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On q-Analogues of Sumudu Transform

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Abstract

The present paper introduces q-analogues of the Sumudu transform and derives some distinct properties, for example its convergence conditions and certain interesting connection theorems involving q-Laplace transforms. Furthermore, certain fundamental properties of q-Sumudu transforms like, linearity, shifting theorems, differentiation and integration etc. have also been investigated. An attempt has also been made to obtain the convolution theorem for the q-Sumudu transform of a function which can be expressed as a convergent infinite series.

1 Introduction

Differential equations play a major role in Mathematics, Physics and Engineering. There are lots of different techniques to solve differential equations. Integral transforms are widely used and thus a lot of work has already been done on the theory and its applications. The most popular integral transforms have been contributed largely by Laplace, Fourier, Mellin and Hankel. The Laplace transform is of great importance among these transforms. For further detail, one may refer to the recent papers [12]-[15] on the subject. The Sumudu transform plays a curious role in the solution of ordinary differential equations and other branches of Mathematics and Physics. The Sumudu transform was proposed originally by Watugala [16] and he applied it to the solution of ordinary differential equations to control engineering problems. Nevertheless,

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this new transform rivals the Laplace transform in problem solving. Its main advantage is the fact that it may be used to solve problems without resorting to a new frequency domain, because it preserves scale and unit properties.

The theory of q-analysis (developed in 18^{th} century), has been applied to many areas of Mathematics and Physics like ordinary fractional calculus, optimal control problems, q-transform analysis and in finding solutions of the q-difference and q-integral equations (for instance, see [2],[3],[7],[9] and [10]). In 1910, Jackson [6] presented a precise definition of so-called q-Jackson integral and developed q-calculus in a systematic way. It is the well known that, in the literature, there are two types of q-Laplace transforms and they have been studied in detail by many authors. ([1], [5], [11], etc.)

The aim of this paper is to give q-analogues of Sumudu transform and to investigate its basic properties. This paper is organized in the following manner. Section 2, introduces q-analogues of Sumudu transform and presents its basic properties. Section 3, gives the q-images of power function $x^{\alpha-1}$ under the q-Sumudu transforms as illustrated from the results of the previous section. The next section, introduces the convolution theorem for q-Sumudu transform. As for prerequisites, the reader is expected to be familiar with notations of q-calculus. The basic definitions and facts from the q-calculus is necessary for the understanding of this study. Throughout this paper we will assume that q satisfies the condition $0 < |q| < 1$. The q-derivative $D_q f$ of an arbitrary function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

where $x \neq 0$. Clearly, if f is differentiable, then

$$\lim_{q \to 1^-} (D_q f)(x) = \frac{df(x)}{dx}.$$

n times repeated application of the D_q operator is denoted by $D_q^{(n)}$.

Let us introduce some notation that is used in the remainder of the paper. For any real number α,

$$[\alpha] := \frac{q^\alpha - 1}{q - 1}.$$

In particular, if $n \in \mathbb{Z}^+$, we denote

$$[n] = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1.$$

and q-binomial coefficient is defined by

$$\begin{bmatrix} \\ n \end{bmatrix}\Big[k = \frac{[n]!}{[k]!\,[n-k]!},$$

where $[n]! = [n]\,[n-1]\cdots[2]\,[1]$.
For $x, y, \nu, a, t \in \mathbb{R}$, the following usual notations is used:

$$(a;q)_n = \begin{cases} \prod\limits_{k=0}^{n-1}\left(1-aq^k\right) & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0 \end{cases}, \tag{1}$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right), \tag{2}$$

$$(a;q)_t = \frac{(a;q)_\infty}{(aq^t;q)_\infty}, \tag{3}$$

$$(x-y)_q^\nu = x^\nu \prod_{k=0}^{\infty}\frac{\left(1-(y/x)\,q^k\right)}{\left(1-(y/x)\,q^{\nu+k}\right)}. \tag{4}$$

The q-analogues of the classical exponential functions are defined as

$$e_q(t) = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}, \tag{5}$$

$$E_q(t) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2} t^n}{(q;q)_n} = (t;q)_\infty, \tag{6}$$

and q-exponential functions have the following properties

$$\lim_{q\to 1^-} e_q\left((1-q)\,x\right) = e^x,$$
$$\lim_{q\to 1^-} E_q\left((1-q)\,x\right) = e^x.$$

