DOI: 10.2478/auom-2013-0034
An. Şt. Univ. Ovidius Constanţa
VERSITA
Vol. 21(2),2013, 207–222

Altering distances, some generalizations of Meir - Keeler theorems and applications

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Abstract

The purpose of this paper is to prove a general theorem of Meir - Keeler type using the notion of altering distance for occasionally weakly compatible mappings satisfying an implicit relation.

As application, a problem of Meir - Keeler type satisfying a condition of integral type becomes a special case of a problem of Meir - Keeler type with an altering distance.

1 Introduction

Let f and g be self mappings of a metric space (X, d). We say that $x \in X$ is a coincidence point of f and g if $fx = gx$.

We denote by $C(f, g)$ the set of all coincidence points of f and g.

A point w is a point of coincidence of f and g if there exists an $x \in X$ such that $w = fx = gx$.

Jungck [10] defined f and g to be compatible if $\lim d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some $t \in X$.

In 1994, Pant introduced the notion of pointwise R - weakly commuting mappings. It is proved in [25] that pointwise R - weakly commuting is equivalent to commutativity in coincidence points.

Key Words: Fixed point, Meir - Keeler type, altering distance, integral type, occasionally weakly compatible.
2010 Mathematics Subject Classification: Primary 54H25; Secondary 47H10.
Received: January 2013
Accepted: February 2013

Definition 1.1 ([11])**.** *Two self mappings of a metric space (X, d) are said to be weakly compatible if $fgu = gfu$ for each $u \in C(f, g)$.*

Al - Thagafi and Naseer Shahzad [3] introduced the notion of occasionally weakly compatible mappings.

Definition 1.2. *Two self mappings f and g of a metric space (X, d) are said to be occasionally weakly compatible (owc) mappings if there exists a point $x \in X$ which is a coincidence point of f and g at which f and g commute.*

Remark 1.1. *Two weakly compatible mappings having coincidence points are owc. The converse is not true, as shown in the Example of [3].*

Some fixed point theorems for occasionally weakly compatible mappings are proved in [2], [12], [34] and in other papers.

Lemma 1.1 ([12])**.** *Let X be a nonempty set and let f and g be owc self maps of X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

2 Preliminaries

In 1969, Meir and Keeler [19], established a fixed point theorem for self mappings of a metric space (X, d) satisfying the following condition:

for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(fx, fy) < \varepsilon. \tag{2.1}$$

There exists a vast literature which generalizes the result of Meir-Keeler. In [18], Mati and Pal proved a fixed point theorem for a self mapping of a metric space (X, d) satisfying the following condition which is a generalization of (2.1):

for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < \max\{d(x, y), d(x, fx), d(y, fy)\} < \varepsilon + \delta \text{ implies } d(fx, fy) < \varepsilon. \tag{2.2}$$

In [29] and [35], Park-Rhoades, respectively Rao-Rao extend this result for two self mappings f and g of a metric space (X, d) satisfying the following condition:

$$\varepsilon < \max\left\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{1}{2}\left[d(fx, gy) + d(fy, gx)\right]\right\} < \varepsilon + \delta \tag{2.3}$$

implies $d(gx, gy) < \varepsilon$.

In 1986, Jungck [10] and Pant [21] extend these results for four mappings. It is known by Jungck [10], Pant [22], [24], [25] and other papers that, in the case of theorems for four mappings A, B, S and $T : (X, d) \to (X, d)$, a condition of Meir-Keeler type does not assure the existence of a fixed point. The following theorem is stated in [9].

Theorem 2.1. *Let (A, S) and (B, T) be compatible pairs of self mappings of a complete metric space (X, d) such that*

(1) $AX \subset TX$ and $BX \subset SX$,

(2) given an $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \text{ implies } d(Ax, By) < \varepsilon,$$

where

$$M(x,y) = \max \left\{ d(Sx,Ty), d(Sx,Ax), d(Ty,By), \frac{1}{2}\left[d(Sx,By) + d(Ty,Ax)\right] \right\}$$

and

(3)

$$\begin{aligned} d(Ax,By) \quad < \quad & k(d(Sx,Ty) + d(Sx,Ax) + d(Ty,By) + \\ & +d(Sx,By) + d(Ty,Ax)) \end{aligned}$$

for all $x, y \in X$, where $k \in \left[0, \frac{1}{3}\right)$.

