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Hereditary right Jacobson radicals of type-1(e) and 2(e) for right near-rings

Ravi Srinivasa Rao and K. Siva Prasad

Abstract

Near-rings considered are right near-rings. In this paper two more radicals, the right Jacobson radicals of type-1(e) and 2(e), are introduced for near-rings. It is shown that they are Kurosh-Amitsur radicals (KA-radicals) in the class of all near-rings and are ideal-hereditary radicals in the class of all zero-symmetric near-rings. Different kinds of examples are also presented.

1 Introduction

Near-rings considered are right near-rings and not necessarily zero-symmetric, and R is a near-ring. The (left) Jacobson radicals $J_{2(0)}$ and $J_{3(0)}$ introduced by Veldsman [14] and the (right) Jacobson radical $J^r_{0(e)}$ introduced by the authors with T. Srinivas [13] are the only known Jacobson-type radicals which are Kurosh-Amitsur in the class of all near-rings and ideal-hereditary in the class of all zero-symmetric near-rings. It is also known that (Corollary 6 of [15]) there is no non-trivial ideal-hereditary radical in the class of all near-rings.

In [5] and [6] the first author has shown that as in rings, matrix units determined by right ideals identify matrix near-rings. The importance of the right Jacobson radicals of type-ν, $\nu \in \{0, 1, 2, \text{s}\}$ of near-rings introduced by the authors in [7], [8] and [9], in the extension of a form of the Wedderburn-Artin

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theorem of rings involving the matrix rings to near-rings, is established in [12]. In [10] and [11] the authors with T. Srinivas have shown that the right Jacobson radicals of type-0, 1 and 2 are Kurosh-Amitsur radicals (KA-radicals) in the class of all zero-symmetric near-rings but they are not ideal-hereditary in that class.
In this paper right R-groups of type-ν(e), right ν(e)-primitive ideals and right ν(e)-primitive near-rings are introduced, $\nu \in \{1, 2\}$. Using them the right Jacobson radical of type-ν(e) is introduced for near-rings and is denoted by $J^r_{\nu(e)}$, $\nu \in \{1, 2\}$. A right ν(e)-primitive ideal of R is an equiprime ideal of R. It is shown that $J^r_{\nu(e)}$ is a Kurosh-Amitsur radical in the class of all near-rings and is an ideal-hereditary radical in the class of all zero-symmetric near-rings, $\nu \in \{1, 2\}$. Moreover, for any ideal I of R, $J^r_{\nu(e)}(I) \subseteq J^r_{\nu(e)}(R) \cap I$ with equality, if I is left invariant, $\nu \in \{1, 2\}$.

2 Preliminaries

Near-rings considered are right near-rings and not necessarily zero-symmetric. Unless otherwise specified R stands for a right near-ring. Near-ring notions not defined here can be found in Pilz [4].
R_0 and R_c denotes the zero-symmetric part and constant part of R respectively. Now we give here some definitions of [7] and [8].
A group (G, +) is called a *right R-group* if there is a mapping ((g, r) $\to$ gr) of G$\times$R into G such that (1) (g + h)r = gr + hr, (2) g(rs) = (gr)s, for all g, h $\in$ G and r, s $\in$ R. A subgroup (normal subgroup) H of a right R-group G is called an *R-subgroup (ideal)* of G if hr $\in$ H for all h $\in$ H and r $\in$ R.
Let G be a right R-group. An element $g_0 \in G$ is called a *generator* of G if $g_0R = G$ and $g_0(r + s) = g_0r + g_0s$ for all r, s $\in$ R. G is said to be *monogenic* if G has a generator. G is said to be *simple* if G $\neq$ {0} and G, and {0} are the only ideals of G.
A monogenic right R-group G is said to be a *right R-group of type-0* if G is simple.
The *annihilator* of G denoted by (0 : G) is defined as (0 : G) = {a $\in$ R | Ga = {0}}.
A right R-group G of type-0 is said to be of *type-1* if G has exactly two R-subgroups, namely {0} and G.
A right R-group G of type-0 is said to be of *type-2* if gR = G for all $g \in G\setminus\{0\}$.
Note that a right R-group of type-2 is of type-1 and a right R-group of type-1 is of type-0.
Let $\nu \in \{0, 1, 2\}$. A right modular right ideal K of R is called *right ν-modular* if R/K is a right R-group of type-ν.
An ideal P of R is called *right ν-primitive* if P is the largest ideal of R contained