If a function $f(x)$ has a series expansion as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n x^n,$$

then the following function is well defined

$$f[x-y] = \sum_{n=-\infty}^{\infty} a_n\,(x-y)_q^n\,. \tag{7}$$

The q-integrals are defined as [7]

$$\int_0^x f(t)d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k), \tag{8}$$

$$\int_x^{\infty} f(t)d_qt = x(1-q)\sum_{k=1}^{\infty} q^{-k} f(xq^{-k}), \tag{9}$$

$$\int_0^{\infty/A} f(t)d_qt = (1-q)\sum_{k\in\mathbb{Z}} \frac{q^k}{A} f(\frac{q^k}{A}). \tag{10}$$

The q-analogue of the integration theorem by a change variable is expressed when $u(x) = \alpha x^{\beta}, \alpha \in \mathbb{C}$ and $\beta > 0$, as

$$\int_{u(a)}^{u(b)} f(u)\, d_qu = \int_a^b f(u(x))\, D_{q^{1/\beta}} u(x)\, d_{q^{1/\beta}}x.$$

A q-analogue of the integration by parts formula is given by

$$\int_a^b f(x)\, D_q g(x)\, d_qx = f(x)\ \ g(x)|_{x=a}^{b} - \int_a^b g(qx)\, D_q f(x)\, d_qx \tag{11}$$

q-analogues of gamma and beta functions are defined as follow (see [8])

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(q(1-q)x)\, d_qx \quad (\alpha > 0), \tag{12}$$

$$\Gamma_q(\alpha) = K(A;\alpha) \int_0^{\infty/A(1-q)} x^{\alpha-1} e_q(-(1-q)x)\, d_qx \ \ (\alpha > 0), \tag{13}$$

$$B_q(t;s) = \int_0^1 x^{t-1} (1-qx)_q^{s-1}\, d_qx \ \ (t,s > 0), \tag{14}$$

where

$$K(A;t) = A^{t-1} \frac{(-q/A;q)_{\infty}}{(-q^t/A;q)_{\infty}} \frac{(-A;q)_{\infty}}{(-Aq^{1-t};q)_{\infty}} \quad (t \in \mathbb{R}). \tag{15}$$

q-gamma function has the following property:

$$\Gamma_q(x) = \frac{(q;q)_{\infty}}{(q^x;q)_{\infty}} (1-q)^{1-x} = \frac{(q;q)_{x-1}}{(1-q)^{x-1}}, \tag{16}$$

and also

$$\lim_{q\to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha).$$

Hahn [5] defined the q-analogues of the well-known classical Laplace transform by means of the following q-integrals

$$L_q\{f(t);s\} = \frac{1}{1-q}\int_0^{s^{-1}} E_q(qst)f(t)d_qt \qquad (\mathrm{Re}\,(s)>0)\,, \tag{17}$$

and

$$\mathcal{L}_q\{f(t);s\} = \frac{1}{1-q}\int_0^{\infty} e_q(-st)f(t)d_qt \qquad (\mathrm{Re}\,(s)>0)\,, \tag{18}$$

where the q-analogues of the classical exponential functions are defined by (5) and (6). By virtue of (8) and (10), q-Laplace transforms can be expressed as

$$L_q\{f(t);s\} = \frac{(q;q)_\infty}{s}\sum_{k=0}^{\infty}\frac{q^k f\left(s^{-1}q^k\right)}{(q;q)_k},$$

and

$$\mathcal{L}_q\{f(t);s\} = \frac{1}{(-s;q)_\infty}\sum_{k=0}^{\infty} q^k f\left(q^k\right)(-s;q)_k\,.$$

Next section, defines a new type of q-integral transforms and investigates certain fundamental properties of these integral transforms.

2 Definitions and Preliminary Results

In 1993, Watugala [16] introduced the Sumudu transform as follow

$$G(u) = S\{f(t);u\} = \int_0^\infty f(ut)\,e^{-t}dt, \qquad u\in(-\tau_1,\tau_2)$$
$$= \frac{1}{u}\int_0^\infty f(t)\,e^{-t/u}dt \qquad u\in(-\tau_1,\tau_2)$$

over the set of functions

$$A = \left\{f(t)|\exists M,\tau_1,\tau_2>0, |f(t)|<Me^{|t|/\tau_j},\ t\in(-1)^j\times[0,\infty)\right\}.$$

Now, the following two q-Sumudu transforms which may be regarded as q-extensions of the Sumudu transform are introduced:

Definition 1. *The first type of q-analogue of Sumudu transform will be denoted by S_q and defined as follow*

$$S_q\{f(t);s\} = \frac{1}{1-q}\int_0^1 E_q(qt)\,f(st)\,d_qt, \qquad s\in(-\tau_1,\tau_2) \tag{19}$$
$$= \frac{1}{(1-q)\,s}\int_0^s E_q\left(\frac{q}{s}t\right)f(t)\,d_qt, \qquad s\in(-\tau_1,\tau_2) \tag{20}$$

over the set of functions

$$A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0,\ |f(t)| < M E_q \left(|t| / \tau_j \right),\ t \in (-1)^j \times [0, \infty) \right\}.$$