If one of mappings A, B, S and T is continuous then A, B, S and T have a unique common fixed point.

Some similar theorems are proved in [8], [27], [28] and in other papers. Recently, Theorem 2.1 was improved and extended for weakly compatible pairs in [4].

Theorem 2.2. *Let (A, S) and (B, T) be weakly compatible pairs of self mappings of a complete metric space (X, d) such that the following conditions hold:*

1) $AX \subset TX$ and $BX \subset SX$,

2) one of AX, BX, SX and TX is closed, *3) for each* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$\varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } d(Ax,By) \leq \varepsilon,$$

4) $x, y \in X$*,* $M(x,y) > 0$ *implies* $d(Ax,By) < M(x,y)$*,*

5)

$$\begin{aligned} d(Ax,By) \quad < \quad & k\,[\, d(Sx,Ty) + d(Sx,Ax) + d(Ty,By) \\ & +d(Sx,By) + d(Ty,Ax)\,] \end{aligned}$$

for all $x, y \in X$*, where* $k \in \left[0, \frac{1}{3}\right)$*.*

Then A*,* B*,* S *and* T *have a unique common fixed point.*

Other generalizations of Theorem 2.1 are proved in [5].

Theorem 2.3. *Let* (A,S) *and* (B,T) *be weakly compatible pairs of self mappings of a complete metric space* (X,d) *such that*

1) $AX \subset TX$ *and* $BX \subset SX$*,*

2) one of AX*,* BX*,* SX *and* TX *is closed,*

3) given an $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that for all* x, y *in* X

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \text{ implies } d(Ax,By) < \varepsilon,$$

4)

$$\begin{aligned} d(Ax,By) \quad \leq \quad & k \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \\ & d(Sx,By), d(Ty,Ax)\} \end{aligned}$$

for all x, y *in* X*, where* $k \in [0,1)$*.*

Then A*,* B*,* S *and* T *have a unique common fixed point.*

Remark 2.1. *Because if* (X,d) *is complete and* AX*,* BX*,* SX*,* TX *are closed, then* AX*,* BX*,* SX*,* TX *are complete subspaces of* X*, in Theorems 2.2, 2.3 the conditions that* X *is complete and* AX*,* BX*,* SX*,* TX *are closed should be replaced by the statement that one of* AX*,* BX*,* SX*,* TX *is a complete subspace of* X*.*

In [6], Branciari established the following result.

Theorem 2.4. *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ such that*

$$\int_0^{d(fx,fy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt, \tag{2.4}$$

whenever $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e. with a finite integral) on each compact subset of $[0,\infty)$ such that for $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. Then, f has a unique fixed point z such that, for each $x \in X$, $\lim f x_n = z$.

Theorem 2.4 is extended to compatible, weakly compatible, occasional weakly compatible in [1], [15], [16], [20], [34] and in other papers.

Quite recently, Gairola and Rawat [7] proved a fixed point theorem for two pairs of maps satisfying a new contractive condition of integral type, using the concept of occasionally weakly compatible mappings, which generalize Theorem 2.1.

Theorem 2.5. *Let A, B, S and T be self mappings of a metric space (X,d) satisfying the following conditions:*

1) $AX \subset TX$ and $BX \subset SX$,

2) given $\varepsilon > 0$ there exists $\delta > 0$ such that for all x, y in X,

$$\int_0^{M(x,y)} h(t)dt < \varepsilon + \delta \text{ implies } \int_0^{d(Ax,By)} h(t)dt \le \varepsilon,$$

where $h(t)$ is as in Theorem 2.4 and $\int_0^{M(x,y)} h(t)dt > 0$ implies

$$\int_0^{d(Ax,By)} h(t)dt < \int_0^{M(x,y)} h(t)dt,$$

3) $$\int_0^{d(Ax,By)} h(t)dt < k \left[\int_0^{d(Sx,Ty)} h(t)dt + \int_0^{d(Sx,Ax)} h(t)dt + \int_0^{d(Ty,By)} h(t)dt + \int_0^{d(Sx,By)} h(t)dt + \int_0^{d(Ty,Ax)} h(t)dt \right],$$

for all x, y in X and $k \in \left[0, \frac{1}{3}\right)$.