in a right ν-modular right ideal of R. R is called a *right ν-primitive near-ring* if $\{0\}$ is a right ν-primitive ideal of R.
$J_\nu^r(R)$ denotes the intersection of all right ν-primitive ideals of R. If R has no right ν-primitive ideals, then $J_\nu^r(R)$ is defined as R. J_ν^r is called the *right Jacobson radical of type-ν*.
A near-ring R is called an *equiprime near-ring* ([1]) if $0 \neq a \in R$, x, y $\in$ R and arx = ary for all r $\in$ R, implies x = y. An ideal I of R is called *equiprime* if R/I is an equiprime near-ring.
It is known that a near-ring R is equiprime if and only if ([1])
1. x, y $\in$ R and xRy = $\{0\}$ implies x = 0 or y = 0.
2. If $\{0\} \neq$ I is an invariant subnear-ring of R, x, y $\in$ R and ax = ay for all a $\in$ I implies x = y.
Moreover, an equiprime near-ring is zero-symmetric.
If I is an ideal of R, then we denote it by I $\lhd$ R. A subset S of R is *left invariant* if RS $\subseteq$ S. By a radical class we mean a radical class in the sense of Kurosh-Amitsur. Let $\mathcal{E}$ be a class of near-rings. $\mathcal{E}$ is called *regular* if $\{0\} \neq I \lhd R \in \mathcal{E}$ implies that $\{0\} \neq I/K \in \mathcal{E}$ for some K $\lhd$ I. A class $\mathcal{E}$ is called hereditary if I $\lhd$ R $\in \mathcal{E}$ implies I $\in \mathcal{E}$. $\mathcal{E}$ is called *c-hereditary* if I is a left invariant ideal of R $\in \mathcal{E}$ implies I $\in \mathcal{E}$. It is clear that a hereditary class is a regular class. If I $\lhd$ R and for every non zero ideal J of R, J $\cap$ I $\neq \{0\}$, then I is called an *essential ideal* of R and is denoted by I $\lhd\cdot$ R. A class of near-rings $\mathcal{E}$ is called closed *under essential extensions (essential left invariant extensions)* if I $\in \mathcal{E}$, I $\lhd\cdot$ R (I is an essential ideal of R which is left invariant) implies R $\in \mathcal{E}$. A class of near-rings $\mathcal{E}$ is said to *satisfy condition (F_l)* whenever K $\lhd$ I $\lhd$ R, and I is left invariant in R and I/K $\in \mathcal{E}$, it follows that K $\lhd$ R.
In [2], G. L. Booth and N. J. Groenewald defined special radicals for near-rings. A class $\mathcal{E}$ consisting of equiprime near-rings is called a *special class* if it is hereditary and closed under left invariant essential extensions. If $\mathcal{R}$ is the upper radical in the class of all near-rings determined by a special class of near-rings, then $\mathcal{R}$ is called a special radical. If $\mathcal{R}$ is a radical class, then the class $\mathcal{SR} = \{R \mid \mathcal{R}(R) = \{0\}\}$ is called the *semisimple class* of $\mathcal{R}$.
We also need the following Theorem:

Theorem 2.1. *(Theorem 2.4 of [14]) Let $\mathcal{E}$ be a class of zero-symmetric near-rings. If $\mathcal{E}$ is regular, closed under essential left invariant extensions and satisfies condition (F_l), then $\mathcal{R} := \mathcal{UE}$ is a c-hereditary radical class in the variety of all near-rings, $\mathcal{SR} = \overline{\mathcal{E}}$ and $\mathcal{SR}$ is hereditary. So, $\mathcal{R}(R) = \cap \{I \lhd R \mid R/I \in \mathcal{E}\}$ for any near-ring R.*

Remark 2.2. Since all ideals in a zero-symmetric near-ring are left invariant, under the hypothesis of Theorem 2.1, in the variety of zero-symmetric near-rings both $\mathcal{R}$ and $\mathcal{SR}$ are hereditary and hence the radical is ideal-hereditary, that is, if $I \lhd R$, then $\mathcal{R}(I) = I \cap \mathcal{R}(R)$.

Proposition 2.3. *(Proposition 3.3 of [1]) The class of all equiprime near-rings is closed under essential left invariant extensions.*

Proposition 2.4. *(Corollary 2.4 of [1]) The class of all equiprime near-rings satisfies condition (F_l).*

We need the following results of [11].

Theorem 2.5. *(Theorems 3.1 and 3.2 of [11]) Let G be a right R-group of type-ν, $\nu \in \{1,2\}$. If S is an invariant subnear-ring of R and $GS \neq \{0\}$, then G is also a right S-group of type-ν.*

Theorem 2.6. *(Theorems 3.9 and 3.11 of [11]) Let S be an invariant subnear-ring of R. If G is a right S-group of type-ν, $\nu \in \{1,2\}$, then G is a right R-group of type-ν.*

3 The right Jacobson radical of type-$\nu(e)$, $\nu \in \{1, 2\}$.

Throughout this section $\nu \in \{1, 2\}$. In this section first we introduce right R-groups of type-$\nu(e)$ and study some of their properties. Using them we introduce right Jacobson radical of type-$\nu(e)$ and study its properties.
We begin with some basic properties of right R-groups of type-ν.
The following Proposition is proved in [11] (Corollary 3.4).
We give here a different proof.