In view of (8), (19) *can be written as*

$$S_q \left\{ f\left(t\right); s \right\} = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k f\left(sq^k\right)}{(q;q)_k}. \tag{21}$$

Theorem 1. *Let* $G_1(s) = S_q\{f(x); s\}$. *The* q*-integral defined by* (20) *converges if* $x^\alpha E_q\left(\frac{q}{s}x\right) f(x)$ *is bounded on* $(0, s]$ *with some* $\alpha < 1$. *Furthermore, if* $G_1(0) = 0$ *then the function* $G_1(s) = S_q\{f(x); s\}$ *is continuous at* $s = 0$.

Proof. Suppose $\left| x^\alpha E_q\left(\frac{q}{s}x\right) f(x) \right| < M$ on $(0, s]$ for some $\alpha < 1$. For any $k \geq 0$ we have

$$\left| E_q\left(q^{k+1}\right) f\left(sq^k\right) \right| < M\left(sq^k\right)^{-\alpha} = M s^{-\alpha} q^{-k\alpha}.$$

Thus, for $0 < x = sq^k \leq s$ and $0 < q < 1$, we have

$$\begin{aligned} \left| \sum_{k=0}^{\infty} q^k E_q\left(q^{k+1}\right) f\left(sq^k\right) \right| &< \sum_{k=0}^{\infty} q^k \left| E_q\left(q^{k+1}\right) f\left(sq^k\right) \right| \\ &< \sum_{k=0}^{\infty} M s^{-\alpha} \left(q^{1-\alpha}\right)^k. \end{aligned}$$

Since $1 - \alpha > 0$ and $0 < q < 1$, it is implicit our series is bounded by a convergent geometric series. Hence, the right hand side of (21) converges pointwise to $G_1(s)$. Continuity of $G_1(s)$ is obvious. An effort will be made to show that $\lim_{s \to 0} G_1(s) = G_1(0) = 0$. Let $\epsilon > 0$ be given. For $|s - 0| < \delta$, we can find a $\delta \in \mathbb{R}^+$ such that

$$\begin{aligned} \left| \sum_{k=0}^{\infty} q^k E_q\left(q^{k+1}\right) f\left(sq^k\right) - 0 \right| &< \sum_{k=0}^{\infty} M s^{-\alpha} \left(q^{1-\alpha}\right)^k \\ &= \frac{M s^{-\alpha}}{1 - q^{1-\alpha}} \\ &< \frac{M \delta^{-\alpha}}{1 - q^{1-\alpha}} < \epsilon \ \left(0 < x < s\right). \end{aligned}$$

Thus, $G_1(s)$ is continuous at $s = 0$.

There is a simple relation between the first type q-Laplace transform L_q and the first type q-Sumudu transform S_q.

Theorem 2. *Let* $F_1(s) = L_q\{f(t);s\}$ *and* $G_1(s) = S_q\{f(t);s\}$. *Then we have*

$$G_1(s) = \frac{1}{s}F_1\left(\frac{1}{s}\right).$$

Proof. Making use of (17) and (20), one can easily obtain

$$\begin{aligned} G_1(s) = S_q\{f(t);s\} &= \frac{1}{1-q}\int_0^1 E_q(qt)\, f(st)\, d_qt \\ &= \frac{1}{1-q}\frac{1}{s}\int_0^s E_q\left(\frac{q}{s}t\right) f(t)\, d_qt \\ &= \frac{1}{s}F_1\left(\frac{1}{s}\right). \end{aligned}$$

Definition 2. *The second type of q-analogue of Sumudu transform will be denoted by $\mathbb{S}_q$ and defined as follow*

$$\mathbb{S}_q\{f(t);s\} = \frac{1}{1-q}\int_0^{\infty/s} e_q(-t)\, f(st)\, d_qt, \qquad s\in(-\tau_1,\tau_2) \tag{22}$$

$$= \frac{1}{(1-q)s}\int_0^{\infty} e_q\left(-\frac{1}{s}t\right) f(t)\, d_qt, \qquad s\in(-\tau_1,\tau_2) \tag{23}$$

over the set of functions

$$B = \left\{f(t)|\exists M,\tau_1,\tau_2>0, |f(t)|<Me_q(|t|/\tau_j), t\in(-1)^j\times[0,\infty)\right\}.$$

In view of (10)*, the formula* (23) *can be written as*

$$\mathbb{S}_q\{f(t);s\} = \frac{s^{-1}}{\left(-\frac{1}{s};q\right)_\infty}\sum_{k\in\mathbb{Z}} q^k f(q^k)\left(-\frac{1}{s};q\right)_k. \tag{24}$$

Theorem 3. *The improper q-integral defined by* (23) *converges if* $x^\alpha e_q\left(-\frac{1}{s}x\right)f(x)$ *is bounded in a neighborhood of* $x=0$ *with some* $\alpha<1$ *and for sufficiently large* x *with some* $\alpha>1$.

