and } \text{Fix}(T) \subset H(x, Tx), \text{ for all } x \in H\}$

Remark 1.1. *The class \mathfrak{T} is fundamental because it contains several types of operators commonly found in various areas of applied mathematics and in particular in approximation and optimization theory (see [1, 2] for details).*

Let C be a nonempty closed convex subset of a Hilbert space H . For a mapping $T : C \rightarrow C$, Moudafi [10] and many other researchers (eg.[7, 8, 11, 12, 13, 14]) studied the viscosity approximation method as follow: for given $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad (1)$$

where $\{\alpha_n\} \subset (0, 1)$ and $f : C \rightarrow C$ is a contraction. It was proved in [10] (also see Xu [13]) that the sequence $\{x_n\}$ generated by (1) converges strongly to the unique solution of the variational inequality problem $VI(I - f, \text{Fix}(T))$: find x^* in $\text{Fix}(T)$ such that

$$\forall v \in \text{Fix}(T), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

A special case of (1) was considered by Halpern [5] who introduced following iterative process:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

where $u, x_0 \in C$ are arbitrary (but fixed) and $\{\alpha_n\} \subset (0, 1)$.

Recently, Maingé [9] studied following algorithm for a quasi-nonexpansive mapping T :

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_w x_n, \quad (2)$$

where $\{\alpha_n\} \subset (0, 1)$, $T_w = (1 - w)I + wT$, $w \in (0, 1)$. He proposed a new analysis of the viscosity approximation method to prove the convergence of the algorithm (2).

Inspired by Maingé [9] and others (e.g. [1, 2, 3, 6]), in this paper we firstly discuss the following viscosity projection method for a sequence of class \mathfrak{T} mappings $T_n : H \rightarrow H$ as follow:

$$x_{n+1} = \alpha_n P_{H(x_n, S_n x_n)} f(x_n) + (1 - \alpha_n)S_n x_n, \quad (3)$$

where $\{\alpha_n\} \subset (0, 1)$, $S_n = (1 - w)I + wT_n$, $w \in (0, 1)$, I is the identity mapping on H and P_K denotes the metric projection from H onto a closed convex subset K of H (see below Lemma 1.3 for the definition). We prove that the sequence $\{x_n\}$ generated by (3) converges strongly to the unique solution of the variational inequality problem $VI(I - f, \bigcap_{n=1}^{\infty} \text{Fix}(T_n))$: find x^* in $\bigcap_{n=1}^{\infty} \text{Fix}(T_n)$ such that

$$\forall v \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0. \quad (4)$$

We will use the following notations:

1. \rightharpoonup for weak convergence and \rightarrow for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω -limit of $\{x_n\}$.

We need some facts and tools in a real Hilbert space H which are listed below.

Definition 1.1. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two iterations which converge to a point p . Then $\{x_n\}_{n=1}^{\infty}$ is said to converge locally faster than $\{y_n\}_{n=1}^{\infty}$ if $x_n = y_n$ implies

$$\|x_{n+1} - p\| \leq \|y_{n+1} - p\|$$

for any $n \in \mathbb{N}$.

Lemma 1.1. Let H be a Hilbert space and I be the identity operator of H .

- (i) If $\text{dom } T = H$, then $2T - I$ is quasi-nonexpansive if and only if $T \in \mathfrak{T}$,
- (ii) If $T \in \mathfrak{T}$, then $\lambda I + (1 - \lambda)T \in \mathfrak{T}$, $\forall \lambda \in [0, 1]$.
- (iii) If $T \in \mathfrak{T}$, then T is quasi-nonexpansive.
- (iv) If $T \in \mathfrak{T}$, then $\|x - Tx\|^2 \leq \langle x - Tx, x - u \rangle$ for all $x \in H$ and $u \in \text{Fix}(T)$.
- (v) If $T \in \mathfrak{T}$ and $S = wI + (1 - w)T$, $w \in (0, 1)$, then $H(x, Tx) \subset H(x, Sx)$, $\forall x \in H$.

Proof. The proof of (i)-(iv) can be found in [1]. Here we just prove (v). For any $y \in H(x, Tx)$, we have

$$\langle y - Tx, x - Tx \rangle \leq 0.$$

So, we get

$$\langle y - Sx, x - Sx \rangle = (1 - w)\langle y - Tx, x - Tx \rangle - (1 - w)w\|x - Tx\|^2 \leq 0,$$

which implies $y \in H(x, Sx)$.

