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Functional Model of Dissipative Fourth Order Differential Operators

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Abstract

In this paper, maximal dissipative fourth order operators with equal deficiency indices are investigated. We construct a self adjoint dilation of such operators. We also construct a functional model of the maximal dissipative operator which based on the method of Pavlov and define its characteristic function. We prove theorems on the completeness of the system of eigenvalues and eigenvectors of the maximal dissipative fourth order operators.

1 Introduction

The spectral analysis of non-selfadjoint operators is based on ideas of the functional model and dilation theory rather than on traditional resolvent analysis and Riesz integrals. Using a functional model of a non-selfadjoint operator as a principal tool, spectral properties of such operators are investigated. The functional model of non-selfadjoint dissipative operators plays an important role within both the abstract operator theory and its more specialized applications in other disciplines. The construction of functional models for dissipative operators, natural analogues of spectral decompositions for selfadjoint operators is based on Sz. Nagy-Foias dilation theory [19] and Lax-Phillips scattering theory [18]. Pavlov's approach ([21-23]) to the model construction of dissipative extensions of symmetric operators was followed by Allahverdiev in his works

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[1-5] and some others, and by the group of authors [6-8], where the theory of the dissipative Schrödinger operator on a finite interval was applied to the problems arising in the semiconductor physics. In [9-12], Pavlov's functional model was extended to (general) dissipative operators which are finite dimensional extensions of a symmetric operator, and the corresponding dissipative and Lax-Phillips scattering problems were investigated in some detail.

The organization of this document is as follows: In Section 2, we construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive, selfadjoint and other extensions of dissipative fourth order differential operators in terms of boundary conditions. Furthermore, we construct a selfadjoint dilation of the dissipative fourth order differential operator. In Section 3, we present its incoming and outgoing spectral representations which makes it possible to determine the scattering matrix of the dilation according to the Lax and Phillips scheme [17,18]. Later, a functional model of the dissipative fourth order differential operator is constructed by methods of Pavlov [21-23] and define its characteristic functions. Finally, in Section 4, we proved a theorem on completeness of the system of eigenvectors and associated vectors of dissipative operators under consideration. In the present paper, we extend the results of [1-5] to the more general eigenvalue problem for fourth order differential operators.

2 Selfadjoint Dilation of Dissipative Fourth Order Differential Operators

We will consider the differential expression

$$l(y) = y^{(4)} + q(x)\, y, \quad 0 \leq x < +\infty \tag{2.1}$$

where $q(x)$ is a real continuous function in $[0,\infty)$.

Let L_0 denote the closure of the minimal operator generated by (2.1) and by D_0 its domain. Besides, we denote by the set of all functions $y(x)$ from $L_2(0,\infty)$ whose first three derivatives are locally absolutely continuous in $[0,\infty)$ and $l(y) \in L_2(0,\infty)$; D is the domain of the maximal operator L. Furthermore $L = L_0^*$ [20].

Suppose that $q(x)$ be a function such that the operator L_0 has defect index $(4,4)$. Let $v_1(x), v_2(x), v_3(x)$ and $v_4(x)$ be four linearly independent solutions of the equation $l(y) = 0$ satisfying the conditions at $x = 0$:

$$\begin{aligned}
v_1(0) &= 1,\ v_1'(0) = 0,\ v_1''(0) = 0,\ v_1'''(0) = 0, \\
v_2(0) &= 0,\ v_2'(0) = 1,\ v_2''(0) = 0,\ v_2'''(0) = 0, \\
v_3(0) &= 0,\ v_3'(0) = 0,\ v_3''(0) = 1,\ v_3'''(0) = 0, \\
v_4(0) &= 0,\ v_4'(0) = 0,\ v_4''(0) = 0,\ v_4'''(0) = 1,
\end{aligned}$$

and their Wronskian equals to one. Since L_0 has defect index $(4,4)$, then $v_1, v_2, v_3, v_4 \in L_2(0,\infty)$.

Let's define by $\Gamma_1,\ \Gamma_2$ the linear maps from D to $\mathbb{C}^4$ by the formula

$$\Gamma_1 f = \begin{pmatrix} f(0) \\ f'(0) \\ [f,v_2]_\infty \\ [f,v_1]_\infty \end{pmatrix},\ \Gamma_2 f = \begin{pmatrix} f'''(0) \\ f''(0) \\ [f,v_4]_\infty \\ [f,v_3]_\infty \end{pmatrix} \tag{2.2}$$

where

$$[y,z]_x = [y'''(x)\, z(x) - y(x)\, z'''(x)] - [y''(x)\, z'(x) - y'(x)\, z''(x)], (0 \le x < \infty)$$

Lemma 1. *For arbitrary $y, z \in D$*

$$(Ly,z)_{L^2} - (y,Lz)_{L^2} = (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4}.$$

Proof. We know that every $f, g \in D$

$$[f,g](x) = \begin{vmatrix} [v_2,g]_x & [g,v_4]_x \\ [v_2,f]_x & [f,v_4]_x \end{vmatrix} + \begin{vmatrix} [v_1,g]_x & [g,v_3]_x \\ [v_1,f]_x & [f,v_3]_x \end{vmatrix} \tag{2.3}$$