Proof. Making use of the formula (10), we have

$$\begin{aligned} \mathbb{S}_q\{f(t);s\} &= \frac{1}{(1-q)s}\int_0^\infty e_q\left(-\frac{1}{s}t\right) f(t)\, d_qt \\ &= \frac{1}{s}\sum_{k\in\mathbb{Z}} q^k e_q\left(-\frac{1}{s}q^k\right) f(q^k) \\ &= \sum_{k=0}^{\infty} q^k e_q\left(-\frac{1}{s}q^k\right) f(q^k) + \sum_{k=1}^{\infty} q^{-k} e_q\left(-\frac{1}{s}q^{-k}\right) f(q^{-k}). \end{aligned} \tag{25}$$

For $0 < q < 1$, an effort has been made to investigate the convergence of the right hand side. Under the hypothesis of the theorem, there exists a positive real number M, provided that $\left|x^\alpha e_q\left(-\frac{1}{s}x\right) f(x)\right| < M$ for all x in the interval $(0, \infty)$. Therefore, for $k \geq 0$, we can write

$$\left|e_q\left(-\frac{1}{s}q^k\right) f\left(q^k\right)\right| < M\left(q^k\right)^{-\alpha}.$$

Thus, for any $k \geq 0, 1-\alpha > 0$ and $0 < q < 1$, we have

$$\begin{aligned}\left|\sum_{k=0}^{\infty} q^k e_q\left(-\frac{1}{s}q^k\right) f\left(q^k\right)\right| &< \sum_{k=0}^{\infty} q^k \left|e_q\left(-\frac{1}{s}q^k\right) f\left(q^k\right)\right| \\ &< \sum_{k=0}^{\infty} M\left(q^{1-\alpha}\right)^k.\end{aligned}$$

As in the proof of the Theorem 1, we see that the first infinite sum in (25) is bounded by a convergent geometric series. For the second infinite sum, we suppose for large x we have

$$\left|x^\alpha e_q\left(-\frac{1}{s}x\right) f(x)\right| < M.$$

Thus we have

$$\left|q^{-k} e_q\left(-\frac{1}{s}q^{-k}\right) f\left(q^{-k}\right)\right| < q^{k(\alpha-1)}\left|q^{-k\alpha} e_q\left(-\frac{1}{s}q^{-k}\right) f\left(q^{-k}\right)\right| < Mq^{k(\alpha-1)}.$$

It follows that

$$\left|\sum_{k=1}^{\infty} q^{-k} e_q\left(-\frac{1}{s}q^{-k}\right) f(q^{-k})\right| < \sum_{k=1}^{\infty} M\left(q^{\alpha-1}\right)^k.$$

Since the second infinite sum is bounded by a convergent geometric series, it can be concluded that it converges. This proves the theorem.

Theorem 4. *Let* $F_2(s) = \mathcal{L}_q\{f(t); s\}$ *and* $G_2(s) = \mathbb{S}_q\{f(t); s\}$. *Then we have*

$$G_2(s) = \frac{1}{s}F_2\left(\frac{1}{s}\right).$$

Proof. Using the definitions (18) of $\mathcal{L}_q$-transform and (23) of $\mathbb{S}_q$-transform, one can easily obtain

$$\begin{aligned} G_2(s) = \mathbb{S}_q\{f(t);s\} &= \frac{1}{1-q}\int_0^{\infty/s} e_q(-t)\, f(st)\, d_qt \\ &= \frac{1}{1-q}\frac{1}{s}\int_0^{\infty} e_q\left(-\frac{1}{s}t\right) f(t)\, d_qt \\ &= \frac{1}{s}F_2\left(\frac{1}{s}\right). \end{aligned}$$