Remark 1.2. Let $T \in \mathfrak{T}$ with $\text{Fix}(T) \neq \emptyset$ and set $T_w := (1-w)I + wT$ for $w \in (0, 1)$. Then the following statements are reached:

(a1) $\text{Fix}(T) = \text{Fix}(T_w)$ if $w \neq 0$;

(a2) $\text{Fix}(T)$ is a closed convex subset of H .

(a3) $\langle x - T_w x, x - q \rangle \geq w\|x - Tx\|^2$ for all $x \in H, q \in \text{Fix}(T)$.

From Lemma 1.1 (i) and (ii), it is an easy matter to show (a1)-(a3) by using Remarks 1.2 and 2.1 in [9].

Definition 1.2. A sequence of mappings $\{T_n\}$ having common fixed points is said to satisfy the condition (Z) if every bounded sequence $\{x_n\}$ with $\|x_n - T_n x_n\| \rightarrow 0$ satisfies $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$.

Definition 1.3. A mapping T is called demiclosed at $y \in H$ if $Tx = y$ whenever $\{x_n\} \subset H, x_n \rightharpoonup x$ and $Tx_n \rightarrow y$.

Next Lemma shows that nonexpansive mappings are demiclosed at 0.

Lemma 1.2. [4] Let C be a closed convex subset of a real Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \rightarrow 0$, then $z = Tz$.

Lemma 1.3. [4] Let K be a closed convex subset of real Hilbert space H and let P_K be the (metric or nearest point) projection from H onto K (i.e., for $x \in H, P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:

$$\langle x - z, y - z \rangle \leq 0, \quad \text{for all } y \in K.$$

Lemma 1.4. [6] Let $C = \{z \in H : \langle x - u, z - u \rangle \leq 0\}$. Assume $x \neq u$ and $x_0 \notin C$. Then

$$P_C x_0 = x_0 - \frac{\langle x - u, x_0 - u \rangle}{\|x - u\|^2} (x - u). \quad (5)$$

Lemma 1.5. Let $F := I - P_{H(x, Tx)} f$, where $x \in H$ and f is the contraction with constant ρ . Then the operator F is $(1 - \rho)$ -strongly monotone, i.e.,

$$\langle Fy - Fz, y - z \rangle \geq (1 - \rho)\|y - z\|^2 \quad \text{for all } y, z \in H.$$

Proof. Note that $P_{H(x, Tx)}$ is a metric projection, so it is firmly nonexpansive and thus is nonexpansive. It is easy to see that, for all $y, z \in H$,

$$\|P_{H(x, Tx)} f(y) - P_{H(x, Tx)} f(z)\| \leq \|f(y) - f(z)\| \leq \rho\|y - z\|. \quad (6)$$

From (6), we have

$$\begin{aligned} \langle Fy - Fz, y - z \rangle &= \|y - z\|^2 - \langle P_{H(x, Tx)}f(y) - P_{H(x, Tx)}f(z), y - z \rangle \\ &\geq \|y - z\|^2 - \|P_{H(x, Tx)}f(y) - P_{H(x, Tx)}f(z)\| \|y - z\| \\ &\geq (1 - \rho) \|y - z\|^2. \end{aligned}$$

Lemma 1.6. ([9] (Lemma 2.1)). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j \geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_{j+1}}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(n)\}_{n \geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \rightarrow \infty} \tau(n) = \infty$ and, for all $n \geq n_0$, it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and we have

$$\Gamma_n \leq \Gamma_{\tau(n)+1}.$$

2 Main results

Lemma 2.1. *Let $T_n \in \mathfrak{T}$ with $\mathcal{F} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset$, $\{\alpha_n\} \subset (0, 1)$ and $w \in (0, 1)$. Let f be a contraction with constant ρ . The sequence $\{x_n\}$ generated by (3) is bounded.*