Proposition 3.1. *Let G be a right R-group of type-ν. Then $GR_c = \{0\}$.*

Proof. Let g_0 be a generator of G. So g_0 is distributive over R, that is, $g_0(r + s) = g_0r + g_0s$ for all r, s $\in$ R and $g_0R = G$. Since g_0 is distributive over R and R_c is an R-subgroup of the right R-group R, g_0R_c is an R-subgroup of the right R-group G. Also since G has no nontrivial right R-subgroups, $g_0R_c = \{0\}$ or G. If $g_0R_c = G$, then $g_0r_c = g_0$ for some $r_c \in R_c$. Therefore, $g_0x = (g_0r_c)x = g_0(r_cx) = g_0r_c = g_0$ for all $x \in R$. So $G = g_0R = \{g_0\}$, a contradiction. Hence, $g_0R_c = \{0\}$. Let $g \in G$. We have $g = g_0s$ for some $s \in R$. Now $gr_c = (g_0s)r_c = g_0(sr_c) = 0$, as $sr_c \in R_c$. So, $GR_c = \{0\}$. □

The following Proposition follows from Proposition 3.7 of [13].

Proposition 3.2. *Let G be a right R-group of type-ν. Then there is a largest ideal of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$.*

Definition 3.3. Let G be a right R-group of type-ν. Suppose that P is the largest ideal of R contained in (0 : G) = {r ∈ R | Gr = {0}}. Then G is said to be a *right R-group of type-ν(e)* if $0 \neq$ g ∈ G, r_1, r_2 ∈ R and $gxr_1 = gxr_2$ for all x ∈ R implies r_1 - r_2 ∈ P.

Proposition 3.4. *Let G be a right R-group of type-ν. Let P be the largest ideal of R contained in (0 : G). Then the following are equivalent.*

1. *G is a right R-group of type-ν(e).*

2. *r_1, r_2 ∈ R and $gr_1 = gr_2$ for all g ∈ G implies r_1 - r_2 ∈ P.*

Proof. Let g_0 be a generator of the right R-group G. (1) implies (2) follows from the definition of a right R-group of type-ν(e) as g_0R = G. Assume (2). Suppose that $0 \neq$ g ∈ G, r_1, r_2 ∈ R and $gxr_1 = gxr_2$ for all x ∈ R. Since g $\neq 0$ and G is a right R-group of type-ν, gR $\neq$ {0} as $\{h \in G \mid hR = \{0\}\}$ is an ideal of G. Let $< gR >_s$ be the subgroup of $(G, +)$ generated by gR. Let h ∈ $< gR >_s$. Now h = $\delta_1 gs_1 + \delta_2 gs_2 + ... + \delta_k gs_k$, s_i ∈ R, $\delta_i \in \{1, -1\}$. hr = $\delta_1 g(s_1 r) + \delta_2 g(s_2 r) + ... + \delta_k g(s_k r) \in < gR >_s$. So $< gR >_s$ is a non-zero R-subgroup of the right R-group G. Since G is of type-ν, $< gR >_s$ = G. Therefore, $hr_1 = hr_2$ for all h ∈ G as $gxr_1 = gxr_2$ for all x ∈ R. So r_1 - r_2 ∈ P. □

We give an example of a right R-group of type-1(e) which is not of type-2(e).

Example 3.5. *Let p be an odd prime number and (G, +) be a group of order p. Consider the near-ring $M_0(G)$. In Example 3.6 of [8], it is shown that $M_0(G)$ is a right $M_0(G)$-group of type-1 but not of type-2. Since $M_0(G)$ is simple, {0} is the largest ideal of $M_0(G)$ contained in (0 : $M_0(G)$). Suppose that $0 \neq$ s, f, h ∈ $M_0(G)$ and stf = sth for all t ∈ $M_0(G)$. Assume that $s(g_0) \neq 0$ and f(g) $\neq$ h(g) for some g_0, g ∈ G. Let h(g) $\neq$ 0. We get t ∈ $M_0(G)$ such that t(f(g)) = 0 and t(h(g)) = g_0. So stf $\neq$ sth, a contradiction. Therefore, f = h, that is, f - h ∈ {0}. Hence, $M_0(G)$ is a right $M_0(G)$-group of type-1(e) but not of type-2(e).*

Example 3.6. *Clearly, a near-field R is a right R-group of type-2(e).*

The following Proposition follows from Proposition 3.12 of [13].

Proposition 3.7. *Let G be right R-group of type-ν(e). Then (0 : G) is an ideal of R.*

Definition 3.8. A right modular right ideal K of R is called *right ν(e)-modular* if R/K is a right R-group of type-ν(e).

Definition 3.9. Let G be a right R-group of type-ν(e). Then (0 : G) is called a *right ν(e)-primitive ideal* of R.

Definition 3.10. Let G be a right R-group of type-ν(e). Then G is called *faithful* if (0 : G) = {0}.

Definition 3.11. A near-ring R is called *right ν(e)-primitive* if {0} is a right ν(e)-primitive ideal of R.

Definition 3.12. The intersection of all ν(e)-primitive ideals of R is called the *right Jacobson radical of R of type-ν(e)* and is denoted by $J^r_{\nu(e)}(R)$. If R has no right ν(e)-primitive ideals, then $J^r_{\nu(e)}$(R) is defined to be R.

Remark 3.13. It is clear that $J^r_{\nu}(R) \subseteq J^r_{\nu(e)}(R)$.

