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Almost Condensed Domains

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Abstract

As an extension of the class of half condensed domains introduced by D.D. Anderson and Dumitrescu, we introduce and study the class of almost condensed domains. An integral domain D is almost condensed if whenever $0 \neq z \in IJ$ with I, J ideals of D, there exist I', J' ideals of D such that $I' \subseteq I_w$, $J' \subseteq J_w$ and $zD = (I'J')_w$.

In 1983, D.F. Anderson and D.E. Dobbs [13] called an integral domain D condensed if for each pair of ideals I, J of D, $IJ = \{ij \mid i \in I, j \in J\}$. A Bézout domain is condensed, cf. [13, Corollary 2.2] and an integrally closed condensed domain is Bézout, cf. [11, Main Theorem]. In 2004, D.D. Anderson and T. Dumitrescu [6] called an integral domain D a half condensed (HC) domain if whenever $0 \neq z \in IJ$ with I, J ideals of D, there exist I', J' (invertible) ideals of D such that $I' \subseteq I$, $J' \subseteq J$ and $zD = I'J'$. In this paper we study the following extension of the concept of HC domain. We call an integral domain D an *almost condensed (AC) domain* if whenever $0 \neq z \in IJ$ with I, J ideals of D, there exist I', J' (w-invertible) ideals of D such that $I' \subseteq I_w$, $J' \subseteq J_w$ and $zD = (I'J')_w$ (the definition of the w-closure of an ideal as well as other basic facts are recalled below). The following implications hold.

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$$\begin{array}{ccccc}
\text{Bézout domain} & \Rightarrow_1 & \text{Prüfer domain} & \Rightarrow_2 & \text{PVMD} \\
\Downarrow_3 & & \Downarrow_4 & & \Downarrow_5 \\
\text{Condensed domain} & \Rightarrow_6 & \text{HC domain} & \Rightarrow_7 & \text{AC domain}
\end{array}$$

The implications "1" and "2" are obvious, "3" is proved in [13, Corollary 2.2], "4" is proved in [6, Proposition 1.2], "5" is proved in Theorem 4 of this paper and "6", "7" are clear from definitions.

This new concept of AC domain depends upon the notion of star operation, v-operation, t-operation and w-operation. A reader in need of a quick review on this topic may consult sections 32 and 34 of Gilmer's book [18]. For the reader's convenience we give a working introduction here for the notions involved. Let D be an integral domain with quotient field K and let $F(D)$ denote the set of nonzero fractional ideals of D. A function $A \mapsto A^* : F(D) \to F(D)$ is called a *star operation* on D if $*$ satisfies the following three conditions for all $0 \neq a \in K$ and for all $I, J \in F(D)$: (1) $D^* = D$ and $(aI)^* = aI^*$, (2) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$, (3) $(I^*)^* = I^*$. An ideal $I \in F(D)$ is called a $*$-ideal if $I^* = I$. For all $I, J \in F(D)$, we have $(IJ)^* = (I^*J)^* = (I^*J^*)^*$. These equations define the so-called $*$-*multiplication.* If $\{I_\alpha\}$ is a subset of $F(D)$ such that $\cap I_\alpha \neq 0$, then $\cap I_\alpha^*$ is a $*$-ideal. Also, if $\{I_\alpha\}$ is a subset of $F(D)$ such that $\sum I_\alpha$ is a fractional ideal, then $(\sum I_\alpha)^* = (\sum I_\alpha^*)^*$. A star operation $*$ is said to be a *stable* star operation if $(I \cap J)^* = I^* \cap J^*$ for all $I, J \in F(D)$. The function $*_f : F(D) \to F(D)$ given by $I^{*_f} = \cup J^*$, where J ranges over all nonzero finitely generated sub-ideals of I, is also a star operation; $*$ is said to be a star operation of *finite character* if $* = *_f$. Clearly $(*_f)_f = *_f$. Let $Max_*(D)$ denote the set of maximal $*$-ideals, that is, ideals maximal among proper integral $*$-ideals of D. Every maximal $*$-ideal is a prime ideal. If $*$ is of finite character, then every proper $*$-ideal is contained in some maximal $*$-ideal, and $*$ is stable if and only if $I^* = \cap_{P \in Max_*(D)} ID_P$ for all $I \in F(D)$, cf. [1, Corollary 4.2]. A $*$-ideal I is of *finite type* if $I = (a_1, ..., a_n)^*$ for some $a_1, ..., a_n \in I$. An ideal $I \in F(D)$ is said to be $*$-*invertible* if $(II^{-1})^* = D$, where $I^{-1} = (D : I) = \{x \in K \mid xI \subseteq D\}$. If $*$ is of finite character, then I is $*$-invertible if and only if II^{-1} is not contained in any maximal $*$-ideal of D; in this case $I^* = (a_1, ..., a_n)^*$ for some $a_1, ..., a_n \in I$.