Proof. By $T_n \in \mathfrak{T}$ and Lemma 1.1 (v), $\text{Fix}(T_n) \subset H(x, S_n x)$, for all $x \in H$, therefore, we have $P_{H(x, S_n x)}p = p$, for all $p \in \mathcal{F}$. So, using Lemma 1.1 (ii)-(iii) and (6), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n P_{H(x_n, S_n x_n)}f(x_n) + (1 - \alpha_n)S_n x_n - p\| \\ &\leq \alpha_n \|P_{H(x_n, S_n x_n)}f(x_n) - p\| + (1 - \alpha_n) \|S_n x_n - p\| \\ &\leq \alpha_n \|P_{H(x_n, S_n x_n)}f(x_n) - P_{H(x_n, S_n x_n)}f(p)\| \\ &\quad + \alpha_n \|P_{H(x_n, S_n x_n)}f(p) - P_{H(x_n, S_n x_n)}p\| + (1 - \alpha_n) \|x_n - p\| \\ &\leq \alpha_n \|f(p) - p\| + [1 - \alpha_n(1 - \rho)] \|x_n - p\| \\ &= \alpha_n(1 - \rho) \frac{\|f(p) - p\|}{1 - \rho} + [1 - \alpha_n(1 - \rho)] \|x_n - p\|. \end{aligned}$$

Thus, by induction on n ,

$$\|x_n - p\| \leq \max \left\{ \frac{\|f(p) - p\|}{1 - \rho}, \|x_0 - p\| \right\},$$

for every $n \in \mathbb{N}$. This shows that $\{x_n\}$ is bounded, and hence, $\{P_{H(x_n, S_n x_n)}f(x_n)\}$ is also bounded.

Lemma 2.2. *Assume a sequence of mappings $T_n \in \mathfrak{T} : H \rightarrow H$ satisfies the condition (Z). If x^* is the solution of (4) and $\{x_n\}$ is a bounded sequence such that $\|T_n x_n - x_n\| \rightarrow 0$, then*

$$\liminf_{n \rightarrow \infty} \langle (I - P_{H(x_n, T_n x_n)}) f x^*, x_n - x^* \rangle \geq 0. \quad (7)$$

Proof. Since the sequence $\{T_n\}$ satisfies the condition (Z) and $\{x_n\}$ is a bounded sequence, $\omega_w(x_n) \subset \mathcal{F}$. It is also a simple matter to see that there exists \bar{x} and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \rightarrow \infty$ (hence $\bar{x} \in \mathcal{F}$) and such that

$$\liminf_{n \rightarrow \infty} \langle (I - f)x^*, x_n - x^* \rangle = \lim_{k \rightarrow \infty} \langle (I - f)x^*, x_{n_k} - x^* \rangle,$$

which by (4) obviously leads to

$$\liminf_{n \rightarrow \infty} \langle (I - f)x^*, x_n - x^* \rangle = \langle (I - f)x^*, \bar{x} - x^* \rangle \geq 0.$$

So,

$$\liminf_{n \rightarrow \infty} \langle (I - f)x^*, x_n - x^* \rangle \geq 0. \quad (8)$$

If $f(x^*) \in H(x_n, T_n x_n)$, then $P_{H(x_n, T_n x_n)} f(x^*) = f(x^*)$ and (8) implies (7). Otherwise, assume $f(x^*) \notin H(x_n, T_n x_n)$. Then, by definition of $H(x_n, T_n x_n)$, we have

$$\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle > 0. \quad (9)$$

By $x^* \in \mathcal{F} \subset H(x_n, T_n x_n)$, we get

$$\langle x_n - T_n x_n, x_n - x^* \rangle = \|x_n - T_n x_n\|^2 + \langle x_n - T_n x_n, T_n x_n - x^* \rangle > 0. \quad (10)$$

From (5), it follows

$$P_{H(x_n, T_n x_n)} f(x^*) = f(x^*) - \frac{\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle}{\|x_n - T_n x_n\|^2} (x_n - T_n x_n). \quad (11)$$

Combining (9), (10) and (11), we obtain

$$\begin{aligned} \langle (I - P_{H(x_n, T_n x_n)}) f x^*, x_n - x^* \rangle &= \langle (I - f)x^*, x_n - x^* \rangle \\ &\quad + \frac{\langle x_n - T_n x_n, f(x^*) - T_n x_n \rangle}{\|x_n - T_n x_n\|^2} \langle x_n - T_n x_n, x_n - x^* \rangle \\ &> \langle (I - f)x^*, x_n - x^* \rangle, \end{aligned} \quad (12)$$

which together with (8) implies

$$\liminf_{n \rightarrow \infty} \langle (I - P_{H(x_n, T_n x_n)}) f x^*, x_n - x^* \rangle \geq \liminf_{n \rightarrow \infty} \langle (I - f)x^*, x_n - x^* \rangle \geq 0.$$

Therefore, we obtain the desired result.