(see [13]). For every $y,\ z \in D$, we have Green's formula

$$(Ly,z)_{L^2} - (y,Lz)_{L^2} = [y,\overline{z}]_\infty - [y,\overline{z}]_0.$$

Then

$$\begin{aligned} (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4} \quad = \quad & y(0)\, \overline{z}'''(0) - \overline{z}(0)\, y'''(0) \\ & + y''(0)\, \overline{z}'(0) - \overline{z}''(0)\, y'(0) \\ & + [y,v_2]_\infty [\overline{z},v_4]_\infty - [\overline{z},v_2]_\infty [y,v_4]_\infty \\ & + [y,v_1]_\infty [\overline{z},v_3]_\infty - [\overline{z},v_1]_\infty [y,v_3]_\infty. \end{aligned}$$

From the conditions (2.3), we have

$$(\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4} = [y,\overline{z}]_\infty - [y,\overline{z}]_0.$$

Hence

$$(Ly,z)_{L^2} - (y,Lz)_{L^2} = (\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}^4} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}^4}.$$

∎

Lemma 2. *For any complex numbers $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_0,\ \beta_1,\ \beta_2$ and β_3, there is a function $y \in D$ satisfying*

$$\begin{aligned} y(0) &= \alpha_0,\ y'(0) = \alpha_1,\ y''(0) = \alpha_2,\ y'''(0) = \alpha_3, \\ [y,v_1]_\infty &= \beta_0,\ [y,v_2]_\infty = \beta_1,\ [y,v_3]_\infty = \beta_2,\ [y,v_4]_\infty = \beta_3. \end{aligned} \tag{2.4}$$

Proof. Let f be an arbitrary element of $L_2(0,\infty)$ satisfying followings:

$$\begin{aligned}(f,v_1)_{L^2} &= \beta_0-\alpha_3,\ (f,v_3)_{L^2}=\beta_2-\alpha_1, \\ (f,v_2)_{L^2} &= \beta_1+\alpha_2,\ (f,v_4)_{L^2}=\beta_3+\alpha_0\ .\end{aligned} \tag{2.5}$$

There is such a f, even among the linear combinations of $v_1,\ v_2,\ v_3$ and v_4. If we set $f=c_1v_1+c_2v_2+c_3v_3+c_4v_4$ then conditions (2.5) are a system of equations in the constants $c_1,\ c_2,\ c_3,\ c_4$ whose determinant is the Gram determinant of the linearly independent functions $v_1,\ v_2,\ v_3,\ v_4$ and is therefore nonzero. Let $y(x)$ denote the solution of $l(y)=f$ satisfying the initial conditions $y(0)=\alpha_0,\ y'(0)=\alpha_1,\ y''(0)=\alpha_2,\ y'''(0)=\alpha_3$. We suppose that $y(x)$ is the desired element. Applying Green' formula to $y(x)$ and v_j, we get

$$(f,v_j)_{L^2}=(l(y),v_j)_{L^2}=[y,v_j]_\infty-[y,v_j]_0,\ j=1,\ 2,\ 3,\ 4.$$

But $l(v_j)=0\ \ (j=1,\ 2,\ 3,\ 4)$. Since $y(0)=\alpha_0,\ y'(0)=\alpha_1,\ y''(0)=\alpha_2,\ y'''(0)=\alpha_3$, we have

$$[y,v_j]_0=\left\{\begin{array}{cc}\alpha_3, & \text{for } j=1\\ -\alpha_2, & \text{for } j=2\\ \alpha_1, & \text{for } j=3\\ -\alpha_0, & \text{for } j=4\end{array}\right\}.$$

Hence

$$\begin{aligned}(f,v_1)_{L^2} &= [y,v_1]_\infty-\alpha_3,\ (f,v_2)_{L^2}=[y,v_2]_\infty+\alpha_2 \\ (f,v_3)_{L^2} &= [y,v_3]_\infty-\alpha_1,\ (f,v_4)_{L^2}=[y,v_4]_\infty+\alpha_0.\end{aligned}$$

According to (2.5), we have

$$[y,v_1]_\infty=\beta_0,\ [y,v_2]_\infty=\beta_1,\ [y,v_3]_\infty=\beta_2,\ [y,v_4]_\infty=\beta_3.$$

∎

We recall that a triple $(\mathbb{H},\Gamma_1,\Gamma_2)$ is called a space of boundary values of a closed symmetric operator A on a Hilbert space H if Γ_1,Γ_2 are linear maps from $D(A^*)$ to $\mathbb{H}$ with equal deficiency numbers such that:

i) Green's formula is valid

$$(A^*f,g)_H-(f,A^*g)_H=(\Gamma_1f,\Gamma_2g)_{\mathbb{H}}-(\Gamma_2f,\Gamma_1g)_{\mathbb{H}}\ ,\ f,g\in D(A^*)\,.$$

ii) For any $F_1,F_2\in H$, there is a vector $f\in D(A^*)$ such that $\Gamma_1f=F_1,\ \Gamma_2f=F_2$ ([14], [15]).