If one of AX, BX, SX, TX is a complete subspace of X, then:

a) A and S have a coincidence point,

b) B and T have a coincidence point.

Moreover, if the pairs (A, S) and (B, T) are occasionally weakly compatible mappings, then A, B, S and T have a unique common fixed point.

Definition 2.1. *An altering distance is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies*
(ψ_1): $\psi(t)$ is increasing and continuous,
(ψ_2): $\psi(t) = 0$ if and only if $t = 0$.

Fixed point problem involving an altering distance have been studied in [14], [17], [32], [36], [37] and in other papers.

Lemma 2.1. *The function $\psi(x) = \int_0^x h(t)dt$, where $h(t)$ is as in Theorem 2.4, is an altering distance.*

Proof. By definitions of $\psi(t)$ and $h(t)$ it follows that $\psi(x)$ is increasing and $\psi(x) = 0$ if and only if $x = 0$. By Lemma 2.5 [20], $\psi(x)$ is continuous. □

In [30] and [31] the study of fixed points for mappings satisfying implicit relations was initiated. A general fixed point theorem of Meir-Keeler type for noncontinuous weakly compatible mappings satisfying an implicit relation, which generalize (2.1) and others is proved in [33].

The purpose of this paper is to prove a general theorem of Meir-Keeler type using the notion of altering distance for occasionally weakly compatible mappings satisfying an implicit relation.

As an application, a problem of Meir-Keeler type satisfying a condition of integral type becomes a special case of Meir-Keeler type with an altering distance.

3 Implicit relation

Let $\mathcal{F}_{MK}$ be the set of all real continuous mappings $\phi(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$, increasing in t_1 satisfying the following conditions:
(ϕ_1): $\phi(t, 0, 0, t, t, 0) \leq 0$ implies $t = 0$,
(ϕ_2): $\phi(t, 0, t, 0, 0, t) \leq 0$ implies $t = 0$,
(ϕ_3): $\phi(t, t, 0, 0, t, t) > 0$, $\forall t > 0$.

Example 3.1. *$\phi(t_1, ..., t_6) = t_1 - k(t_2 + t_3 + t_4 + t_5 + t_6)$, where $k \in \left[0, \frac{1}{3}\right)$.*

Example 3.2. *$\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where a, b, $c \geq 0$, $b + c < 1$ and $a + 2c < 1$.*

Example 3.3. $\phi(t_1, ..., t_6) = t_1 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, *where* b, $c \geq 0$, $b < 1$ *and* $a + c < 1$.

Example 3.4. $\phi(t_1, ..., t_6) = t_1 - h\max\{t_2, t_3, t_4, t_5, t_6\}$, *where* $h \in (0, 1)$.

Example 3.5. $\phi(t_1, ..., t_6) = t_1 - h\max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, *where* $h \in (0, 1)$.

Example 3.6. $\phi(t_1, ..., t_6) = t_1^2 - at_2^2 - t_3t_4 - bt_5^2 - ct_6^2$, *where* a, b, $c \geq 0$ *and* $a + b + c < 1$.

Example 3.7. $\phi(t_1, ..., t_6) = t_1^3 - k(t_2^3 + t_3^3 + t_4^3 + t_5^3 + t_6^3)$, *where* $k \in \left[0, \frac{1}{3}\right)$.

Example 3.8. $\phi(t_1, ..., t_6) = t_1^3 - \dfrac{t_3^2 \cdot t_4^2 + t_5^2 \cdot t_6^2}{1 + t_2 + t_3 + t_4}$.