Proposition 3.14. *Let G be a right R-group of type-ν(e). Let g_0 be a generator of G and K := (0 : g_0) = $\{r \in R \mid g_0 r = 0\}$. Then K is right ν(e)-modular right ideal of R.*

Proof. Since g_0R = G, $g_0 = g_0$e for some e ∈ R. So $r - er \in K$ for all r ∈ R and hence K is right modular by e. Since the mapping $r \to g_0 r$ is right R-homomorphism of R onto G with kernel K, the right R-group G is isomorphic to the right R-group R/K. So K is a right ν(e)-modular right ideal of R. □

Remark 3.15. Let K be a right ideal of R. Then the ideal {0} of R is contained in K. Since K is a subgroup of (R, +) if I and J are ideals of R contained in K, then I + J ⊆ K. So there is a largest ideal of R contained in K.

The following Proposition follows from Proposition 3.19 of [13].

Proposition 3.16. *Let G be right R-group of type-ν(e) and P := (0 : G) = $\{r \in R \mid Gr = \{0\}\}$. Then P is the largest ideal of R contained in (0 : g_0), g_0 is a generator of the right R-group G.*

Corollary 3.17. *Let P be an ideal of R. P is a right ν(e)-primitive ideal of R if and only if P is the largest ideal of R contained in a right ν(e)-modular right ideal of R.*

We give some more examples of right R-groups of type-2(e).

Proposition 3.18. *If G be a finite group and G has a subgroup of index two, then M_0(G) is a right 2(e)-primitive near-ring.*

Proof. Let G be a finite group and H be a subgroup of G of index 2. So H is a normal subgroup of G. Let R = M_0(G). Then R/K is a right R-group of type-2(e), where K = (H : G) = {r ∈ R | r(g) ∈ H, for all g ∈ G}. To show this we consider the two distinct cosets H and H + a of H in G. Now G = H ∪ H + a, H and H + a are disjoint sets. K is a right ideal of R which is right modular by the identity element of R. So R/K is a monogenic right R-group. Now we show that R/K is a right R-group of type-2. Let $0 \neq$ r + K ∈ R/K. (r + K)R = R/K if and only if there is an s ∈ R such that (r + K)s = 1 + K, that is, 1 - rs ∈ K. Let P_1 = {x ∈ G | r(x) ∈ H} and P_2 = {x ∈ G | r(x) ∈ H + a}. Let b ∈ P_2 and r(b) = h' + a, h' ∈ H. Define s : G → G by s(g) = b, if g ∈ H + a, and 0, if g ∈ H. We have s ∈ R. For y ∈ H, (1 - rs)(y) = y - r(s(y)) = y - r(0) = y ∈ H and for z = h + a ∈ H + a, (1 - rs)(z) = z - r(s(z)) = z - r(b) = (h + a) - (h' + a) = h - h' ∈ H. Therefore, 1 - rs ∈ (H : G) = K and hence R/K is a right R-group of type-2. Since R is simple, {0} is the largest ideal of R contained in (0 : R/K) = (K : R) = {t ∈ R | Rt ⊆ K}. Let u, v ∈ R and (t + K)u = (t + K)v for all t + K ∈ R/K. Now tu - tv ∈ K, for all t ∈ R. Suppose that g ∈ G and u(g) ≠ v(g). We can choose a t ∈ R such that (tu)(g) - (tv)(g) ∈ H + a, a contradiction to the fact that tu - tv ∈ K. Therefore, u = v and hence R/K is a right R-group of type-2(e). Since R is simple, it is a right 2(e)-primitive near-ring. □

Proposition 3.19. *If G is a finite group having no subgroup of index 2, then* $J^r_{2(e)}(M_0(G)) = M_0(G)$.

Proof. Let G be a finite group having no subgroup of index 2. Let R := M_0(G). Suppose that K is a right 2-modular right ideal of R. Now K = (N : G), where N is a normal subgroup of G. By our assumption the index of N in G is greater than or equal to 3. Let N, N + a, N + b be three distinct right cosets of N in G. Since R/K is a right R-group of type-2, for $0 \neq$ t + K ∈ R/K, (t + K)R = R/K. Since 1 + K ∈ R/K, we get s ∈ R such that (t + K)s = 1 + K, and hence 1 - ts ∈ K = (N : G). Define r : G → G by r(a) = b and r(g) = 0 for all $g \in G \setminus \{a\}$. Now r ∈ R. If r ∈ K = (N : G), then r(x) ∈ N for all x ∈ G and in particular b = r(a) ∈ N, a contradiction. So r ∉ K and there is a p ∈ R such that 1 - rp ∈ K = (N : G). Now (1 - rp)(x) ∈ N for all x ∈ G. If p(a) = a, then (1 - rp)(a) = a - b ∈ N and hence N + a = N + b, a contradiction. If p(a) ≠ a, then (1 - rp)(a) = a - 0 = a ∈ N and N = N + a, a contradiction. Therefore, R has no right 2-modular right ideal. So, J^r_2(R) = R and hence $J^r_{2(e)}$(R) = R. □

Proposition 3.20. *If F is a near-field, then* M_n*(F) is a right 2(e)-primitive near-ring.*