Proposition 1. *Let $f_1, g_1 \in A$, $f_2, g_2 \in B$ and $a, b \in \mathbb{R}$. One has the following properties of q-Sumudu transforms;*
a) Linearity properties:

$$\begin{aligned} S_q\{af_1(t)+bg_1(t);s\} &= aS_q\{f_1(t);s\}+bS_q\{g_1(t);s\}, \\ \mathbb{S}_q\{af_2(t)+bg_2(t);s\} &= a\mathbb{S}_q\{f_2(t);s\}+b\mathbb{S}_q\{g_2(t);s\}. \end{aligned}$$

b) Shifting properties:

$$\begin{aligned} S_q\{f_1(at);s\} &= S_q\{f_1(t);as\}, \\ \mathbb{S}_q\{f_2(at);s\} &= \mathbb{S}_q\{f_2(t);as\}. \end{aligned}$$

Proof. We only give here the proof of the linearity and shifting property of S_q, since the proof of the linearity and shifting property of $\mathbb{S}_q$ is similar. By the definition (19) of the S_q-transform, we have

$$\begin{aligned} S_q\{af_1(t)+bg_1(t);s\} &= \frac{1}{1-q}\int_0^1 E_q(qt)\{af_1(st)+bg_1(st)\}\, d_qt \\ &= a\frac{1}{1-q}\int_0^1 E_q(qt)\, f_1(st)\, d_qt + b\frac{1}{1-q}\int_0^1 E_q(qt)\, g_1(st)\, d_qt \\ &= aS_q\{f_1(t);s\}+bS_q\{g_1(t);s\}. \end{aligned}$$

Now assertion follows by making the change of variable $at = u$ and then by using the definition (19) of the S_q-transform

$$\begin{aligned} S_q\{f_1(at);s\} &= \frac{1}{1-q}\int_0^1 E_q(qt)\, f_1(ast)\, d_qt \\ &= S_q\{f_1(t);as\}. \end{aligned}$$

Proposition 2. *If f and $D_qf \in A$, then we have*

$$S_q\{D_qf(t);s\} = \frac{1}{(1-q)\,s}S_q\{f(t);s\} - \frac{1}{1-q}\frac{1}{s}f(0). \tag{26}$$

Proof. Applying the q-integration by parts formula (11) to the function $g(t) = E_q\left(\frac{1}{s}t\right)$ and then making use of the definition (20) of S_q-transform, we have

$$\begin{aligned} S_q\{D_q f(t); s\} &= \frac{1}{1-q}\frac{1}{s}\int_0^s E_q\left(\frac{q}{s}t\right) D_q f(t)\, d_q t \\ &= \frac{1}{1-q}\frac{1}{s}\left[E_q\left(\frac{1}{s}t\right) f(t)\Big|_{t=0}^{s} - \int_0^s D_q E_q\left(\frac{1}{s}t\right) f(t)\, d_q t\right] \\ &= \frac{1}{1-q}\frac{1}{s}\left[-f(0) + \frac{1}{(1-q)s}\int_0^s E_q\left(\frac{q}{s}t\right) f(t)\, d_q t\right] \\ &= \frac{1}{(1-q)s} S_q\{f(t); s\} - \frac{1}{1-q}\frac{1}{s} f(0). \end{aligned}$$

Proposition 3. *If $f, D_q f$ and $D_q^{(2)} f \in A$, then we have*

$$S_q\left\{D_q^{(2)} f(t); s\right\} = \frac{1}{(1-q)^2 s^2} S_q\{f(t); s\} - \frac{1}{(1-q)^2 s^2} f(0) - \frac{1}{1-q}\frac{1}{s} D_q f(0).$$

Proof. Using the formula (26), we get

$$\begin{aligned} S_q\left\{D_q^{(2)} f(t); s\right\} &= S_q\{D_q(D_q f(t)); s\} \\ &= \frac{1}{(1-q)s} S_q\{D_q f(t); s\} - \frac{1}{1-q}\frac{1}{s} D_q f(0), \end{aligned}$$

and then applying the formula (26) again, we obtain

$$\begin{aligned} S_q\left\{D_q^{(2)} f(t); s\right\} &= \frac{1}{(1-q)s}\left[\frac{1}{(1-q)s} S_q\{f(t); s\} - \frac{1}{1-q}\frac{1}{s} f(0)\right] - \frac{1}{1-q}\frac{1}{s} D_q f(0) \\ &= \frac{1}{(1-q)^2 s^2} S_q\{f(t); s\} - \frac{1}{(1-q)^2 s^2} f(0) - \frac{1}{1-q}\frac{1}{s} D_q f(0). \end{aligned}$$

Theorem 5. *Let $f, D_q f, D_q^{(2)} f, ..., D_q^{(n)} \in A$. Then we have the following generalization formula*

$$S_q\left\{D_q^{(n)} f(t); s\right\} = \frac{1}{(1-q)^n s^n} S_q\{f(t); s\} - \sum_{k=1}^{n} \frac{\left(D_q^{(n-k)} f\right)(0)}{(1-q)^k s^k}.$$

Proof. The result of the theorem is an immediate consequence of Propositions 2 and 3.