Example 3.9. $\phi(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* a, b, $c \geq 0$, $c < 1$ *and* $a + b < 1$.

Example 3.10. $\phi(t_1, ..., t_6) = t_1 - \alpha\max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, *where* $\alpha \in [0, 1)$, a, $b \geq 0$ *and* $a + b < 1$.

Example 3.11. $\phi(t_1, ..., t_6) = t_1 - \max\left\{t_2, \dfrac{1}{2}(t_3 + t_4), \dfrac{1}{2}[(t_5 + t_6)k]\right\}$, *where* $k \in [0, 1)$.

Example 3.12. $\phi(t_1, ..., t_6) = t_1 - \max\left\{k_1t_2, \dfrac{k_2}{2}(t_3 + t_4), \dfrac{t_5 + t_6}{2}\right\}$, *where* $k_1 \in [0, 1)$, $k_2 \in [1, 2)$.

Example 3.13. $\phi(t_1, ..., t_6) = t_1 - \max\left\{k_1(t_2 + t_3 + t_4), \dfrac{k_2}{2}(t_5 + t_6)\right\}$, *where* $k_1 \in [0, 1)$, $k_2 \in [0, 2)$.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying the following conditions:
(φ_1) : φ is continuous,
(φ_2) : φ is nondecreasing on $\mathbb{R}_+$,
(φ_3) : $0 < \varphi(t) < t$ for $t > 0$.

Example 3.14. $\phi(t_1, ..., t_6) = t_1 - \varphi\max\left\{t_2, t_3, t_4, t_5, \dfrac{t_6}{2}\right\}$.

Example 3.15. $\phi(t_1, ..., t_6) = t_1 - \varphi\max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}$.

Example 3.16. $\phi(t_1, ..., t_6) = t_1 - \varphi\max\left\{t_2, t_3, t_4, \dfrac{k}{2}(t_5 + t_6)\right\}$, *where* $k \in [0, 2)$.

Example 3.17. $\phi(t_1, ..., t_6) = t_1 - \varphi\max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}$.

4 Main results

Theorem 4.1. *Let A, B, S, T be self mappings of a metric space (X,d) satisfying the inequality:*

$$\begin{aligned}\phi(\psi(d(Ax,By)),\psi(d(Sx,Ty)),\psi(d(Ax,Sx)),\\ \psi(d(Ty,By)),\psi(d(Sx,By)),\psi(d(Ty,Ax)))\leq 0\end{aligned} \tag{4.1}$$

for all x, y in X, where ϕ satisfies property (ϕ_3) and ψ is an altering distance. If there exist $u,v\in X$ such that $Su=Au$ and $Tv=Bv$, then A and S have a unique point of coincidence and B and T have a unique point of coincidence.

Proof. First we prove that $Su=Tv$. If $Su\neq Tv$, using (4.1), we have, successively:

$$\begin{aligned}\phi(\psi(d(Au,Bv)),\psi(d(Su,Tv)),\psi(d(Au,Su)),\\ \psi(d(Tv,Bv)),\psi(d(Su,Bv)),\psi(d(Tv,Au)))\leq 0,\end{aligned}$$

$$\phi(\psi(d(Su,Tv)),\psi(d(Su,Tv)),0,0,\psi(d(Su,Tv)),\psi(d(Su,Tv)))\leq 0,$$

a contradiction of (ϕ_3) if $d(Su,Tv)>0$. Hence $\psi(d(Su,Tv))=0$, which implies that $Su=Tv$.