Theorem 1. *The triple* $\left(\mathbb{C}^4, \Gamma_1, \Gamma_2\right)$ *defined by (2.2) is a boundary spaces of the operator* L_0.

Proof. First condition of the definition of a space of boundary value follows from Lemma 1 and second condition follows from Lemma 2. ∎

Corollary 1. *For any contraction* K *in* $\mathbb{C}^4$ *the restriction of the operator* L *to the set of functions* $y \in D$ *satisfying either*

$$(K - I)\,\Gamma_1 y + i\,(K + I)\,\Gamma_2 y = 0 \tag{2.6}$$

or

$$(K - I)\,\Gamma_1 y - i\,(K + I)\,\Gamma_2 y = 0 \tag{2.7}$$

is respectively the maximal dissipative and accretive extension of the operator L_0. *Conversely, every maximal dissipative (accretive) extension of the operator* L_0 *is the restriction of* L *to the set of functions* $y \in D$ *satisfying (2.6) ((2.7)), and the extension uniquely determines the contraction* K. *Conditions (2.6) ((2.7)), in which* K *is an isometry describe the maximal symmetric extensions of* L_0 *in* $L_2(0,\infty)$. *If* K *is unitary, these conditions define selfadjoint extensions.*

In particular, the boundary conditions

$$y'''(0) + h_1 y(0) = 0, \tag{2.8}$$

$$y'(0) + h_2\, y''(0) = 0, \tag{2.9}$$

$$[y, v_2]_\infty - h_3 [y, v_4]_\infty = 0, \tag{2.10}$$

$$[y, v_1]_\infty - h_4 [y, v_3]_\infty = 0 \tag{2.11}$$

with $\operatorname{Im} h_1 \geq 0$ or $h_1 = \infty$, $\operatorname{Im} h_2 \geq 0$ or $h_2 = \infty$, $\operatorname{Im} h_3 \geq 0$ or $h_3 = \infty$ and $\operatorname{Im} h_4 \geq 0$ or $h_4 = \infty$ ($\operatorname{Im} h_1 = 0$ or $h_1 = \infty$, $\operatorname{Im} h_2 = 0$ or $h_2 = \infty$, $\operatorname{Im} h_3 = 0$ or $h_3 = \infty$ and $\operatorname{Im} h_4 = 0$ or $h_4 = \infty$) describe the maximal dissipative (selfadjoint) extensions of L_0 with separated boundary conditions.

Now, we study the maximal dissipative operator L_K, where K is the strict contraction in $\mathbb{C}^4$ generated by the expression $l(y)$ and boundary condition (2.6).

Let us define the "incoming" and "outgoing" subspaces $D_- = L^2(-\infty, 0)$ and $D_+ = L^2(0,\infty)$. The orthogonal sum H=$D_- \oplus H \oplus D_+$ is called *main Hilbert space of the dilation*.

In the space $\mathcal{H}$, we consider the operator $\mathcal{L}_G$ on the set $D(\mathcal{L}_G)$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+ \rangle$, generated by the expression

$$\mathcal{L}_G \langle \varphi_-, \widehat{y}, \varphi_+ \rangle = \langle i\frac{d\varphi_-}{d\xi}, l(y), i\frac{d\varphi_+}{d\xi} \rangle \tag{2.12}$$

satisfying the conditions: $\varphi_- \in W_2^1(-\infty,0)$, $\varphi_+ \in W_2^1(0,\infty)$, $y \in H$, $y'''(0)+h_1 y(0) = 0$, $y'(0) + h_2\, y''(0) = 0$, $[y,v_2]_\infty - h_3[y,v_4]_\infty = 0$, $[y,v_1]_\infty - G[y,v_3]_\infty = C\varphi_-(0)$, $[y,v_1]_\infty - \overline{G}[y,v_3]_\infty = C\varphi_+(0)$, where W_2^1 are Sobolev spaces and $C^2 := 2\operatorname{Im} G$, $C > 0$.

Theorem 2. *The operator $\mathcal{L}_G$ is selfadjoint in $\mathcal{H}$ and it is a selfadjoint dilation of the operator $\widetilde{L}_G\,(= L_K)$.*

Proof. Let $f, g \in D(\mathcal{L}_G)$, $f = \langle \varphi_-, y, \varphi_+ \rangle$ and $g = \langle \psi_-, z, \psi_+ \rangle$. Then we have

$$
\begin{aligned}
(\mathcal{L}_G f, g)_{\mathcal{H}} - (f, \mathcal{L}_G g)_{\mathcal{H}} \quad &= \quad \big(\mathcal{L}_G\langle \varphi_-, y, \varphi_+\rangle, \langle \psi_-, z, \psi_+\rangle\big) \\
&\quad - \big(\langle \varphi_-, y, \varphi_+\rangle, \mathcal{L}_G\langle \psi_-, z, \psi_+\rangle\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} i\varphi_-' \overline{\psi}_- d\xi + (l(y), z)_H + \int_0^{\infty} i\varphi_+' \overline{\psi}_+ d\xi \\
&\quad - \int_{-\infty}^{0} i\psi_-' \overline{\varphi}_- d\xi - (y, l(z))_H - \int_0^{\infty} i\psi_+' \overline{\varphi}_+ d\xi \\
&= \int_{-\infty}^{0} i\varphi_-' \overline{\psi}_- d\xi + [y, \overline{z}]_\infty + \int_0^{\infty} i\varphi_+' \overline{\psi}_+ d\xi \\
&\quad - \int_{-\infty}^{0} i\psi_-' \overline{\varphi}_- d\xi - [y, \overline{z}]_0 - \int_0^{\infty} i\psi_+' \overline{\varphi}_+ d\xi \\
&= i\psi_-(0)\,\overline{\varphi}_-(0) - i\varphi_+(0)\,\overline{\psi}_+(0) + [y,\overline{z}]_\infty - [y,\overline{z}]_0.
\end{aligned}
$$