Let $*_1, *_2$ be star operations on D. We write $*_1 \leq *_2$, if $I^{*_1} \subseteq I^{*_2}$ for all $I \in F(D)$. In this case we get $(I^{*_1})^{*_2} = I^{*_2} = (I^{*_2})^{*_1}$ and every $*_1$-invertible ideal is $*_2$-invertible. Some well-known star operations are: the *d-operation* (given by $I \mapsto I$), the *v-operation* (given by $I \mapsto I_v = (I^{-1})^{-1}$) and the *$t$-operation* (defined by $t = v_f$). The *w-operation* is the star operation given by $I \mapsto I_w = \{x \in K \mid Jx \subseteq I$ for some finitely generated ideal J of D with $J^{-1} = D\}$. The w-operation is a stable star operation of finite character. For every $I \in F(D)$, we have $I \subseteq I_w \subseteq I_t \subseteq I_v$; so a v-ideal is a t-ideal and a t-ideal is a w-ideal. It is known that $Max_w(D) = Max_t(D)$, cf. [3, Corollary 2.17] and $I_w = \cap_{P \in Max_t(D)} ID_P$ by [3, Corollary 2.13]. So, a nonzero fractional ideal is w-invertible if and only if it is t-invertible. Recall from [15] that the quotient group of t-invertible t- fractional ideals modulo the subgroup of principal fractional ideals is called the t-class group denoted by $Cl_t(D)$.

Let $*$ be a finite character star operation on an integral domain D. Recall from [2] that D is called *$*$-Noetherian* if D satisfies the ascending chain condition for $*$-ideals (equivalently, if for every ideal I there exists a finitely generated ideal $J \subseteq I$ with $I^* = J^*$). For instance, the t-Noetherian domains are the Mori domains.

A domain is *independent of finite character $\mathcal{F}$ (or an $\mathcal{F}$-IFC domain)* if it has a defining family $\mathcal{F}$ of primes that is independent and of finite character [9]. An ideal I of a domain is called *unidirectional* if it belongs to a unique member of the defining family $\mathcal{F}$ of primes [9]. A domain D is called a *weakly Matlis domain* if the intersection $D = \cap\{D_P \mid P \in Max_t(D)\}$ is independent of finite character [10]. For an integral domain D, $X^1(D)$ will denote the set of height-one prime ideals of D. A domain D is said to be a *weakly Krull domain* if D is a locally finite intersection of its localizations at members of $X^1(D)$. A domain D is called a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible [20].

Let D be an integral domain and I, J nonzero ideals of D. We say that I, J is an *almost condensed (AC) pair*, if for each $0 \neq z \in IJ$, there exists a pair I', J' of (w-invertible) ideals of D such that $I' \subseteq I_w$, $J' \subseteq J_w$ and $zD = (I'J')_w$. We say that D is an *almost condensed (AC) domain*, if I, J is an AC pair for all nonzero ideals I and J of D. If I or J is w-invertible (equivalently t-invertible), then I, J is an AC pair (Remark 3). A domain D is AC if and only if every pair of finitely generated ideals of D is AC. Hence a PVMD is AC (Proposition 4). Since a Prüfer domain is a PVMD, studying AC domains covers a vast area of commutative ring theory of interest. An HC domain is clearly an AC domain. There exist simple examples of AC domains which are not HC. For instance, the domain $\mathbb{Z}[X]$ is an AC domain but not HC. In general a non-Prüfer PVMD is an AC domain which is not HC. We show that every t-linked overring of an AC domain is AC (Proposition 2). If D is an AC domain in which every maximal ideal is a t-ideal, then D is an HC domain (Proposition 6). If D is an AC domain with $Cl_t(D) = 0$, then every pair of w-ideals of D is condensed (Proposition 8). If D is an AC domain, then D_M is condensed for each $M \in Max_t(D)$ (Proposition 9). A domain D is a PVMD if and only if it is integrally closed and AC (Proposition 10). If D is an $\mathcal{F}$-IFC domain such that D_M is condensed for each $M \in \mathcal{F}$, then D is an AC domain (Proposition 12). A weakly Matlis domain D is an AC domain if and only if D_M is condensed for each $M \in Max_t(D)$ (Corollary 13). A Noetherian domain D is an AC domain if and only if D_M is condensed for each $M \in Max_t(D)$ if and only if D is weakly Krull and D_M is condensed for each $M \in X^1(D)$ (Theorem 14). If $\{D_i\}_{i\in\Lambda}$ is an ascending chain of AC domains such that D_i is t-linked over D_j for every $D_j \subseteq D_i$, then $D = \cup_{i\in\Lambda} D_i$ is an AC domain (Proposition 15). If D is a Noetherian weakly Krull domain whose integral closure D' is two-generated as a D-module, then $D[\{X_\lambda\}]$ is an AC domain for every set of indeterminates $\{X_\lambda\}_{\lambda\in\Lambda}$ over D (Proposition 16).