Proposition 4. *If f and $D_q f \in B$, then we have*

$$\mathbb{S}_q\left\{D_q f(t);s\right\} = \frac{1}{(1-q)s}\mathbb{S}_q\left\{f(t);qs\right\} - \frac{1}{1-q}\frac{1}{s}f(0). \tag{27}$$

Proof. Applying the q-integration by parts formula (11) to the function $g(t) = e_q\left(-\frac{1}{qs}t\right)$ and then making use of the definition (23) of $\mathbb{S}_q$-transform, we have

$$\begin{aligned}
\mathbb{S}_q\left\{D_q f(t);s\right\} &= \frac{1}{1-q}\frac{1}{s}\int_0^\infty e_q\left(-\frac{1}{s}t\right)D_q f(t)\,d_q t \\
&= \frac{1}{1-q}\frac{1}{s}\left[e_q\left(-\frac{1}{qs}t\right)f(t)\Big|_{t=0}^{\infty} - \int_0^\infty D_q e_q\left(-\frac{1}{qs}t\right)f(t)\,d_q t\right] \\
&= \frac{1}{1-q}\frac{1}{s}\left[-f(0) + \frac{1}{(1-q)qs}\int_0^\infty e_q\left(-\frac{1}{qs}t\right)f(t)\,d_q t\right] \\
&= \frac{1}{(1-q)s}\mathbb{S}_q\left\{f(t);qs\right\} - \frac{1}{1-q}\frac{1}{s}f(0).
\end{aligned}$$

Proposition 5. *If $f, D_q f$ and $D_q^{(2)} f \in B$, then we have*

$$\mathbb{S}_q\left\{D_q^{(2)} f(t);s\right\} = \frac{1}{(1-q)^2 s^2 q}\mathbb{S}_q\left\{f(t);q^2 s\right\} - \frac{1}{(1-q)^2 s^2 q}f(0) - \frac{1}{1-q}\frac{1}{s}D_q f(0).$$

Proof. Using formula (27), we get

$$\begin{aligned}
\mathbb{S}_q\left\{D_q^{(2)} f(t);s\right\} &= \mathbb{S}_q\left\{D_q\left(D_q f(t)\right);s\right\} \\
&= \frac{1}{(1-q)s}\mathbb{S}_q\left\{D_q f(t);qs\right\} - \frac{1}{1-q}\frac{1}{s}D_q f(0),
\end{aligned}$$

and then re-applying the formula (27), we obtain

$$\begin{aligned}
\mathbb{S}_q\left\{D_q^{(2)} f(t);s\right\} &= \frac{1}{(1-q)s}\left[\frac{1}{(1-q)qs}\mathbb{S}_q\left\{f(t);q^2 s\right\} - \frac{1}{1-q}\frac{1}{qs}f(0)\right] - \frac{1}{1-q}\frac{1}{s}D_q f(0) \\
&= \frac{1}{(1-q)^2 s^2 q}\mathbb{S}_q\left\{f(t);q^2 s\right\} - \frac{1}{(1-q)^2 s^2 q}f(0) - \frac{1}{1-q}\frac{1}{s}D_q f(0).
\end{aligned}$$

Theorem 6. *Let $f, D_q f, D_q^{(2)} f, ..., D_q^{(n)} \in B$. Then we have the following generalization formula*

$$\mathbb{S}_q\left\{D_q^{(n)} f(t);s\right\} = \frac{1}{(1-q)^n s^n q^{\frac{n(n-1)}{2}}}\mathbb{S}_q\left\{f(t);q^n s\right\} - \sum_{k=1}^{n}\frac{\left(D_q^{(n-k)} f\right)(0)}{(1-q)^k s^k q^{k-1}}.$$

Proof. The result of the theorem is an immediate consequence of Propositions 4 and 5.