By direct computation, we obtain

$$
i\psi_-(0)\,\overline{\varphi}_-(0) - i\varphi_+(0)\,\overline{\psi}_+(0) + [y,\overline{z}]_\infty - [y,\overline{z}]_0 = 0.
$$

Thus, $\mathcal{L}_G$ is a symmetric operator. If we show that $\mathcal{L}_G \subseteq \mathcal{L}_G^*$, we prove that $\mathcal{L}_G$ is selfadjoint. Take $g = \langle \psi_-, z, \psi_+ \rangle \in D(\mathcal{L}_G^*)$. Let $\mathcal{L}_G^* g = g^* = \langle \psi_-^*, z^*, \psi_+^* \rangle \in \mathcal{H}$, so that

$$
(\mathcal{L}_G f, g)_{\mathcal{H}} = (f, \mathcal{L}_G^* g)_{\mathcal{H}} = (f, g^*)_{\mathcal{H}}\,. \tag{2.13}
$$

It is easy to show that $\psi_- \in W_2^1(-\infty,0)$, $\psi_+ \in W_2^1(0,\infty)$, $g \in D(\mathcal{L}_G)$ and $g^* = \mathcal{L}_G g$, by choosing elements with suitable components as the $f \in D(\mathcal{L}_G)$ in (2.13). Then we have

$$
(\mathcal{L}_G f, g)_{\mathcal{H}} = (f, \mathcal{L}_G g)_{\mathcal{H}}
$$

for all $f \in D(\mathcal{L}_G^*)$. Furthermore, $g \in D(\mathcal{L}_G^*)$ satisfies the conditions

$$\begin{aligned} [y, v_1]_\infty - G[y, v_3]_\infty &= C\varphi_-(0), \\ [y, v_1]_\infty - \overline{G}[y, v_3]_\infty &= C\varphi_+(0). \end{aligned}$$

Hence, $D(\mathcal{L}_G^*) \subseteq D(\mathcal{L}_G)$, *i.e.*, $\mathcal{L}_G = \mathcal{L}_G^*$.

The selfadjoint operator $\mathcal{L}_G$ generates on $\mathcal{H}$ a unitary group $U_t = \exp(i\mathcal{L}_G t)$ $(t \in \mathbb{R}_+ = (0, \infty))$. Let denote by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mapping acting according to the formulae $P : \langle \varphi_-, y, \varphi_+ \rangle \to \widehat{y}$ and $P_1 : y \to \langle 0, y, 0 \rangle$. Let $Z_t := PU_tP_1,$ $t \geq 0$, by using U_t. The family $\{Z_t\}$ $(t \geq 0)$ of operators is a strongly continuous semigroup of completely nonunitary contraction on H. Let us denote by B_G the generator of this semigroup: $B_G\, y = \lim\limits_{t \to +0} (it)^{-1} (Z_t y - y)$. The domain of B_G consists of all the vectors for which the limit exists. The operator B_G is dissipative. The operator $\mathcal{L}_G$ is called the selfadjoint dilation of B_G (see [5, 16, 19]). We show that $B_G = \widetilde{L}_G$, hence $\mathcal{L}_G$ is selfadjoint dilation of B_G. To show this, it is sufficient to verify the equality

$$P(\mathcal{L}_G - \lambda I)^{-1} P_1 y = \left(\widetilde{L}_G - \lambda I\right)^{-1} y, y \in H, \ \operatorname{Im} h < 0. \tag{2.14}$$

For this purpose, we set $(\mathcal{L}_G - \lambda I)^{-1} P_1 y = g = \langle \psi_-, z, \psi_+ \rangle$ implies that $(\mathcal{L}_G - \lambda I) g = P_1 y$, and hence $l(z) - \lambda z = y$, $\psi_-(\xi) = \psi_-(0) e^{-i\lambda\xi}$ and $\psi_+(\xi) = \psi_+(0) e^{-i\lambda\xi}$. Since $g \in D(\mathcal{L}_G)$, then $\psi_- \in W_2^1(-\infty, 0)$, it follows that $\psi_-(0) = 0$, and consequently z satisfies the boundary condition $[y, v_1]_\infty - G[y, v_3]_\infty = 0$. Therefore $z \in D\left(\widetilde{L}_G\right)$, and since point λ with $\operatorname{Im} \lambda < 0$ cannot be an eigenvalue of dissipative operator, then $z = \left(\widetilde{L}_G - \lambda I\right)^{-1} y$. Thus