Throughout this paper all rings are (commutative unitary) integral domains.

For a domain D, denote the complete integral closure by $\bar{D}$ and the integral closure by D'. Our standard reference for any undefined notation or terminology is [18].

Recall that an extension $R \subseteq T$ of domains is said to be *t-linked* if, whenever J is a nonzero finitely generated ideal of R with $J^{-1} = R$, we have $(JT)^{-1} = T$. Such extensions were introduced in [17]. In [6, Proposition 1.1], it is proved that every overring of an HC domain is HC. In the next proposition, we shall prove that a t-linked overring of an AC domain is AC. A lemma is in order.

Lemma 1. *Let $R \subseteq T$ be a t-linked extension of domains. Then $(I_wT)_w = (IT)_w$ for any nonzero ideal I of R.*

Proof. Let $y \in I_w$. Then $yA \subseteq I$ for some finitely generated ideal A with $A^{-1} = R$. This implies that $yAT \subseteq IT$ with $(AT)^{-1} = T$, because T is a t-linked overring of R. Hence $y \in (IT)_w$ and thus $I_w \subseteq (IT)_w$. This implies that $(I_wT)_w \subseteq (IT)_w$. The opposite inclusion is clear. □

Proposition 2. *Let D be an AC domain and E a t-linked overring of D. Then E is also an AC domain. In particular, every fraction ring of an AC domain is AC.*

Proof. Let I, J be two nonzero ideals of E and $0 \neq z \in IJ$. Then $z = x_1y_1 + \cdots + x_ny_n$ for some $x_i \in I$ and $y_i \in J$. Let $0 \neq d \in D$ such that $dx_i, dy_i \in D$ for all i. Then $d^2z \in (dx_1D + \cdots + dx_nD)(dy_1D + \cdots + dy_nD)$. Since D is AC, $d^2zD = (I'J')_w$ for some ideals $I' \subseteq (dx_1D + \cdots + dx_nD)_w$ and $J' \subseteq (dy_1D + \cdots + dy_nD)_w$. Then $I'' = I'/d \subseteq (x_1D + \cdots + x_nD)_w$,

$J'' = J'/d \subseteq (y_1D+\cdots+y_nD)_w$ are fractional ideals of D and $zD = (I''J'')_w$. This implies that $zE = (zE)_w = ((I''J'')_wE)_w$. Since E is t-linked, we get that $zE = (I''J''E)_w = ((I''E)(J''E))_w$ and $I''E \subseteq I_wE \subseteq (I_wE)_w = (IE)_w$, $J''E \subseteq J_wE \subseteq (J_wE)_w = (JE)_w$, cf. Lemma 1. The in particular statement follows from [7, Proposition 2.3(3)]. □

Remark 3. *Let D be a domain and I, J be nonzero ideals of D. If I or J is w-invertible (equivalently t-invertible, cf. [3, Corollary 2.17]), then I, J is an AC pair. Indeed, if J is w-invertible and $0 \neq z \in IJ$, then $zD = ((zJ^{-1})J)_w$ with $zJ^{-1} \subseteq I \subseteq I_w$.*

Proposition 4. *A domain D is an AC domain if and only if every pair of finitely generated ideals of D is an AC pair. In particular, every PVMD is an AC domain.*