Assume that there exists a $p\in X$ such that $Ap=Sp$. Then by (4.1) we have successively:

$$\begin{aligned}\phi(\psi(d(Ap,Bv)),\psi(d(Sp,Tv)),\psi(d(Sp,Ap)),\\ \psi(d(Tv,Bv)),\psi(d(Sp,Bv)),\psi(d(Tv,Ap)))\leq 0,\end{aligned}$$

$$\phi(\psi(d(Sp,Tv)),\psi(d(Sp,Tv)),0,0,\psi(d(Sp,Tv)),\psi(d(Sp,Tv)))\leq 0,$$

a contradiction of (ϕ_3) if $d(Sp,Tv)>0$. Therefore $Sp=Tv$ and $z=Au=Su$ is the unique point of coincidence of A and S. Similarly, $w=Tv=Bv$ is the unique point of coincidence of B and T. □

Lemma 4.1. *Let A, B, S, T be self mappings of a metric space (X,d) such that $AX\subset TX$ and $BX\subset SX$ and ψ is an altering distance. For each $\varepsilon>0$ there exists a $\delta>0$ such that*

$$\varepsilon<\psi(M(x,y))<\varepsilon+\delta \text{ implies } \psi(d(Ax,By))\leq\varepsilon, \tag{4.2}$$

$$\psi(M(x,y))>0 \text{ implies } \psi(d(Ax,By))<\psi(M(x,y)). \tag{4.3}$$

For $x_0\in X$ and $\{y_n\}$ defined by $y_{2n-1}=Tx_{2n-1}=Ax_{2n-2}$ and $y_{2n}=Sx_{2n}=Bx_{2n-1}$ for $n\in\mathbb{N}^*$ and $d_n=d(y_n,y_{n+1})$, then $\lim d_n=0$.

Proof. First we prove that if, for some $k \in \mathbb{N}^*$, $d_{k+1} > 0$, then

$$\psi(d_{k+1}) < \psi(d_k). \tag{4.4}$$

a) Assume that $d_{2k} > 0$ for some $k \in \mathbb{N}^*$. Then $M(x_{2k}, x_{2k-1}) > 0$ otherwise $Ax_{2k} = Bx_{2k-1}$, i.e. $y_{2k} = y_{2k+1}$ so $d_{2k} = 0$, a contradiction. Hence $d_{2k} = d(Ax_{2k}, Bx_{2k-1}) < M(x_{2k}, x_{2k-1})$ which implies by (4.3) that $0 < \psi(d_{2k}) < \psi(M(x_{2k}, x_{2k-1})) \leq \psi(\max\{d_{2k-1}, d_{2k}\}) = \psi(d_{k-1})$.

b) If $d_{2k+1} > 0$ for some $k \in \mathbb{N}^*$, using a similar argument as in a), one can verify that $\psi(d_{2k+1}) < \psi(d_{2k})$.

c) Combining the results of a) and b) we may conclude that

$$\psi(d_{k-1}) < \psi(d_{2k}) \text{ for } k \in \mathbb{N}^*. \tag{4.5}$$

Moreover, if for some $k \in \mathbb{N}^*$, $\psi(d_k) = 0$, then $d_k = 0$ which implies $d_{k+1} = 0$ because, if $d_{k+1} > 0$, then $\psi(d_{k+1}) > 0$ which implies by a) and b) that $\psi(d_{k+1}) < \psi(d_k) = 0$, a contradiction. Hence, for $n \geq k$ we have $y_n = y_k$ and hence $\lim d(y_n, y_{n+1}) = 0$.

We prove that $\lim d(y_n, y_{n+1}) = 0$ for $\psi(d_k) > 0$.

By (4.5) it follows that $\psi(d_n)$ is strictly decreasing, hence convergent to some $\ell \in \mathbb{R}_+$. Suppose that $\ell > 0$. Then by (4.2) for $\varepsilon = \ell$, there exists a $\delta > 0$ such that $\ell < \psi(d_n) < \ell + \delta$ for $n \geq k$. In particular $\ell < \psi(M(x_{2k}, x_{2k+1})) < \ell + \delta$, since $M(x_{2k}, x_{2k+1}) = \max\{d_{2k}, d_{2k+1}\} \leq \ell$. Hence $\ell < \psi(d_{2k}) \leq \ell$, a contradiction, and $\ell = 0$. Let $a_n = \psi(d(y_n, y_{n+1}))$, $n \geq 0$. Then by the continuity of ψ we obtain