$$(\mathcal{L}_G - \lambda I)^{-1} P_1 y = \langle 0, \ \left(\widetilde{L}_G - \lambda I\right)^{-1} y, \ C^{-1}\left([y, v_1]_\infty - \overline{G}[y, v_3]_\infty\right) e^{-i\lambda\xi} \rangle$$

for y and $\operatorname{Im} \lambda < 0$. On applying the mapping P, we obtain (2.14), and

$$\begin{aligned} \left(\widetilde{L}_G - \lambda I\right)^{-1} &= P(\mathcal{L}_G - \lambda I)^{-1} P_1 = -iP \int_0^\infty U_t e^{-i\lambda t} dt P_1 \\ &= -i \int_0^\infty Z_t e^{-i\lambda t} dt = (B_G - \lambda I)^{-1}, \ \operatorname{Im} \lambda < 0, \end{aligned}$$

so this clearly shows that $\widetilde{L}_G = B_G$. ∎

3 Functional Model of Dissipative Fourth Order Operator

The unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips [18]. It has orthogonal incoming and outgoing subspaces $D_- = \langle L^2(-\infty,0), 0, 0\rangle$ and $D_+ = \langle 0, 0, L^2(0,\infty)\rangle$ having the following properties:

(1) $U_t D_- \subset D_-$, $t \leq 0$ and $U_t D_+ \subset D_+$, $t \geq 0$;

(2) $\bigcap_{t\leq 0} U_t D_- = \bigcap_{t\geq 0} U_t D_+ = \{0\}$;

(3) $\overline{\bigcup_{t\geq 0} U_t D_-} = \overline{\bigcup_{t\leq 0} U_t D_+} = \mathcal{H}$;

(4) $D_- \perp D_+$.

Property (4) is clear. To be able to prove property (1) for D_+ (the proof for D_- is similar), we set $\mathcal{R}_\lambda = (\mathcal{L}_G - \lambda I)^{-1}$. For all λ, with $\operatorname{Im}\lambda < 0$ and for any $f = \langle 0, 0, \varphi_+\rangle \in D_+$, we have

$$\mathcal{R}_\lambda f = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\varphi_+(s)\,ds\rangle.$$

As $\mathcal{R}_\lambda f \in D_+$. Therefore, if $g \perp D_+$, then

$$0 = (\mathcal{R}_\lambda f, g)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}(U_t f, g)_{\mathcal{H}}\,dt,\ \ \operatorname{Im}\lambda < 0.$$

which implies that $(U_t f, g)_{\mathcal{H}} = 0$ for all $t \geq 0$. Hence, for $t \geq 0$, $U_t D_+ \subset D_+$, and property (1) has been proved.

In order to prove property (2), we define the mappings $P^+ : \mathcal{H} \to L^2(0,\infty)$ and $P_1^+ : L^2(0,\infty) \to D_+$ as follows $P^+ : \langle \varphi_-, \widehat{y}, \varphi_+\rangle \to \varphi_+$ and $P_1^+ : \varphi \to \langle 0, 0, \varphi\rangle$, respectively. We take into consider that the semigroup of isometries $U_t^+ := P^+ U_t P_1^+ \ \ (t \geq 0)$ is a one-sided shift in $L^2(0,\infty)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2(0,\infty)$ is the differential operator $i\left(\frac{d}{d\xi}\right)$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator S of the semigroup of isometries U_t^+ $(t \geq 0)$ is the operator $S\varphi = P^+\mathcal{L}_G P_1^+\varphi = P^+\mathcal{L}_G\langle 0, 0, \varphi\rangle = P^+\langle 0, 0, i(\frac{d}{d\xi})\varphi\rangle = i(\frac{d}{d\xi})\varphi$, where $\varphi \in W_2^1(0,\infty)$ and $\varphi(0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and, hence,

$$\bigcap_{t\geq 0} U_t D_+ = \langle 0, 0, \bigcap_{t\leq 0} V_t L^2(0,\infty)\rangle = \{0\},$$

so the proof is completed.

Definition 1. *The linear operator A with domain $D(A)$ acting in the Hilbert space H is called completely nonselfadjoint (or simple) if there is no invariant subspace $M \subseteq D(A)$ $(M \neq \{0\})$ of the operator A on which the restriction A to M is selfadjoint.*

To prove property (3) of the incoming and outgoing subspaces, let us prove following lemma.