Theorem 2.1. *Suppose that a sequence $\{T_n\} \subset \mathfrak{T}$ satisfies $\mathcal{F} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset$ and the condition (Z). Let f be a contraction with constant $\rho \in [0, 1)$. Assume $w \in (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ generated by (3) converges strongly to $x^* \in \mathcal{F}$ verifying*

$$x^* = (P_{\mathcal{F}} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \mathcal{F}, \quad \text{and} \quad (\forall v \in \mathcal{F}), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0. \quad (13)$$

Proof. Let x^* be the solution of (13). From (3) we obviously have

$$x_{n+1} - x_n + \alpha_n(x_n - P_{H(x_n, S_n x_n)} f(x_n)) = (1 - \alpha_n)(S_n x_n - x_n), \quad (14)$$

hence

$$\langle x_{n+1} - x_n + \alpha_n(x_n - P_{H(x_n, S_n x_n)} f(x_n)), x_n - x^* \rangle = -(1 - \alpha_n) \langle x_n - S_n x_n, x_n - x^* \rangle. \quad (15)$$

Moreover, by $x^* \in \mathcal{F}$, and using Remark 1.2 (a3), we have

$$\langle x_n - S_n x_n, x_n - x^* \rangle \geq w \|x_n - T_n x_n\|^2,$$

which together with (15) entails

$$\langle x_{n+1} - x_n + \alpha_n(x_n - P_{H(x_n, S_n x_n)} f(x_n)), x_n - x^* \rangle \leq -w(1 - \alpha_n) \|x_n - T_n x_n\|^2,$$

or equivalently

$$\begin{aligned} -\langle x_n - x_{n+1}, x_n - x^* \rangle &\leq -\alpha_n \langle x_n - P_{H(x_n, S_n x_n)} f(x_n), x_n - x^* \rangle \\ &\quad - w(1 - \alpha_n) \|x_n - T_n x_n\|^2. \end{aligned} \quad (16)$$

Setting $\Gamma_n := \frac{1}{2} \|x_n - x^*\|^2$, we have

$$\langle x_n - x_{n+1}, x_n - x^* \rangle = -\Gamma_{n+1} + \Gamma_n + \frac{1}{2} \|x_n - x_{n+1}\|^2.$$

So that (16) can be equivalently rewritten as

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n - \frac{1}{2} \|x_n - x_{n+1}\|^2 &\leq -\alpha_n \langle x_n - P_{H(x_n, S_n x_n)} f(x_n), x_n - x^* \rangle \\ &\quad - w(1 - \alpha_n) \|x_n - T_n x_n\|^2. \end{aligned} \quad (17)$$

Now using (14) again, we have

$$\|x_{n+1} - x_n\|^2 = \|\alpha_n(P_{H(x_n, S_n x_n)} f(x_n) - x_n) + (1 - \alpha_n)(S_n x_n - x_n)\|^2.$$

Hence it is a classical matter to see that

$$\|x_{n+1} - x_n\|^2 \leq 2\alpha_n^2 \|P_{H(x_n, S_n x_n)} f(x_n) - x_n\|^2 + 2(1 - \alpha_n)^2 \|S_n x_n - x_n\|^2,$$

which by $\|S_n x_n - x_n\| = w \|T_n x_n - x_n\|$ and $(1 - \alpha_n)^2 \leq (1 - \alpha_n)$ yields

$$\frac{1}{2} \|x_{n+1} - x_n\|^2 \leq \alpha_n^2 \|P_{H(x_n, S_n x_n)} f(x_n) - x_n\|^2 + w^2 (1 - \alpha_n) \|T_n x_n - x_n\|^2. \quad (18)$$

Then from (17) and (18) we obtain

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n + (1 - w)w(1 - \alpha_n) \|x_n - T_n x_n\|^2 \\ \leq \alpha_n (\alpha_n \|P_{H(x_n, S_n x_n)} f(x_n) - x_n\|^2 - \langle x_n - P_{H(x_n, S_n x_n)} f(x_n), x_n - x^* \rangle). \end{aligned} \quad (19)$$

The rest of the proof will be divided into two parts:

Case 1. Suppose that there exists n_0 such that $\{\Gamma_n\}_{n \geq n_0}$ is nonincreasing. In this situation, $\{\Gamma_n\}$ is then convergent because it is also nonnegative (hence it is bounded from below), so that $\lim_{n \rightarrow \infty} (\Gamma_{n+1} - \Gamma_n) = 0$, hence, in light of (19) together with $\alpha_n \rightarrow 0$, and the boundedness of $\{x_n\}$ (hence, thanks Lemma 2.1, $\{P_{H(x_n, S_n x_n)} f(x_n)\}$ is also bounded), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0,$$

which together with $S_n = (1 - w)I + wT_n$, $w \in (0, 1)$, implies

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \quad (20)$$

From (19) again, we have

$$\alpha_n (-\alpha_n \|P_{H(x_n, S_n x_n)} f(x_n) - x_n\|^2 + \langle x_n - P_{H(x_n, S_n x_n)} f(x_n), x_n - x^* \rangle) \leq \Gamma_n - \Gamma_{n+1}.$$

Then, by $\sum_n \alpha_n = \infty$, we obviously deduce that

$$\liminf_{n \rightarrow \infty} (-\alpha_n \|P_{H(x_n, S_n x_n)} f(x_n) - x_n\|^2 + \langle x_n - P_{H(x_n, S_n x_n)} f(x_n), x_n - x^* \rangle) \leq 0,$$

or equivalently (as $\alpha_n \|P_{H(x_n, S_n x_n)} f(x_n) - x_n\|^2 \rightarrow 0$)

$$\liminf_{n \rightarrow \infty} \langle x_n - P_{H(x_n, S_n x_n)} f(x_n), x_n - x^* \rangle \leq 0. \quad (21)$$

Moreover, by Lemma 1.5, we have

$$2(1 - \rho)\Gamma_n + \langle x^* - P_{H(x_n, S_n x_n)} f(x^*), x_n - x^* \rangle \leq \langle x_n - P_{H(x_n, S_n x_n)} f(x_n), x_n - x^* \rangle, \quad (22)$$

which by (21) entails

$$\liminf_{n \rightarrow \infty} (2(1 - \rho)\Gamma_n + \langle x^* - P_{H(x_n, S_n x_n)} f(x^*), x_n - x^* \rangle) \leq 0.$$

Hence, recalling that $\lim_{n \rightarrow \infty} \Gamma_n$ exists, we equivalently obtain

$$2(1 - \rho) \lim_{n \rightarrow \infty} \Gamma_n + \liminf_{n \rightarrow \infty} \langle x^* - P_{H(x_n, S_n x_n)} f(x^*), x_n - x^* \rangle \leq 0,$$

namely,

$$2(1 - \rho) \lim_{n \rightarrow \infty} \Gamma_n \leq - \liminf_{n \rightarrow \infty} \langle x^* - P_{H(x_n, S_n x_n)} f(x^*), x_n - x^* \rangle. \quad (23)$$

From (20) and invoking Lemma 2.2, we have

$$\liminf_{n \rightarrow \infty} \langle x^* - P_{H(x_n, S_n x_n)} f(x^*), x_n - x^* \rangle \geq 0,$$

which by (23) yields $\lim_{n \rightarrow \infty} \Gamma_n = 0$, so that $\{x_n\}$ converges strongly to x^* .

Case 2. Suppose there exists a subsequence $\{\Gamma_{n_k}\}_{k \geq 0}$ of $\{\Gamma_n\}_{n \geq 0}$ such that $\Gamma_{n_k} < \Gamma_{n_{k+1}}$ for all $k \geq 0$. In this situation, we consider the sequence of indices $\{\tau(n)\}$ as defined in Lemma 1.6. It follows that $\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} > 0$, which by (19) amounts to

$$\begin{aligned} (1 - w)w(1 - \alpha_{\tau(n)}) \|x_{\tau(n)} - T_{\tau(n)} x_{\tau(n)}\|^2 \\ < \alpha_{\tau(n)} (\alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)} x_{\tau(n)})} f(x_{\tau(n)}) - x_{\tau(n)}\|^2 \\ - \langle x_{\tau(n)} - P_{H(x_{\tau(n)}, S_{\tau(n)} x_{\tau(n)})} f(x_{\tau(n)}), x_{\tau(n)} - x^* \rangle). \end{aligned} \quad (24)$$

Hence, by the boundedness of $\{x_n\}$ and $\{P_{H(x_n, S_n x_n)} f(x_n)\}$, and $\alpha_n \rightarrow 0$, we immediately obtain

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - T_{\tau(n)} x_{\tau(n)}\| = 0, \quad (25)$$

which together with $S_{\tau(n)} = (1 - w)I + wT_{\tau(n)}$, $w \in (0, 1)$, implies