Proof. Let F be a near-field. Let M_n(F) be the near-ring of n×n-matrices over F. Let 1 ≤ i ≤ n. Now from the proof of the Theorem 3.15 of [6], we have that $f^1_{ii}M_n$(F) is a right M_n(F)-group of type-2. Since M_n(F) is simple, {0} is the largest ideal of M_n(F) contained in (0 : $f^1_{ii}M_n$(F)). We show now that

$f^1_{ii}M_n(F)$ is a right $M_n(F)$-group of type-2(e). Let B, C $\in M_n(F)$ and $(f^1_{ii}A)B = (f^1_{ii}A)C$, for all A $\in M_n(F)$. Suppose that B $\neq$ C. We get $(x_1, x_2, \ldots, x_n) \in F^n$ such that $B(x_1, x_2, \ldots, x_n) \neq C(x_1, x_2, \ldots, x_n)$. Let $B(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ and $C(x_1, x_2, \ldots, x_n) = (z_1, z_2, \ldots, z_n)$. We get $1 \leq j \leq n$ such that $y_j \neq z_j$. Now $(f^1_{ii}f^1_{ij})B(x_1, x_2, \ldots, x_n) = (f^1_{ii}f^1_{ij})C(x_1, x_2, \ldots, x_n)$ and that $y_j = z_j$, a contradiction. Therefore B = C and hence $f^1_{ii}M_n(F)$ is a right $M_n(F)$-group of type-2(e). Since F is simple, $M_n(F)$ is also simple. So, we get that $M_n(F)$ is a right 2(e)-primitive near-ring. □

Now we give a right R-group of type-2(e), where R is a near-ring with trivial multiplication.

Example 3.21. *Let (R, +) be a group and let K be a subgroup of (R, +) of index 2. The trivial multiplication on (R, +) determined by R - K is given by a.b = a if b ∈ R - K and 0 if b ∈ K. Now (R, +, .) is a near-ring. It is clear that K is a maximal right ideal of R and also R/K is a right R-group of type-2. Now we show that R/K is a right R-group of type-2(e). K is an ideal of R and it is the largest ideal of R contained in K and hence in (K : R) =* $\{r \in R \mid Rr \subseteq K\}$. *Let x, y ∈ R and (r + K)x = (r + K)y for all r ∈ R. Now rx - ry ∈ K for all r ∈ R. So, either both x and y are in K or both in R - K. Therefore, x - y ∈ K as K is of index 2 in (R, +). Hence, R/K is a right R-group of type-2(e).*

Now we give an example of a right R-group of type-ν which is not of type-ν(e).
This example was considered in [3] and [13].

Example 3.22. *Consider G :=* Z_8*, the group of integers under addition modulo 8. Now T : G → G defined by T(g) = 5g, for all g ∈ G is an automorphism of G. T fixes 0, 2, 4, 6 and maps 1 to 5, 5 to 1, 7 to 3 and 3 to 7. A := {I, T} is an automorphism group of G. {0}, {2}, {4}, {6}, {1, 5} and {3, 7} are the orbits. Let R be the centralizer near-ring* $M_A(G)$*, the near-ring of all self maps of G which fix 0 and commute with T. An element of R is completely determined by its action on {1, 2, 3, 4, 6}. Note that for f ∈ R we have f(2), f(4), f(6) are arbitrary in 2G and f(1), f(3) are arbitrary in G. In [3] it is proved that I := (0 : 2G) =* $\{f \in R \mid f(h) = 0$, *for all* $h \in 2G\}$ *is the only non-trivial ideal of R. Let K := (2G : G) =* $\{t \in R \mid t(G) \subseteq 2G\} \neq R$*. Let* t_0 *be the identity element in R. Now* $t_0 + K$ *is a generator of the right R-group R/K. Let h ∈ R - K. We show now that (h + K)R = R/K. Since h ∉ K, there is an a ∈ G - 2G such that b := h(a) ∉ 2G. We construct an element s ∈ R such that s(1) = s(3) = a, so that s(5) = s(7) = a + 4, and s = 0 on 2G. Since s maps G - 2G to G - 2G, we get that* t_0 *- hs ∈ K and hence (h + K)s =* t_0 *+ K. So (h + K)R = R/K. Therefore, R/K is a right R-group of type-*ν*.*

Moreover, $(R/K)I \neq \{K\}$. *Therefore,* $\{0\}$ *is the largest ideal of R contained in* $(K : R)$ *and hence* $J^r_\nu(R) = \{0\}$. *Consider* s_1, $s_1 \in R$, *where* $s_1(1) = 1$ *and* 0 *on* $G - \{1, 5\}$ *and* $s_2(1) = 5$ *and* 0 *on* $G - \{1, 5\}$. *Clearly* $(h + K)s_1 = (h + K)s_2$ *for all* $h \in R$ *as* $h(1) - h(5) \in 2G$ *for all* $h \in R$. *But* $s_1 - s_2 \notin \{0\}$. *Therefore, by Proposition 3.4, R/K is not a right R-group of type-*$\nu(e)$.