Lemma 3. *The operator $\widetilde{L}_G$ is completely noselfadjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which $\widetilde{L}_G$ induces a selfadjoint operator $\widetilde{L}'_G$ with domain $D\left(\widetilde{L}'_G\right) = H' \cap D\left(\widetilde{L}_G\right)$. If $f \in D\left(\widetilde{L}'_G\right)$,then $f \in D\left(\widetilde{L}^*_G\right)$ and

$$\begin{aligned} 0 &= \frac{d}{dt}\|e^{i\widetilde{L}'_G t}f\|^2_H = \frac{d}{dt}\left(e^{i\widetilde{L}'_G t}f, e^{i\widetilde{L}'_G t}f\right)_H \\ &= -C^2\left([e^{i\widetilde{L}'_G t}f, v_3]\right). \end{aligned}$$

Consequently, we have $[e^{i\widetilde{L}'_G t}f, v_3] = 0$. Using this result with boundary condition $[y, v_1]_\infty - G[y, v_3]_\infty = 0$, we have $[y, v_1]_\infty = 0$, i.e., $y(\lambda) = 0$. Since all solutions of $l(y) = \lambda y$ belong to $L^2(0, \infty)$, from this it can be concluded that the resolvent $R_\lambda\left(\widetilde{L}_G\right)$ is a compact operator, and the spectrum of $\widetilde{L}_G$ is purely discrete. Consequently, by the theorem on expansion in the eigenvectors of the selfadjoint operator $\widetilde{L}'_G$, we obtain $H' = \{0\}$. Hence the operator $\widetilde{L}_G$ is simple. The proof is completed. ∎

Let us define $H_- = \overline{\bigcup_{t \geq 0} U_t D_-}$, $H_+ = \overline{\bigcup_{t \leq 0} U_t D_+}$.

Lemma 4. *The equality $H_- + H_+ = \mathcal{H}$ holds.*

Proof. Considering property (1) of the subspace D_+, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ (and hence also H') were nontrivial, then the unitary group $\{U'_t\}$ restricted to this subspace would be a unitary part of the group $\{U_t\}$, and hence, the restriction $\widetilde{L}'_G$ of $\widetilde{L}_G$ to H' would be a selfadjoint operator in H'. Since the operator $\widetilde{L}_G$ is simple, it follows that $H' = \{0\}$. The lemma is proved. ∎

Assume that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of $l(y)=\lambda y$ satisfying the conditions

$$\begin{aligned}
\varphi'(0,\lambda) &= 1,\ \varphi(0,\lambda)=0,\ \psi'(0,\lambda)=0,\ \psi(0,\lambda)=1,\\
[\varphi,v_4]_\infty &= -\frac{1}{\sqrt{1+h_3^2}},\ [\varphi,v_2]_\infty=\frac{h_3}{\sqrt{1+h_3^2}},\\
[\psi,v_4]_\infty &= \frac{h_3}{\sqrt{1+h_3^2}},\ [\psi,v_2]_\infty=\frac{1}{\sqrt{1+h_3^2}}.
\end{aligned}$$

Let us adopt the following notations:

$$k(\lambda):=-\frac{[\varphi,v_1]_\infty}{[\psi,v_3]_\infty},\ M(\lambda)=-\frac{[\psi,v_3]_\infty}{[\varphi,v_3]_\infty},$$

$$S_G(\lambda)=\frac{M(\lambda)k(\lambda)-G}{M(\lambda)k(\lambda)-\overline{G}}. \tag{3.1}$$

$M(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $M(\lambda)$ possesses the following properties: $\operatorname{Im} M(\lambda)\le 0$ for all $\operatorname{Im}\lambda\neq 0$, and $M^*(\lambda)=M(\overline{\lambda})$ for all $\lambda\in\mathbb{C}$, except the real poles $M(\lambda)$.

We set

$$U_\lambda^-(x,\xi,\zeta)=$$
$$\langle e^{-i\lambda\xi},\alpha M(\lambda)\left[(M(\lambda)k(\lambda)-G)[\psi,v_3]_\infty\right]^{-1}\overline{\varphi(x,\lambda)},\overline{S_G}(\lambda)e^{-i\lambda\zeta}\rangle.$$

We note that the vectors $U_\lambda^-(x,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_\lambda^-(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}U=\lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_h$.

By means of vector $U_\lambda^-(x,\xi,\zeta)$, we define the transformation $F_-: f\to \widetilde{f}_-(\lambda)$ by

$$(F_-f)(\lambda):=\widetilde{f}_-(\lambda):=\frac{1}{\sqrt{2\pi}}(f,U_{\overline{\lambda}})_{\mathcal{H}}$$

on the vectors $f=\langle\varphi_-,\widehat{y},\varphi_+\rangle$ in which $\varphi_-(\xi),\ \varphi_+(\zeta),\ y(x)$ are smooth, compactly supported functions

Lemma 5. *The transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$. For all vectors $f,\ g\in H_-$ the Parseval equality and the inversion formulae hold:*

$$(f,g)_{\mathcal{H}}=\left(\widetilde{f}_-,\widetilde{g}_-\right)_{L^2}=\int_{-\infty}^{\infty}\widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda,\ f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_-(\lambda)U_{\overline{\lambda}}d\lambda,$$

where $\widetilde{f}_-(\lambda)=(F_-f)(\lambda)$ and $\widetilde{g}_-(\lambda)=(F_-g)(\lambda)$.