$$0 = \lim a_n = \lim \psi(d(y_n, y_{n+1})) = \psi(\lim d(y_n, y_{n+1})).$$

Hence $\lim d(y_n, y_{n+1}) = 0$. □

Theorem 4.2. *Let A, B, S and T be self mappings of a metric space (X, d) satisfying the following conditions:*

a) $AX \subset TX$ *and* $BX \subset SX$*,*

b) given an $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $x, y \in X$

$$\varepsilon < \psi(M(x,y)) < \varepsilon + \delta \text{ implies } \psi(d(Ax, By)) \leq \varepsilon,$$

c) $\psi(M(x,y)) > 0$ *implies* $\psi(d(Ax, By)) < \psi(M(x,y))$*,*

d) the inequality (4.1) holds for all x, y *in* X*, where* $\phi \in \mathcal{F}_{MK}$ *and* ψ *is an altering distance.*

If one of AX*,* BX*,* SX*,* TX *is a complete subspace of* X*, then:*

e) A and S have a coincidence point,

f) B and T have a coincidence point.

Moreover, if the pairs (S, A) and (T, B) are occasionally weakly compatible mappings, then A, B, S and T have a unique common fixed point.

Proof. First we prove that $\{y_n\}$ is a Cauchy sequence. Since by Lemma 4.1, $\lim d(y_n, y_{n+1}) = 0$ it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ such that for even integer $2k$, there exists even integers $2m(k)$ and $2n(k)$ such that $d(y_{2m(k)}, y_{2n(k)}) > \varepsilon$ with $2m(k) > 2n(k) \geq 2k$. For even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ such that $d(y_{2n(k)}, y_{2m(k)}) < \varepsilon$.

As in Theorem 2.2 [12] we deduce that

$$\lim d(y_{2n(k)}, y_{2m(k)}) = \varepsilon,$$
$$\lim d(y_{2n(k)}, y_{2m(k)-1}) = \varepsilon,$$
$$\lim d(y_{2n(k)+1}, y_{2m(k)-1}) = \varepsilon.$$

On the other hand we have successively

$$d(y_{2n(k)}, y_{2m(k)}) \leq d_{2n(k)} + d(Ax_{2n(k)}, Bx_{2m(k)-1}),$$
$$d(y_{2n(k)}, y_{2m(k)}) - d_{2n(k)} \leq d(y_{2n(k)+1}, y_{2m(k)}).$$

Hence

$$\psi(d(y_{2n(k)}, y_{2m(k)}) - d_{2n(k)}) \leq \psi(d(y_{2n(k)+1}, y_{2m(k)})).$$

Setting in (4.1) $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$ we obtain

$$\phi(\psi(d(y_{2n(k)}, y_{2m(k)})), \psi(d(y_{2n(k)}, y_{2m(k)-1})), \psi(d(y_{2n(k)}, y_{2n(k)+1})),$$
$$\psi(d(y_{2m(k)-1}, y_{2m(k)})), \psi(d(y_{2n(k)}, y_{2m(k)})), \psi(d(y_{2m(k)-1}, y_{2n(k)})) \leq 0,$$

$$\phi(\psi(d(y_{2n(k)}, y_{2m(k)}) - d_{2n(k)}), \psi(d(y_{2n(k)}, y_{2m(k)-1})), \psi(d(y_{2n(k)}, y_{2n(k)+1})),$$
$$\psi(d(y_{2m(k)-1}, y_{2m(k)})), \psi(d(y_{2n(k)}, y_{2m(k)})), \psi(d(y_{2m(k)-1}, y_{2n(k)})) \leq 0.$$

Letting n tend to infinity we obtain

$$\phi(\psi(\varepsilon), \psi(\varepsilon), 0, 0, \psi(\varepsilon), \psi(\varepsilon)) \leq 0,$$

a contradiction of (ϕ_3). Hence, $\{y_{2n}\}$ is a Cauchy sequence. It follows that $\{y_n\}$ is a Cauchy sequence.

Assume that least one of AX or TX is a complete subspace of X.