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - S_{\tau(n)} x_{\tau(n)}\| = 0. \quad (26)$$

Using (3), we have

$$\begin{aligned} \|x_{\tau(n)+1} - x_{\tau(n)}\| &\leq \alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)} x_{\tau(n)})} f(x_{\tau(n)}) - x_{\tau(n)}\| \\ &\quad + (1 - \alpha_{\tau(n)}) \|x_{\tau(n)} - S_{\tau(n)} x_{\tau(n)}\|, \end{aligned}$$

which together with (26) and $\alpha_n \rightarrow 0$ yields

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \quad (27)$$

Now by (24), we clearly have

$$\begin{aligned} & \langle x_{\tau(n)} - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}f(x_{\tau(n)}), x_{\tau(n)} - x^* \rangle \\ & \leq \alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}f(x_{\tau(n)}) - x_{\tau(n)}\|^2, \end{aligned}$$

which in the light of (22) yields

$$\begin{aligned} & 2(1 - \rho)\Gamma_{\tau(n)} + \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}f(x^*), x_{\tau(n)} - x^* \rangle \\ & \leq \alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}f(x_{\tau(n)}) - x_{\tau(n)}\|^2. \end{aligned}$$

Hence (as $\alpha_{\tau(n)} \|P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}f(x_{\tau(n)}) - x_{\tau(n)}\|^2 \rightarrow 0$) it follows that

$$2(1 - \rho) \limsup_{n \rightarrow \infty} \Gamma_{\tau(n)} \leq - \liminf_{n \rightarrow \infty} \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}f(x^*), x_{\tau(n)} - x^* \rangle. \quad (28)$$

From (26) and invoking Lemma 2.2, we have

$$\liminf_{n \rightarrow \infty} \langle x^* - P_{H(x_{\tau(n)}, S_{\tau(n)}x_{\tau(n)})}f(x^*), x_{\tau(n)} - x^* \rangle \geq 0,$$

which by (28) yields $\limsup_{n \rightarrow \infty} \Gamma_{\tau(n)} = 0$, so that $\lim_{n \rightarrow \infty} \Gamma_{\tau(n)} = 0$. Applying (27), we have $\lim_{n \rightarrow \infty} \Gamma_{\tau(n)+1} = 0$. Then, recalling that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ (by Lemma 1.6), we get $\lim_{n \rightarrow \infty} \Gamma_n = 0$, so that $x_n \rightarrow x^*$ strongly.

Remark 2.1. Assume that $f(x_n) \notin H(x_n, S_n x_n)$. From Lemma 1.4, we have

$$P_{H(x_n, S_n x_n)}f(x_n) = f(x_n) - \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n \rangle}{\|x_n - S_n x_n\|^2} (x_n - S_n x_n). \quad (29)$$

So, the algorithm (3) can be rewritten as the form:

$$x_{n+1} = \begin{cases} \alpha_n f(x_n) + (1 - \alpha_n) S_n x_n, & \text{if } f(x_n) \in H(x_n, S_n x_n) \\ \alpha_n P_{H(x_n, S_n x_n)}f(x_n) + (1 - \alpha_n) S_n x_n, & \text{if } f(x_n) \notin H(x_n, S_n x_n) \end{cases} \quad (30)$$

where $P_{H(x_n, S_n x_n)}f(x_n)$ is given by (29). From (30), we know the algorithm (3) can be easily realized although there is a metric projection.

From (2), the classical viscosity method for class \mathfrak{T} mappings $\{T_n\}$ is

$$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) S_n y_n, \quad (31)$$

where $S_n = (1 - w)I + wT_n$.

Next, we will compare the convergence rate of the viscosity projection method with the viscosity method.

Theorem 2.2. *Suppose that a sequence $\{T_n\} \subset \mathfrak{T}$ satisfies $\mathcal{F} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset$. Take the same parameters $\{\alpha_n\}$ and w in (3) and (31). Let $y_n = x_n$ and $p \in \mathcal{F}$. Then it holds*

$$\|x_{n+1} - p\| \leq \|y_{n+1} - p\|. \quad (32)$$

Proof. From $T_n \in \mathfrak{T}$ and Lemma 1.1 (v), it follows $\mathcal{F} \in H(x_n, S_n x_n)$. If $f(x_n) \in H(x_n, S_n x_n)$ and then $P_{H(x_n, S_n x_n)} f(x_n) = f(x_n)$, then, it is obvious that $y_{n+1} = x_{n+1}$ and (32) follows.