Proposition 3.23. *Let R be the near-ring considered in the Example 3.22 and let Z be a right ideal of R. Then* $H_1 := \{f(g) \mid f \in Z, g \in G\} \subseteq G$ *and* $H_2 := \{f(g) \mid f \in Z, g \in 2G\} \subseteq 2G$ *are (normal) subgroups of G and 2G respectively.*

Proof. We show that H_1 is a subgroup of G. Since $0 \in H_1$, H_1 is non-empty. Let $h_1, h_2 \in H_1$. We get $f_1, f_2 \in Z$ and $g_1, g_2 \in G$ such that $h_1 = f_1(g_1)$ and $h_2 = f_2(g_2)$. Clearly, $-h_1 = (-f_1)(g_1) \in H_1$ as $-f_1 \in Z$. Suppose that one of the g_i is in G - 2G. With out loss of generality, suppose that $g_1 \in G - 2G$. We get $f_3 \in R$ such that $f_3(g_1) = g_2$. Now $f_1 - f_2f_3 \in Z$ and $h_1 - h_2 = (f_1 - f_2f_3)(g_1) \in H_1$. Assume now that $g_1, g_2 \in 2G$. So, $h_1, h_2 \in 2G$. If $g_1 = 0$, then $h_1 - h_2 = -h_2 \in H_1$. Suppose that $g_1 \neq 0$. So, we get $f_4 \in R$ such that $f_4(g_1) = g_2$. Now $f_1 - f_2f_4 \in Z$ and $h_1 - h_2 = (f_1 - f_2f_4)(g_1) \in H_1$. Therefore, H_1 is a subgroup of G. Similarly, we get that H_2 is a subgroup of 2G. □

Proposition 3.24. *Let R, Z,* H_1 *and* H_2 *be as defined in Proposition 3.23. If* $H_1 = G$ *and* $H_2 = 2G$, *then* $Z = R$.

Proof. Suppose that $H_1 = G$ and $H_2 = 2G$. We have $1, 3 \in H_1$. So, for $i \in \{1, 3\}$, we get $f_i \in Z$ such that $f_i(g_i) = i$, where $g_i \in \{1, 3, 5, 7\} = G - 2G$. For $i = 1, 3$ we also get $m_i \in R$ such that $m_i(i) = g_i$, so that $m_i(i + 4) = g_i + 4$ and $m_i = 0$ on $G - \{i, i + 4\}$. Now $f_im_i \in Z$, $i = 1, 3$. Clearly, $f_1m_1 + f_3m_3$ fixes all the elements of G - 2G and maps all the elements of 2G to 0. We have $2, 4, 6 \in H_2 = 2G = \{0, 2, 4, 6\}$. For $i = 2, 4, 6$ we get $f_i \in Z$ such that $f_i(g_i) = i$, $g_i \in 2G$. So, for $i = 2, 4, 6$ we get $m_i \in R$ such that $m_i(i) = g_i$ and m_i is 0 on $G - \{i\}$. Now $f_im_i \in Z$, $i = 2, 4, 6$. $f_2m_2 + f_4m_4 + f_6m_6$ fixes all the elements of 2G and maps all the elements of G -2G to 0. Therefore, the identity map I of G can be expressed as $I = f_1m_1 + f_2m_2 + f_3m_3 + f_4m_4 + f_6m_6 \in Z$. Hence, $Z = R$. □

Proposition 3.25. *Let R, Z,* H_1 *and* H_2 *be as defined in Proposition 3.23. If Z is a maximal right ideal of R, then* $Z = (2G : G) = \{f \in R \mid f(G) \subseteq 2G\}$ *or* $(4G : 2G) = \{f \in R \mid f(2G) \subseteq 4G\}$

Proof. Suppose that Z is a maximal right ideal of R. Clearly, if H and T are (normal) subgroups of G and 2G respectively, then $(H : G) = \{f \in R \mid f(G) \subseteq H\}$ and $(T : 2G) = \{f \in R \mid f(2G) \subseteq T\}$ are right ideals of R. Now 2G and 4G are the maximal (normal) subgroups of G and 2G respectively. We have

$Z \subseteq (H_1 : G)$ and $Z \subseteq (H_2 : 2G)$. Since Z is a maximal right ideal of R, by Proposition 3.24, either $H_1 \neq G$ or $H_2 \neq 2G$.
Case(i) Suppose that $H_2 \neq 2G$. Since Z is a maximal right ideal of R and Z $\subseteq (H_2 : 2G) \neq R$, we get that $H_2 = 4G$ and Z = (4G : 2G).
case(ii) Suppose that $H_1 \neq G$. Since Z is a maximal right ideal of R and Z $\subseteq$ $(H_1 : G) \neq R$, we get that $H_1 = 2G$ and Z = (2G : G).
Therefore, either Z = (2G : G) or (4G : 2G). □

Proposition 3.26. *Let R be the near-ring considered in the Example 3.22. Let U = (4G : 2G) = $\{f \in R \mid f(2G) \subseteq 4G\}$. Then U is a maximal right ideal of R and R/U is a right R-group of type-2(e).*