Proof. For $f,g\in D_-$, $f=\langle\varphi_-,0,0\rangle$, $g=\langle\psi_-,0,0\rangle$, with Paley-Wiener theorem, we have

$$\widetilde{f_-}(\lambda)=\frac{1}{\sqrt{2\pi}}(f,U_{\overline{\lambda}})_{\mathcal{H}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi_-(\xi)\,e^{-i\lambda\xi}d\xi\in H_-^2$$

and by using usual Parseval equality for Fourier integrals,

$$(f,g)_{\mathcal{H}}=\int_{-\infty}^{\infty}\varphi_-(\xi)\,\overline{\psi_-(\xi)}d\xi=\int_{-\infty}^{\infty}\widetilde{f_-}(\lambda)\,\overline{\widetilde{g_-}(\lambda)}d\lambda=(F_-f,F_-g)_{L^2}\,.$$

Here, $H_\pm^2$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively.

We now extend to the Parseval equality to the whole of H_-. We consider in H_- the dense set of H'_- of the vectors obtained as follows from the smooth, compactly supported functions in D_- : $f\in H'_-$ if $f=U_T\ f_0$, $f_0=\langle\varphi_-,0,0\rangle$, $\varphi_-\in C_0^\infty(-\infty,0)$, where $T=T_f$ is a nonnegative number depending on f. If $f,g\in H'_-$, then for $T>T_f$ and $T>T_g$ we have $U_{-T}f,U_{-T}g\in D_-$, moreover, the first components of these vectors belong to $C_0^\infty(-\infty,0)$. Therefore, since the operators U_t $(t\in\mathbb{R})$ are unitary, by the equality

$$F_-U_tf=\left(U_tf,U_\lambda^-\right)_{\mathcal{H}}=e^{i\lambda t}\left(f,U_\lambda^-\right)_{\mathcal{H}}=e^{i\lambda t}F_-f,$$

we have

$$(f,g)_{\mathcal{H}}=(U_{-T}\ f,U_{-T}\ g)_{\mathcal{H}}=(F_-U_{-T}\ f,F_-U_{-T}\ g)_{L^2}$$

and

$$(e^{i\lambda T}F_-f,e^{i\lambda T}F_-g)_{L^2}=\left(\widetilde{f},\widetilde{g}\right)_{L^2}. \tag{3.2}$$

By taking the closure (3.2), we obtain the Parseval equality for the space H_-. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally $F_-H_-=\overline{\underset{t\geq0}{\cup}F_-U_tD_-}=\overline{\underset{t\geq0}{\cup}e^{i\lambda t}H_-^2}=L^2(\mathbb{R})$, that is F_- maps H_- onto the whole of $L^2(\mathbb{R})$. The lemma is proved. ∎

We set

$$U_\lambda^+(x,\xi,\zeta)=$$
$$\langle S_G(\lambda)\,e^{-i\lambda\xi},\alpha M(\lambda)\left[\left(M(\lambda)\,k(\lambda)-\overline{G}\right)[\psi,v_3]_\infty\right]^{-1}\varphi(x,\lambda)\,,e^{-i\lambda\zeta}\rangle.$$

We note that the vectors $U_\lambda^+(x,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_\lambda^+(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}U=\lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_h$. With the help of vector $U_\lambda^+(x,\xi,\zeta)$, we define the transformation $F_+ : f \to \widetilde{f}_+(\lambda)$ by $(F_+f)(\lambda) := \widetilde{f}_+(\lambda) := \frac{1}{\sqrt{2\pi}}\left(f,U_\lambda^+\right)_{\mathcal{H}}$ on the vectors $f=\langle\varphi_-,\widehat{y},\varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$ and $y(x)$ are smooth, compactly supported functions.

Lemma 6. *The transformation F_+ isometrically maps H_+ onto $L^2(\mathbb{R})$. For all vectors $f,g\in H_+$ the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathcal{H}}=\left(\widetilde{f}_+,\widetilde{g}_+\right)_{L^2}=\int_{-\infty}^{\infty}\widetilde{f}_+(\lambda)\overline{\widetilde{g}_+(\lambda)}d\lambda,\qquad f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_+(\lambda)U_\lambda^+d\lambda,$$

where $\widetilde{f}_+(\lambda)=(F_+f)(\lambda)$ and $\widetilde{g}_+(\lambda)=(F_+g)(\lambda)$.

Proof. The proof is analogous to the Lemma 5. ∎

It is obvious that the matrix-valued function $S_G(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (3.1), $|S_G(\lambda)|\leq 1$ for $\operatorname{Im}\lambda>0$; and $S_G(\lambda)$ is the unitary matrix for all $\lambda\in\mathbb{R}$. Therefore, it explicitly follows from the formulae for the vectors U_λ^- and U_λ^+ that

$$U_\lambda^+=S_G(\lambda)\,U_\lambda^-. \tag{3.3}$$

It follows from Lemmas 5 and 6 that $H_-=H_+$. Together with Lemma 4, this shows that $H_-=H_+=\mathcal{H}$, therefore property (3) above has been proved for the incoming and outgoing subspaces.

Thus, the transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that F_- is the incoming spectral representation for the group $\{U_t\}$. Similarly, F_+ is the outgoing spectral representation for the group $\{U_t\}$. It follows from (3.3) that the passage from the F_- representation of an element $f\in\mathcal{H}$ to its F_+ representation is accomplished as $\widetilde{f}_+(\lambda)=S_G^{-1}(\lambda)\,\widetilde{f}_-(\lambda)$. Consequently, according to [18], we have proved the following.