Next, assume $f(x_n) \notin H(x_n, S_n x_n)$, then it is easy to verify $P_{H(x_n, S_n x_n)} f(x_n) \in \partial H(x_n, S_n x_n)$. Actually, from (29), it follows

$$\begin{aligned} & \langle P_{H(x_n, S_n x_n)} f(x_n) - S_n x_n, x_n - S_n x_n \rangle \\ &= \langle f(x_n) - S_n x_n - \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n \rangle}{\|x_n - S_n x_n\|^2} (x_n - S_n x_n), x_n - S_n x_n \rangle \\ &= \langle f(x_n) - S_n x_n, x_n - S_n x_n \rangle - \\ & \quad \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n \rangle}{\|x_n - S_n x_n\|^2} \langle x_n - S_n x_n, x_n - S_n x_n \rangle \\ &= 0, \end{aligned}$$

which yields

$$\begin{aligned} & \langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), S_n x_n - P_{H(x_n, S_n x_n)} f(x_n) \rangle \\ &= \frac{\langle x_n - S_n x_n, f(x_n) - S_n x_n \rangle}{\|x_n - S_n x_n\|^2} \langle x_n - S_n x_n, P_{H(x_n, S_n x_n)} f(x_n) - S_n x_n \rangle \quad (33) \\ &= 0. \end{aligned}$$

On the other hand, since $p \in \mathcal{F} \subset H(x_n, S_n x_n)$, using Lemma 1.3, we get

$$\langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), P_{H(x_n, S_n x_n)} f(x_n) - p \rangle \leq 0. \quad (34)$$

Applying (33), (34) and $x_n = y_n$, we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n P_{H(x_n, S_n x_n)} f(x_n) + (1 - \alpha_n) S_n x_n - p\|^2 \\ &= \|\alpha_n (P_{H(x_n, S_n x_n)} f(x_n) - f(y_n)) + (y_{n+1} - p)\|^2 \\ &\leq \|y_{n+1} - p\|^2 + 2\alpha_n \langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), x_{n+1} - p \rangle \\ &= \|y_{n+1} - p\|^2 + 2\alpha_n \langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), P_{H(x_n, S_n x_n)} f(x_n) - p \rangle \\ & \quad + 2\alpha_n (1 - \alpha_n) \langle P_{H(x_n, S_n x_n)} f(x_n) - f(x_n), S_n x_n - P_{H(x_n, S_n x_n)} f(x_n) \rangle \\ &\leq \|y_{n+1} - p\|^2, \end{aligned}$$

which implies $\|x_{n+1} - p\| \leq \|y_{n+1} - p\|$.

Remark 2.2. From the Definition 1.1 and Theorem 2.2, it follows that the viscosity projection method converges locally faster than viscosity method.

Remark 2.3. In [3], Dong et al proved the strong convergence theorem of the shrinking projection methods under the assumption that a sequence of class \mathfrak{T} mappings $\{T_n\}$ is coherent (see definition 1.1 in [3]). In Theorem 2.1, the condition (Z) is needed for a sequence of class \mathfrak{T} mappings $\{T_n\}$. Comparing the definition of coherent and condition (Z), it is obvious that a sequence $\{T_n\}$ satisfies condition (Z) if it is coherent. So, in order to obtain strong convergence results, in this paper we just need a weaker condition than that in [3].

3 Deduced results

In this section, using Theorem 2.1, we obtain some strong convergence results for a class \mathfrak{T} mapping, a quasi-nonexpansive mapping and a nonexpansive mapping in a Hilbert space.

Theorem 3.1. Assume $T \in \mathfrak{T}$ with $Fix(T) \neq \emptyset$ satisfies that $I - T$ is demiclosed at 0. Let f be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{x_n\}$ as follow:

$$x_{n+1} = \alpha_n P_{H(x_n, Sx_n)} f(x_n) + (1 - \alpha_n) Sx_n, \quad (35)$$

where $S = (1 - w)I + wT$, $w \in (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $x^* \in Fix(T)$ verifying

$$x^* = (P_{Fix(T)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in Fix(T), \quad \text{and} \quad (\forall v \in Fix(T)), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

Proof. Let $T_n = T$ in (3) for all $n \in \mathbb{N}$. From Lemma 2.1, it follows that $\{x_n\}$ is bounded. Using the definition of demiclosed, we get that T satisfies condition (Z). From Theorem 2.1, the desired result follows.