Proof. One implication is clear. For the converse, let I, J be an arbitrary pair of ideals of D and $0 \neq z \in IJ$. Then $z = i_1j_1 + \cdots + i_nj_n$, where $i_k \in I$ and $j_k \in J$ for $k = 1, ..., n$. Then $z \in I'J'$, where the ideals I' and J' are given by $I' = (i_1, ..., i_n) \subseteq I$ and $J' = (j_1, ..., j_n) \subseteq J$. By hypothesis, I', J' is an AC pair, so there exist ideals $I'' \subseteq I'_w \subseteq I_w$ and $J'' \subseteq J'_w \subseteq J_w$ such that $zD = (I''J'')_w$. The "in particular" assertion follows from Remark 3 because a nonzero fractional ideal is w-invertible if and only if it is t-invertible. □

Example 5. *Consider the domain $D = F[[X^2, X^3]]$ where F is a field. Then by [13, Example 2.3], D is condensed domain and hence AC. Clearly D is not PVMD because it is not integrally closed.*

Proposition 6. *Let D be an AC domain in which every maximal ideal is a t-ideal. Then D is an HC domain.*

Proof. If every maximal ideal is a t-ideal, then $w = d$, cf. [24, Proposition 2.2]. □

Example 7. *Since a quasi-local HC domain is condensed [6], Proposition 6 shows that, a quasi-local one-dimensional domain is AC if and only if it is condensed. So, the rings $\mathbb{R} + X\mathbb{R}(Y,Z)[[X]]$ and $\mathbb{R}[[X^3, X^5, X^6, X^8, ...]]$ are not AC, cf. [13, Example 2.11].*

Proposition 8. *Let D be an AC domain with $Cl_t(D) = 0$. Then every pair of w-ideals of D is condensed.*

Proof. Let I, J be a pair of w-ideals of D with $0 \neq z \in IJ$. Since D is AC, there exist I', J' (t-invertible) ideals of D such that $I' \subseteq I_w = I$, $J' \subseteq J_w = J$ and $zD = (I'J')_w$. As I', J' are w-invertible and $Cl_t(D) = 0$, we get $I' = xD$ and $J' = yD$ for some $x \in I$ and $y \in J$. Then $zD = ((xD)(yD))_w = ((xy)D)_w = (xy)D$ implies $z = xyu$ for some unit u. Thus D is condensed. □

Proposition 9. *Let D be an AC domain. Then D_M is condensed for each $M \in Max_t(D)$.*

Proof. Let $M \in Max_t(D)$ and $0 \neq z \in (ID_M)(JD_M)$, where I, J are ideals of D. Then $sz \in IJ$ for some $s \in D - M$. Since D is AC, there exist I', J' ideals of D such that $I' \subseteq I_w, J' \subseteq J_w$ and $szD = (I'J')_w$. Then $zD_M = ((I'J')_w)D_M = (I'J')D_M$ and $I'D_M \subseteq ID_M$, $J'D_M \subseteq JD_M$. As $I'D_M$ and $J'D_M$ are principal, D_M is condensed. □

In [6, Proposition 1.2], it is shown that an integral domain is Prüfer if and only if it is integrally closed and HC. We extend this result to AC domains.

Proposition 10. *A domain D is a PVMD if and only if it is integrally closed and an AC domain.*

Proof. A PVMD is integrally closed (cf. [22, Theorem 3.5]) and by Proposition 4 it is an AC domain. Conversely, assume that D is integrally closed and an AC domain. Then by Proposition 9, [11, Main Theorem] and [23, Theorem 63], D_M is a valuation domain for each maximal t-ideal M of D. Hence D is a PVMD, cf. [20, Theorem 5]. □

Corollary 11. *The complete integral closure of an AC domain is a PVMD.*

Proof. Let D be an AC domain. Then $\bar{D}$ is t-linked by [7, Proposition 2.3(5)] and integrally closed by [18, Theroem 13.1(2)]. Now apply Propositions 2 and 10. □