Since $y_{2n+1} \in AX \subset TX$ and $\{y_{2n+1}\}$ is a Cauchy sequence, there exists a $u \in TX$ such that $\lim y_{2n+1} = u$. The sequence $\{y_n\}$ converges to u since it is Cauchy and has the subsequence $\{y_{2n+1}\}$ convergent to u. Let $v \in X$ such that $u = Tv$. Setting $x = x_{2n}$ and $y = v$ in (4.1) we get

$$\phi(\psi(d(Ax_{2n}, Bv)), \psi(d(Sx_{2n}, Tv)), \psi(d(Sx_{2n}, Ax_{2n})), \psi(d(Tv, Bv)), \psi(d(Sx_{2n}, Bv)), \psi(d(Tv, Ax_{2n}))) \leq 0,$$

$$\phi(\psi(d(y_{2n+1}, Bv)), \psi(d(y_{2n}, Tv)), \psi(d(y_{2n}, y_{2n+1})), \psi(d(Tv, Bv)), \psi(d(y_{2n}, Bv)), \psi(d(u, y_{2n+1}))) \leq 0.$$

Letting n tend to infinity in the above inequality we obtain

$$\phi(\psi(d(u, Bv)), 0, 0, \psi(d(u, Bv)), \psi(d(u, Bv)), 0) \leq 0.$$

From (ϕ_1), $\psi(d(u, Bv)) = 0$ which implies that $d(u, Bv) = 0$ i.e. $u = Bv$. Hence $u = Tv = Bv$ and v is a coincidence point of T and B. Since $u = Bv \in BX \subset SX$, there exists a $w \in X$ such that $u = Sw$. Using a similar argument as above we obtain $u = Aw$. Hence, $u = Tv = Bv = Sw = Aw$.

Indeed, setting $x = w$ and $y = x_{2n+1}$ in (4.1) we obtain

$$\phi(\psi(d(Aw, y_{2n+2})), \psi(d(u, y_{2n+2})), \psi(d(u, Aw)), \psi(d(y_{2n+1}, y_{2n+2})), \psi(d(u, y_{2n+2})), \psi(d(y_{2n+1}, Aw))) \leq 0,$$

and letting n tend to infinity we obtain

$$\phi(\psi(d(Aw, u)), 0, \psi(d(u, Aw)), 0, 0, \psi(d(Aw, u))) \leq 0.$$

By (ϕ_1) it follows that $\psi(d(Aw, u)) = 0$, hence $u = Aw$.

By Theorem 4.1 u is the unique point of coincidence of A, S and B, T.

If the pairs (A, S) and (B, T) are occasionally weakly compatible, then by Lemma 1.1 u is the unique common fixed point of A, B, S and T. □

Taking $A = B$ and $S = T$ in Theorem 4.2 we obtain

Theorem 4.3. *Let A and S and T be self mappings of a metric space (X, d) satisfying the following conditions:*

1) *$AX \subset SX$,*
2) *$\varepsilon < \psi(M_1(x, y)) < \varepsilon + \delta$ implies $\psi(d(Ax, By)) \leq \varepsilon$, where $M_1(x, y) = \max\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ty), d(Sx, Ay), d(Sy, Ax)\}$,*
3) *$\psi(M_1(x, y)) > 0$ implies $\psi(d(Ax, By)) < \psi(M_1(x, y))$.*
4) *$\phi(\psi(d(Ax, Ay)), \psi(d(Sx, Sy)), \psi(d(Sx, Ax)), \psi(d(Sy, Ay)), \psi(d(Sx, Ay)), \psi(d(Sy, Ax))) \leq 0$ for all x, y in X, where $\phi \in \mathcal{F}_{MK}$ and ψ is an altering distance.*

   If one of SX and AX is a complete subspace of X, then:

5) A and S have a coincidence point.

Moreover, if the pair (A, S) is occasionally weakly compatible, then A and S have a unique common fixed point.