Theorem 3.2. Assume $U : H \rightarrow H$ is a quasi-nonexpansive mapping with $Fix(U) \neq \emptyset$ and satisfies that $I - U$ is demiclosed at 0. Let f be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{x_n\}$ as follow:

$$x_{n+1} = \alpha_n P_{H(x_n, Vx_n)} f(x_n) + (1 - \alpha_n) Vx_n,$$

where $V = (1 - \gamma)I + \gamma U$, $\gamma \in (0, \frac{1}{2})$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $x^* \in \text{Fix}(U)$ verifying

$$x^* = (P_{\text{Fix}(U)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \text{Fix}(U), \quad \text{and} \quad (\forall v \in \text{Fix}(U)), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

Proof. By Lemma 1.1 (i), $\frac{U+I}{2} \in \mathfrak{T}$. Substitute T in (35) by $\frac{U+I}{2}$. Then,

$$\begin{aligned} S &= (1 - w)I + wT = (1 - w)I + w\frac{U + I}{2} \\ &= (1 - \frac{w}{2})I + \frac{w}{2}U. \end{aligned}$$

Set $\gamma = \frac{w}{2} \in (0, \frac{1}{2})$ and $V = S = (1 - \gamma)I + \gamma U$. Since $I - U$ is demiclosed at 0, $I - \frac{U+I}{2} = \frac{I-U}{2}$ is demiclosed at 0. So we can obtain the result by using Theorem 3.1.

Since a nonexpansive mapping is quasi-nonexpansive and demiclosed (see Lemma 1.2), using Theorem 3.2, we have following theorem.

Theorem 3.3. *Let $U : H \rightarrow H$ be a nonexpansive mapping with $\text{Fix}(U) \neq \emptyset$ and f be a contraction with constant $\rho \in [0, 1)$. Define a sequence $\{\alpha_n\}$ as follow:*

$$x_{n+1} = \alpha_n P_{H(x_n, Vx_n)} f(x_n) + (1 - \alpha_n) Vx_n,$$

where $V = (1 - \gamma)I + \gamma U$, $\gamma \in (0, \frac{1}{2})$, and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $x^* \in \text{Fix}(U)$ verifying

$$x^* = (P_{\text{Fix}(U)} \circ f)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \text{Fix}(U), \quad \text{and} \quad (\forall v \in \text{Fix}(U)), \quad \langle (I - f)x^*, v - x^* \rangle \geq 0.$$

4 Numerical tests

For comparing the convergent rate of viscosity projection with viscosity method, we compute two simple examples. Let $w = \frac{1}{3}$, $\alpha_n = \frac{1}{n}$, and $x_0 = y_0 = -0.3$. Consider two cases:

Case 1. $T_1(x) = \sin(x)$ and $f_1(x) = \cos(\frac{x}{2})$ with constant $\frac{1}{2}$;

Case 2. $T_2(x) = \cos(x)$ and $f_2(x) = \sin(\frac{x}{2})$ with constant $\frac{1}{2}$.

It is obvious T_1 and T_2 are two nonexpansive mappings on \mathbb{R} . From Figure 1, It illustrates that viscosity projection methods converges faster than viscosity methods for the given examples.

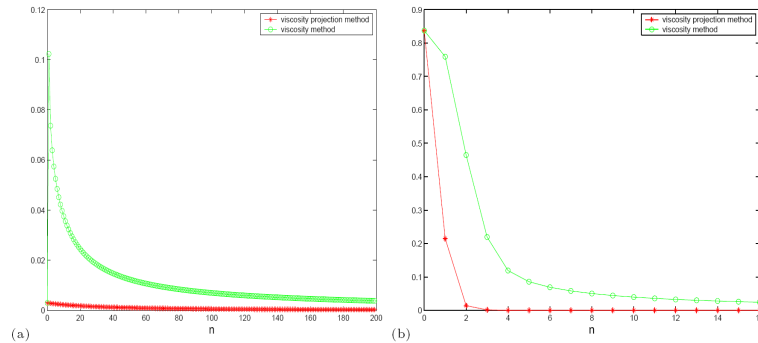


Figure 1: (a) Case 1 $\|x_n - Tx_n\|$; (b) Case 2 $\|x_n - Tx_n\|$.

Remark 4.1. *We just prove that viscosity projection method converges locally faster than viscosity in Theorem 2.2, and don't know if viscosity projection method converges faster than viscosity. It is an open problem.*

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