Proposition 6. *If $f \in A$, $g \in B$ and $a \in \mathbb{R}$, then the following identities hold true*

$$S_q \left\{ \int_0^t f(x)\, d_q x; s \right\} = (1-q)\, s \; S_q \left\{ f(t); s \right\}, \tag{28}$$

$$\mathbb{S}_q \left\{ \int_0^t g(x)\, d_q x; s \right\} = \frac{(1-q)\, s}{q} \mathbb{S}_q \left\{ g(t); s \right\}. \tag{29}$$

Proof. By definition (20) of S_q-transform, we have

$$S_q \left\{ \int_0^t f(x)\, d_q x; s \right\} = \frac{1}{(1-q)\, s} \int_0^s E_q \left(\frac{q}{s} t \right) \left(\int_0^t f(x)\, d_q x \right) d_q t.$$

On the other hand, if we set $g(t) = E_q\left(\frac{1}{s}t\right)$ in (11) of q-analogue of the integration by parts formula, we get

$$\int_0^s F(t) D_q E_q \left(\frac{1}{s} t \right) d_q t = F(t) E_q \left(\frac{1}{s} t \right) \Big|_{t=0}^{s} - \int_0^s E_q \left(\frac{q}{s} t \right) D_q F(t) d_q t,$$

where $F(t) = \int_0^t f(x)\, d_q x$. Using the property $D_q E_q\left(\frac{1}{s}t\right) = -\frac{1}{(1-q)s} E_q\left(\frac{q}{s}t\right)$, we have

$$-\frac{1}{(1-q)s} \int_0^s F(t) E_q \left(\frac{q}{s} t \right) d_q t = F(t) E_q \left(\frac{1}{s} t \right) \Big|_{t=0}^{s} - \int_0^s E_q \left(\frac{q}{s} t \right) D_q F(t) d_q t.$$

Since $E_q(1) = 0$ by (6), we find

$$S_q \left\{ \int_0^t f(x)\, d_q x; s \right\} = ((1-q)\, s)\, S_q \left\{ f(t); s \right\}.$$

By definition (23) of $\mathbb{S}_q$-transform, we have

$$\mathbb{S}_q \left\{ \int_0^t g(x)\, d_q x; s \right\} = \frac{1}{(1-q)\, s} \int_0^\infty e_q \left(-\frac{1}{s} t \right) \left(\int_0^t g(x)\, d_q x \right) d_q t.$$

If we set $g(t) = e_q\left(-\frac{1}{s}t\right)$ in (11) of q-analogue of the integration by parts formula, we get

$$\int_0^\infty G(t) D_q e_q \left(-\frac{1}{s} t \right) d_q t = G(t) e_q \left(-\frac{1}{s} t \right) \Big|_{t=0}^{\infty} - \int_0^\infty e_q \left(-\frac{q}{s} t \right) D_q G(t) d_q t,$$

where $G(t)=\int_0^t g(x)\,d_qx$. Using the property $D_q e_q\left(-\frac{1}{s}t\right)=-\frac{1}{(1-q)s}e_q\left(-\frac{1}{s}t\right)$, we have

$$-\frac{1}{(1-q)s}\int_0^\infty G(t)e_q\left(-\frac{1}{s}t\right)d_qt=G(t)e_q\left(-\frac{1}{s}t\right)\Big|_{t=0}^{\infty}-\int_0^\infty e_q\left(-\frac{q}{s}t\right)D_qG(t)d_qt.$$

Since $e_q(-\infty)=0$, we find

$$\mathbb{S}_q\left\{\int_0^t g(x)\,d_qx;s\right\}=\frac{(1-q)\,s}{q}\mathbb{S}_q\left\{g(t);\frac{s}{q}\right\}.$$

Theorem 7. *If $f\in A$, $g\in B$ and $a\in\mathbb{R}$, then following identities hold true*

$$S_q\left\{\underbrace{\int_0^t\left(\int_0^t\cdots\left(\int_0^t f(t)\,d_qt\right)\ldots d_qt\right)d_qt}_{n\ times};s\right\}=((1-q)\,s)^n\,S_q\{f(t);s\},$$

$$\mathbb{S}_q\left\{\underbrace{\int_0^t\left(\int_0^t\cdots\left(\int_0^t g(t)\,d_qt\right)\ldots d_qt\right)d_qt}_{n\ times};s\right\}=\left(\frac{(1-q)\,s}{q}\right)^n\mathbb{S}_q\left\{g(t);\frac{s}{q^n}\right\}.$$

Proof. Assertions of the theorem are immediate consequences of Proposition 6.

Proposition 7. *Let $f\in A$, $g\in B$, $S_q\{f(t);s\}=G_1(s)$ and $\mathbb{S}_q\{g(t);s\}=G_2(s)$. Then the following identities hold true*

$$S_q\left\{\frac{f(t)}{t};s\right\}=\frac{1}{(1-q)\,s}\int_0^s\frac{G_1(s)}{s}d_qs, \tag{30}$$

$$\mathbb{S}_q\left\{\frac{g(t)}{t};s\right\}=\frac{1}{(1-q)\,s}\int_0^{qs}\frac{G_2(s)}{s}d_qs. \tag{31}$$