Recall from [9] that a family $\mathcal{F} = \{P_i\}_{i \in L}$ of nonzero prime ideals of D is called a *defining family of primes* for D if $D = \bigcap_{i \in L} D_{P_i}$. If, further, every nonzero nonunit of D belongs to at most finitely many members of $\mathcal{F}$, $\mathcal{F}$ is of finite character, and if no two members of $\mathcal{F}$ contain a nonzero prime ideal, $\mathcal{F}$ is independent. An ideal I of a domain is called *unidirectional* if it belongs to a unique member of the defining family $\mathcal{F}$ of primes. An integral domain D is independent of finite character $\mathcal{F}$ (or an $\mathcal{F}$-IFC-domain) if it has a defining family $\mathcal{F}$ of primes that is independent and of finite character. The corresponding family $\{D_{P_i}\}_{i \in L}$ of overrings of D induces a star operation $*_{\mathcal{F}}$ on D defined by $I \longmapsto I^{*_{\mathcal{F}}} = \bigcap_{i \in L} ID_{P_i}$ for all $I \in F(D)$. We shall often refer to $*_{\mathcal{F}}$ as the star operation induced by $\mathcal{F}$. By [3, Corollary 2.13] and [9, Proposition 3.2], $I^{*_{\mathcal{F}}} \subseteq I_w$.

Proposition 12. *Let D be an $\mathcal{F}$-IFC domain such that D_M is condensed for each $M \in \mathcal{F}$. Then D is an AC domain.*

Proof. Let I, J be nonzero ideals of D. Let $0 \neq x \in IJ$ and let $M_1, ..., M_n$ be the members of $\mathcal{F}$ containing x. Since ID_{M_i}, JD_{M_i} is a condensed pair, $x = a_i b_i$ for some $a_i \in ID_{M_i}$ and $b_i \in JD_{M_i}$. Hence $xD_{M_i} = (A_i)_{M_i}(B_i)_{M_i}$, where $A_i = a_i D_{M_i} \cap D$ and $B_i = b_i D_{M_i} \cap D$ are unidirectional ideals with respect to $\mathcal{F}$ or equal to D, cf. [9, Theorem 3.3]. Set $A = A_1 \cdots A_n$ and $B = B_1 \cdots B_n$. We have $(AB)_{M_i} = (A_i B_i)_{M_i} = (xD)_{M_i}$. Also $(AB)D_M = D_M = xD_M$ for each M in $\mathcal{F}$ distinct from $M_1, ..., M_n$. It follows that $(AB)^{*_{\mathcal{F}}} = (xD)^{*_{\mathcal{F}}} = xD$ and hence $xD = (AB)_w$ with $A \subseteq I^{*_{\mathcal{F}}} \subseteq I_w$ and $B \subseteq J^{*_{\mathcal{F}}} \subseteq J_w$. □

Recall from [10], that a domain D is called a *weakly Matlis domain* if the intersection $D = \cap\{D_P \mid P \in Max_t(D)\}$ is independent of finite character.

Corollary 13. *Let D be a weakly Matlis domain. Then D is an AC domain if and only if D_M is condensed for each $M \in Max_t(D)$.*

Proof. Apply Proposition 12 and 9. □

Theorem 14. *Let D be a Noetherian domain. Then the following are equivalent:*

(1) *D is an AC domain.*
(2) *D_M is condensed for each $M \in Max_t(D)$.*
(3) *D is weakly Krull and D_M is condensed for each $M \in X^1(D)$.*

Proof. (1) $\Rightarrow$ (2). Apply Proposition 9.

(2) $\Rightarrow$ (3). As D is a Mori domain, the intersection $D = \cap_{M \in Max_t(D)} D_M$ has finite character, cf. [14, Theorem 3.3]. Also every maximal t-ideal M of D has height one. Indeed, if $ht(M) > 1$, then $ht(MD_M) > 1$ which is a contradiction because a Noetherian condensed domain has dimension at most one, cf. [13, Corollary 2.9]. Thus $Max_t(D) = X^1(D)$ and we are done.

(3) $\Rightarrow$ (1). By [8, Lemma 2.1], every maximal t-ideal has height one. So D is an $\mathcal{F}$-IFC domain with $\mathcal{F} = X^1(D)$. Hence by Proposition 12, D is an AC domain. □

Proposition 15. *Let $\{D_i\}_{i \in \Lambda}$ be an ascending chain of AC domains such that each D_i is t-linked over D_j for $D_j \subseteq D_i$. Then $D = \cup_{i \in \Lambda} D_i$ is an AC domain.*