For $\psi(t) = t$ by Theorem 4.2 we obtain

Theorem 4.4. *Let A, B, S and T be self mappings of a metric space (X, d) satisfying the following conditions:*

1) $AX \subset SX$,

2) given an $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, y in X, $\varepsilon < M(x, y) < \varepsilon + \delta$ implies $d(Ax, By) \leq \varepsilon$,

3) $M(x, y) > 0$ implies $d(Ax, By) < M(x, y)$,

4) $\phi(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)) \leq 0$ for all x, y in X and $\phi \in \mathcal{F}_{MK}$.

If one of AX, BX, SX, TX is a complete subspace of X, then:

5) A and S have a coincidence point,

6) B and T have a coincidence point.

Moreover, if the pairs (A, S) and (B, T) are occasionally weakly compatible, then A, B, S and T have a unique common fixed point.

Remark 4.1. *1) By Example 3.1 and Theorem 4.4 we obtain a generalization of Theorem 2.2 and Theorem 2.1.*

2) By Example 3.4 and Theorem 4.4 we obtain a generalization of Theorem 2.1 [5].

3) By Example 3.12 and Theorem 4.4 we obtain a generalization of the result from [27].

4) By Example 3.11 and Theorem 4.4 we obtain a generalization of the result from [28] for $k \in [0, 1)$.

5) By Example 3.16 and Theorem 4.4 we obtain a generalization of Theorem 2.1 [26].

6) Theorem 4.4 is a generalization of Theorem 5 [33] for weakly compatible mappings satisfying an implicit relation.

7) By Examples 3.2, 3.3, 3.5 - 3.10, 3.13 - 3.15 we obtain new fixed point theorems of Meir - Keeler type.

5 Applications

Theorem 5.1. *Let A, B, S and T be self mappings of a metric space (X, d) satisfying conditions (a), (b), (c) of Theorem 4.2. Assume that there exists a $\phi \in \mathcal{F}_{MK}$ such that:*

$$\begin{aligned}&\phi\Big(\int_0^{d(Ax,By)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sx,Ax)} h(t)dt,\\ &\int_0^{d(Ty,By)} h(t)dt, \int_0^{d(Sx,By)} h(t)dt, \int_0^{d(Ty,Ax)} h(t)dt\Big) \leq 0\end{aligned} \tag{5.1}$$

for all x, y in X where $h(t)$ is as in Theorem 2.4.

If one of AX, BX, SX, TX is a complete subspace of X then:

1) A and S have a coincidence point,

2) B and T have a coincidence point.

Moreover, if the pairs (A, S) and (B, T) are occasionally weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Let

$$\begin{aligned}\psi(d(Ax, By)) &= \int_0^{d(Ax,By)} h(t)dt, \psi(d(Sx, Ty)) = \int_0^{d(Sx,Ty)} h(t)dt,\\ \psi(d(Sx, Ax)) &= \int_0^{d(Sx,Ax)} h(t)dt, \psi(d(Ty, By)) = \int_0^{d(Ty,By)} h(t)dt,\\ \psi(d(Sx, By)) &= \int_0^{d(Sx,By)} h(t)dt, \psi(d(Ty, Ax)) = \int_0^{d(Ty,Ax)} h(t)dt,\end{aligned}$$

where $h(t)$ is as in Theorem 2.4.

By Lemma 2.1 $\psi(x) = \int_0^x h(t)dt$ is an altering distance.

By (5.1) we obtain

$$\begin{aligned}&\phi(\psi(d(Ax, By)), \psi(d(Sx, Ty)), \psi(d(Sx, Ax)),\\ &\psi(d(Ty, By)), \psi(d(Sx, By)), \psi(d(Ty, Ax))) \leq 0\end{aligned}$$

for all x, y in X, which is inequality (4.1).

Hence, the conditions of Theorem 4.2 are satisfied and the conclusion of Theorem 5.1 it follows from Theorem 4.2. □

Remark 5.1. *If $h(t) = 1$ we obtain Theorem 4.3.*

Remark 5.2. *By Theorem 5.1 and Example 3.1 we obtain Theorem 2.5.*

Examples 3.2 - 3.17 are new results.

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