Proof. Clearly, U is a right ideal of R. Consider the right R-group R/U. We prove that R/U is a right R-group of type-2. Since R has identity I, I + U is a generator of the right R-group R/U and hence R/U is a monogenic right R-group. Let $0 \neq f + U \in R/U$. So, $f \notin U$. We get $0 \neq a \in 2G$ such that b := f(a) $\notin$ 4G. So, 2G = {0, b, 2b, 3b} as 2 and 6 are generators of 2G. Construct $r \in R$ by r(b) = a, r(2b) = 0, r(3b) = a and r = 0 on G - {0, 1, 3, 5, 7}. Now (I - fr)(x) $\in$ 4G for all $x \in 2G$. Therefore, I - fr $\in$ U and hence (f + U)r = I + U. This shows that (f + U)R = R/U. So, R/U is a right R-group of type-2. We know that P := (0 : 2G) is the only non-trivial ideal of R. Therefore, P is the largest ideal of R contained in U = (4G : 2G) and hence P is the largest ideal of R contained in (0 : R/U) = (U : R) = $\{f \in R \mid Rf \subseteq U\}$. Let $0 \neq s +$ U $\in$ R/U and f, h $\in$ R. Suppose that (s + U)rf = (s + U)rh for all $r \in R$. So, srf - srh $\in$ U for all $r \in R$. We show that f - h $\in$ P. If possible, suppose that f - h $\notin$ P. We get $0 \neq a \in 2G$ such that (f - h)(a) = f(a) - h(a) $\neq 0$ with h(a) $\neq$ 0. Let s(c) $\notin$ {0, 4} for some $c \in 2G$. Choose $r \in R$ such that r(f(a)) = 0 and r(h(a)) = c. Now (srf)(a) = 0 and (srh)(a) = s(c). So, (srf - srh)(a) = 0 - s(c) $\notin$ {0, 4}, a contradiction to the fact that srf - srh $\in$ U. Therefore, f(a) = h(a) for all $a \in 2G$. Hence f - h $\in$ P. So, R/U is a right R-group of type-2(e). □

Proposition 3.27. *Let R be the near-ring considered in Example 3.22. Then* $J^r_\nu(R) = \{0\}$ *and* $J^r_{\nu(e)}(R) = (0 : 2G) \neq \{0\}$.

Proof. We know that {0} and I := (0 : 2G) = $\{f \in R \mid f(2G) = \{0\}\}$ are the only proper ideals of R. Let K_1 := (2G : G) = $\{f \in R \mid f(G) \subseteq 2G\}$ and K_2 := (4G : 2G) = $\{f \in R \mid f(2G) \subseteq 4G\}$. By Proposition 3.25, a maximal right ideal of R is either K_1 or K_2. So, a right R-group of type-0 is isomorphic to R/K_1 or R/K_2. By Example 3.22, R/K_1 is a right R-group of type-2 but not of type-2(e). Since {0} is the largest ideal of R contained in K_1, {0} is a right 2-primitive ideal of R but not a right 2(e)-primitive ideal of R. By Proposition 3.26, R/K_2 is a right R-group of type-2(e). Since I = (0 : 2G) is the largest ideal of R contained in K_2, I is a right 2(e)-primitive ideal of R. Therefore, $J^r_\nu(R) = \{0\}$ and $J^r_{\nu(e)}(R) = (0 : 2G)$. □

Now we study some of the properties of the radical $J^r_{\nu(e)}$.

Proposition 3.28. *Let P be an ideal of R. P is a right ν(e)-primitive ideal of R if and only if R/P is a right ν(e)-primitive near-ring.*

A proof similar to the one given for Proposition 3.21 of [13] works here also, which uses Corollary 3.17.

Theorem 3.29. *Let R be a right ν(e)-primitive near-ring. Then R is an equiprime near-ring.*

Proof. Since $\{0\}$ is a right ν(e)-primitive ideal of R, by Proposition 3.7, $\{0\} = (0 : G)$ for a right R-group G of type-ν(e). Let $a \in R \setminus \{0\}$, $r_1, r_2 \in R$ and $axr_1 = axr_2$ for all $x \in R$. Since $(0 : G) = \{0\}$, there is a $g \in G$ such that $ga \neq 0$. Let $h := ga$. Now $hxr_1 = hxr_2$ for all $x \in R$. Since G is a right R-group of type-ν(e), $r_1 - r_2 \in P$, the largest ideal of R contained in $(0 : G) = \{0\}$. Therefore, $r_1 = r_2$ and hence R is an equiprime near-ring. □

Corollary 3.30. *A right ν(e)-primitive ideal of R is an equiprime ideal of R.*

Corollary 3.31. *A right ν(e)-primitive near-ring is a zero-symmetric near-ring.*

Theorem 3.32. *Let G be a right R-group of type-ν(e). Suppose that S is an invariant subnear-ring of R. If $GS \neq \{0\}$, then G is also a right S-group of type-ν(e).*