Proof. Let I, J be two nonzero ideals of D with $0 \neq z \in IJ$. Then $z = x_1 y_1 + \cdots + x_n y_n$, where $x_k \in I$ and $y_k \in J$ for $k = 1, ..., n$. There exist $i_0 \in \Lambda$ such that D_{i_0} contains all elements x_i, y_j and so $z \in D_{i_0}$. Let $I' = x_1 D_{i_o} + \cdots + x_n D_{i_o}$ and $J' = y_1 D_{i_o} + \cdots + y_n D_{i_o}$. Then $z \in I'J'$. As D_{i_0} is an AC domain, $zD_{i_0} = (I''J'')_w$ for some ideals $I'' \subseteq I'_w$ and $J'' \subseteq J'_w$. Since D is t-linked over D_{i_0} (cf. [7, Proposition 2.3(1)]), $zD = ((I''J'')_w D)_w = (I''J''D)_w = ((I''D)(J''D))_w$ and $I''D \subseteq (I'D)_w \subseteq I_w$, $J''D \subseteq (J'D)_w \subseteq J_w$, cf. Lemma 1. □

A domain D is said to have the *two-generator property* or simply is *two-generated* if every ideal of D is generated by two elements.

Proposition 16. *Let D be a Noetherian weakly Krull domain whose integral closure D' is two-generated as a D-module. Then $D[\{X_\lambda\}]$ is an AC domain for every set of indeterminates $\{X_\lambda\}_{\lambda\in\Lambda}$ over D.*

Proof. Since a polynomial extension is t-linked [22, Corollary 2.3], we can assume that $\{X_\lambda\}_{\lambda\in\Lambda}$ is finite, cf. Proposition 15. By induction, on the cardinality of Λ, we can assume that $\{X_\lambda\}_{\lambda\in\Lambda}$ has a single indeterminate X. The domain $D[X]$ is a Noetherian weakly Krull domain (cf. [7, Corollary 4.12]) and $D'[X]$ is two-generated as a $D[X]$-module. Let P be a height-one prime ideal of $D[X]$. By using $(a) \Leftrightarrow (c)$ in [19, Theorem 2.3], $D[X]_P$ is two-generated and by [6, Corollary 1.5] it is condensed. By Theorem 14, $D[X]$ is an AC domain. □

Example 17. *Let $A \subset B$ be an extension of DVRs such that $B_{A-\{0\}}$ is a field, $qf(A)\cap B = A$ and B is two-generated as an A-module. Then $D = A+XB[X]$ is an AC domain. Indeed, D is a Noetherian weakly Krull domain (see [12, Theorem 3.4]) and $D' = B[X]$ is two-generated as a D-module. In particular, the domain $\mathbb{R}[[X]] + Y\mathbb{C}[[X]][Y]$ is an AC domain.*

Remark 18. *A domain D is PVMD if and only if $D[X]$ is PVMD, cf. [22, Theorem 3.7]. Therefore an integrally closed domain D is AC if and only $D[X]$ is AC, cf. Proposition 10.*

Recall from [18, Section 33] that the *Nagata ring* $D(X)$ of a domain D is the fraction ring of $D[X]$ with respect to the multiplicative set S of polynomials whose coefficients generate the unit ideal. Our final example shows that if D is a Noetherian AC domain, then $D[X]$ need not be an AC domain.

Example 19. Let $\mathbf{B}$ be the field of all complex numbers which can be constructed by straight-edge and compass from 0 and 1 (see [21, page 210]). Let $D = \mathbf{B} + X\mathbf{B}(\sqrt[3]{3})[[X]]$. Then D is a local condensed domain (see [4, Example 2.11]) and $D' = \mathbf{B}(\sqrt[3]{3})[[X]]$ is a DVR which is finite over D. The domain $D[Y]$ is a Noetherian weakly Krull domain, cf. [12, Theorem 3.4]. Let $M = X\mathbf{B}(\sqrt[3]{3})[[X]]$. Then $D[Y]_{M[Y]} = D(Y)$. The ring $D(Y)$ is a local ring with residue field $\mathbf{B}(Y)$. Also the integral closure of $D(Y)$ is $D'(Y)$ (see [18, Exercise 2, p. 415]) which is a DVR and is finite over $D(Y)$. The residue field of $D'(Y)$ is $\mathbf{B}(\sqrt[3]{3})(Y)$ and $[\mathbf{B}(\sqrt[3]{3})(Y) : \mathbf{B}(Y)] = 3$. Note that $Z^2 - Y$ is a quadratic polynomial in $\mathbf{B}(Y)[Z]$ which does not split in $\mathbf{B}(Y)$. By [5, Theorem 5], $D(Y)$ is not condensed and hence $D[Y]$ is not AC, cf. Theorem 14.

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