Theorem 3. *The function $\overline{S_G}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the selfadjoint operator $\mathcal{L}_G$).*

Let $S(\lambda)$ be an arbitrary nonconstant inner function (see [19]) on the upper half-plane (the analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S_h(\lambda)|\leq 1$ for all $\lambda\in\mathbb{C}_+$ and $|S_h(\lambda)|=1$ for almost all $\lambda \in \mathbb{R}$). Define $K = H_+^2 \ominus SH_+^2$. Then $K \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators Z_t $(t \geq 0)$ acting in K according to the formula $Z_t\varphi = P\left[e^{i\lambda t}\varphi\right],\ \varphi = \varphi(\lambda) \in K$, where P is the orthogonal projection from H_+^2 onto K. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t\to+0}(it)^{-1}(Z_t\varphi - \varphi),$$

which T is a maximal dissipative operator acting in K and with the domain $D(T)$ consisting of all functions $\varphi \in K$, such that the limit exists. The operator T is called *a model dissipative operator.* Recall that this model dissipative operator, which is associated with the names of Lax-Phillips [18], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [19]. The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $K = \langle 0, H, 0\rangle$, so that $\mathcal{H} = D_- \oplus K \oplus D_+$. It follows from the explicit form of the unitary transformation F_- under the mapping F_-

$$\begin{aligned}
\mathcal{H} &\rightarrow L^2(\mathbb{R}),\ f \rightarrow \widetilde{f}_-(\lambda) = (F_-f)(\lambda),\ D_- \rightarrow H_-^2,\ D_+ \rightarrow S_G H_+^2,\\
K &\rightarrow H_+^2 \ominus S_G H_+^2,\ U_t \rightarrow \left(F_-U_tF_-^{-1}\widetilde{f}_-\right)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda).
\end{aligned} \tag{3.4}$$

The formulas (3.4) show that operator $\widetilde{L}_G$ is a unitarily equivalent to the model dissipative operator with the characteristic function $S_G(\lambda)$. We have thus proved following theorem.

Theorem 4. *The characteristic function of the maximal dissipative operator $\widetilde{L}_G$ coincides with the function $S_G(\lambda)$ defined* (3.1).

4 The spectral properties of dissipative fourth order operators

Using characteristic function, we investigate the spectral properties of the maximal dissipative operator $\widetilde{L}_G$ (L_K). We know that the characteristic function of a maximal dissipative operator carries information about the spectral properties of this operator. In order to prove completeness of the system of eigenvectors and associated vectors of the operator $\widetilde{L}_G$ (L_K) in the space $L_2(0,\infty)$ (see [5, 16, 19]), we must show that there exists no singular factor $s(\lambda)$ of the characteristic funcion $S_G(\lambda)$ in the factorization $detS_G(\lambda) = s(\lambda)B(\lambda)$ $(B(\lambda)$ is a Blaschke product) (see [3, 22, 25]).

The characteristic function $S_G(\lambda)$ of the maximal dissipative operator $\widetilde{L}_G$ has the form
$$S_G(\lambda) := \frac{M(\lambda)\,k(\lambda) - G}{M(\lambda)\,k(\lambda) - \overline{G}},$$
where $\operatorname{Im} G > 0$.

Theorem 5. *For all the values of G with $\operatorname{Im} G > 0$, except possibly for a single value $G = G_0$, the characteristic function $S_G(\lambda)$ of the maximal dissipative operator $\widetilde{L}_G$ is a Blaschke product. The spectrum of $\widetilde{L}_G$ is purely discrete and belongs to the open upper half-plane. The operator $\widetilde{L}_G$ $(G \neq G_0)$has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors of the operator $\widetilde{L}_G$ is complete in the space H.*

Proof. From (3.1), it is clear that $S_G(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane. Therefore, it can be factored in the form
$$S_G(\lambda) = e^{i\lambda c} B_G(\lambda),\ c = c(G) \geq 0, \tag{4.1}$$
where $B_G(\lambda)$ is a Blaschke product. It follows from (4.1) that
$$|S_G(\lambda)| = \left|e^{i\lambda c}\right| |B_G(\lambda)| \leq e^{-c(G)\operatorname{Im}\lambda},\ \operatorname{Im}\lambda \geq 0. \tag{4.2}$$
Further, expressing $A_G(\lambda) := M(\lambda)\,K(\lambda)$ in terms of $S_G(\lambda)$, we find from (3.1) that
$$A_G(\lambda) = \frac{\overline{G} S_G(\lambda) - G}{1 - S_G(\lambda)}. \tag{4.3}$$
For a given value G ($\operatorname{Im} G > 0$), if $c(G) > 0$, then (4.2) implies that $\lim_{t\to+\infty} S_G(it) = 0$, and then (4.3) gives us that $\lim_{t\to+\infty} A_G(it) = G_0$. Since $A_G(\lambda)$ does not depend on G, this implies that $c(G)$ can be nonzero at not more than a single point $G = G_0$ (and further $G_0 = -\lim_{t\to+\infty} A_G(it)$). This completes the proof. ∎

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