Proof. Suppose that $GS \neq \{0\}$. By Theorem 2.5, G is a right S-group of type-ν. Let P be the largest ideal of S contained in $(0 : G)_S = \{s \in S \mid Gs = \{0\}\}$. Let $g \in G \setminus \{0\}$, $s_1, s_2 \in S$ and $gxs_1 = gxs_2$ for all $x \in S$. Let $r \in R$. Fix $x \in S$. We have $g(rx)s_1 = g(rx)s_2$. So $gr(xs_1) = gr(xs_2)$. Since G is a right R-group of type-ν(e), by Proposition 3.7, $xs_1 - xs_2 \in (0 : G) = \{r \in R \mid Gr = \{0\}\}$ which is an ideal of R. Let g_0 be a generator of the right S-group G. Now $g_0(xs_1 - xs_2) = 0$ and hence $g_0xs_1 = g_0xs_2$. Since $g_0S = G$, we have $g_0R = G$. So $g_0rs_1 = g_0rs_2$, for all $r \in R$. Since G is a right R-group of type-ν(e), by Proposition 3.7, $s_1 - s_2 \in (0 : G)$. We have $(0 : G)_S = (0 : G) \cap S$ is an ideal of S and hence $P = (0 : G)_S$. Now $s_1 - s_2 \in (0 : G) \cap S = P$. Therefore, G is a right S-group of type-ν(e). □

Theorem 3.33. *If R is a right ν(e)-primitive near-ring and I is a nonzero ideal (or a nonzero invariant subnear-ring) of R, then I is a right ν(e)-primitive near-ring.*

Theorem 3.34. *The class of all right $\nu(e)$-primitive near-rings is hereditary.*

Corollary 3.35. *The class of all right $\nu(e)$-primitive near-rings is regular.*

Theorem 3.36. *Let I be an essential left invariant ideal of R. If I is a right $\nu(e)$-primitive near-ring, then R is also a right $\nu(e)$-primitive near-ring.*

Proof. Suppose that I is a right ν(e)-primitive near-ring and G is a faithful right I-group of type-ν(e). Let r, s ∈ R. Let g_0 be a generator of the right I-group G. Define gr := g_0(ar), if g = g_0a, a ∈ I. By Theorem 2.6, G is a right R-group of type-ν. Suppose that $g \in G \setminus \{0\}$, r, s ∈ R and gxr = gxs, for all x ∈ R. Fix a ∈ I. Now g((ba)r) = g((ba)s) and hence g(b(ar)) = g(b(as)) for all b ∈ I. Since G is a faithful right I-group of type-ν(e), ar - as = 0, that is, ar = as. Now ar = as for all a ∈ I. Since I is a right ν(e)-primitive near-ring, by Theorem 3.33, I is an equiprime near-ring. Also, since I is an essential left invariant ideal of R, by Proposition 2.3, we get that R is an equiprime near-ring. Since R is equiprime and ar = as for all a ∈ I and I is a left invariant ideal of R, we get that r = s. So, 0 = r - s ∈ P, where P is the largest ideal of R contained in (0 : G) = {r ∈ R | Gr = {0}}. Therefore G is a right R-group of type-ν(e). Let t ∈ (0 : G). Now Gt = 0. So g_0(at) = 0, for all a ∈ I and hence 0 = g_0((ba)t) = g_0(b(at)) = (g_0b)at for all a, b ∈ I. Since g_0I = G, we have G(at) = 0 for all a ∈ I and hence It = 0, as $(0 : G)_I = 0$. Also, since at = 0 = a0 for all a ∈ I and I is an invariant subnear-ring of R and R is an equiprime near-ring, we get that t = 0. Therefore, G is a faithful right R-group of type-ν(e) and hence R is a right ν(e)-primitive near-ring. □

Theorem 3.37. *The class of all right $\nu(e)$-primitive near-rings is closed under essential left invariant extensions.*

Remark 3.38. By Proposition 2.4, the class of all equiprime near-rings satisfy condition F_l. So, the class of all ν(e)-primitive near-rings which is also a class of all equiprime near-rings also satisfy condition F_l.

By Theorem 2.1, Corollaries 3.31, and 3.35, Theorem 3.37 and Remark 3.38, we get the following:

Theorem 3.39. *Let $\mathcal{E}$ be the class of all right $\nu(e)$-primitive near-rings and $\mathcal{UE}$ be the upper radical class determined by $\mathcal{E}$. Then $\mathcal{UE}$ is a c-hereditary Kurosh-Amitsur radical class in the variety of all near-rings with hereditary semisimple class $\mathcal{SUE} = \overline{\mathcal{E}}$. So, $\mathcal{J}^r_{\nu(e)}$ is a Kurosh-Amitsur radical in the class of all near-rings and for any ideal I of R, $\mathcal{J}^r_{\nu(e)}(I) \subseteq \mathcal{J}^r_{\nu(e)}(R) \cap I$ with equality if I is left invariant.*

Corollary 3.40. *$\mathcal{J}^r_{\nu(e)}$ is an ideal-hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

Corollary 3.41. *$\mathcal{J}^r_{\nu(e)}$ is a special radical in the class of all near-rings.*

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Department of Mathematics,
R. V. R. & J. C. College of Engineering,
Chandramoulipuram, Chowdavaram,
Guntur-522019, Andhra Pradesh, India.
Email: dr_ rsrao@yahoo.com

Department of Mathematics,
Acharya Nagarjuna University,
Nagarjunanagar-522510,
Guntur (Dist.), Andhra Pradesh, India.
Email: siva235prasad@yahoo.co.in