Proof. Writing the definition (8) of q-integral, we find

$$\begin{aligned}\frac{1}{(1-q)}\int_0^s\frac{G_1(s)}{s}d_qs&=\frac{1}{(1-q)}(1-q)\,s\sum_{n=0}^\infty q^n\frac{G_1(q^ns)}{q^ns}\\&=\sum_{n=0}^\infty G_1(q^ns)\\&=\sum_{n=0}^\infty(q;q)_\infty\sum_{k=0}^\infty\frac{q^kf\left(sq^{n+k}\right)}{(q;q)_k}.\end{aligned}$$

An easy computation shows that,

$$\begin{aligned}\frac{1}{(1-q)}\int_0^s \frac{G_1(s)}{s}d_qs &= (q;q)_\infty \sum_{k=0}^{\infty} f\left(sq^k\right)\sum_{n=0}^{k}\frac{q^n}{(q;q)_n} \\ &= (q;q)_\infty \sum_{k=0}^{\infty}\frac{f\left(sq^k\right)}{(q;q)_k} \\ &= s\left((q;q)_\infty \sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\frac{f\left(sq^k\right)}{sq^k}\right) \\ &= sS_q\left\{\frac{f(t)}{t};s\right\}.\end{aligned}$$

To prove (31), we start with the definition (8) of q-integral

$$\begin{aligned}\frac{1}{(1-q)}\int_0^{qs}\frac{G_2(s)}{s}d_qs &= \frac{1}{1-q}(1-q)\,qs\sum_{n=0}^{\infty}q^n\frac{G_2\left(q^{n+1}s\right)}{q^nqs} \\ &= \sum_{n=0}^{\infty}G_2\left(q^{n+1}s\right) \\ &= \sum_{n=0}^{\infty}\frac{q^{-1-n}s^{-1}}{\left(-\frac{1}{q^{n+1}s};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^kg(q^k)\left(-\frac{1}{q^{n+1}s};q\right)_k.\end{aligned}$$

Changing the order of summation, we have

$$\frac{1}{(1-q)}\int_0^{qs}\frac{G_2(s)}{s}d_qs = \sum_{k\in\mathbb{Z}}q^kg(q^k)\sum_{n=0}^{\infty}\frac{q^{-1-n}s^{-1}}{\left(-\frac{1}{q^{n+1}s};q\right)_\infty}\left(-\frac{1}{q^{n+1}s};q\right)_k. \tag{32}$$

If we set $a=-\frac{1}{q^{n+1}s}$ and $a=-\frac{1}{qs}$ respectively, in the following well-known formulas due to [4, p. 6]

$$(a;q)_\infty = (aq^n;q)_\infty (a;q)_n \quad\text{and}\quad \left(aq^{-n};q\right)_k = \frac{(a;q)_k\left(qa^{-1};q\right)_n}{\left(a^{-1}q^{1-k};q\right)_n}q^{-nk},$$

and introducing them in (32), we find that

$$\begin{aligned}\frac{1}{(1-q)}\int_0^{qs}\frac{G_2(s)}{s}d_qs &= \sum_{k\in\mathbb{Z}}q^kg(q^k)\sum_{n=0}^{\infty}\frac{q^{-1-n}s^{-1}}{\left(-\frac{1}{qs};q\right)_\infty\left(-\frac{1}{q^{n+1}s};q\right)_n}\frac{\left(-\frac{1}{qs};q\right)_k\left(-sq^2;q\right)_n}{\left(-sq^{2-k};q\right)_n}q^{-nk} \\ &= \sum_{k\in\mathbb{Z}}\frac{q^kg(q^k)\left(-\frac{1}{qs};q\right)_k}{\left(-\frac{1}{qs};q\right)_\infty}\sum_{n=0}^{\infty}\frac{q^{-1-n}s^{-1}}{\left(-\frac{1}{q^{n+1}s};q\right)_n}\frac{\left(-sq^2;q\right)_n}{\left(-sq^{2-k};q\right)_n}q^{-nk}.\end{aligned}$$

Similarly, if we set $a=-\frac{1}{qs}$ in

$$\left(aq^{-n};q\right)_n=\left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}\left(\frac{q}{a};q\right)_n,$$

we have

$$\begin{aligned}\frac{1}{(1-q)}\int_0^{qs}\frac{G_2(s)}{s}d_qs&=\sum_{k\in\mathbb{Z}}\frac{q^kg(q^k)\left(-\frac{1}{qs};q\right)_k}{\left(-\frac{1}{qs};q\right)_\infty}\sum_{n=0}^{\infty}\frac{q^{-1-n}s^{-1}}{(-sq^2;q)_n\left(\frac{1}{q^2s}\right)^n q^{-\binom{n}{2}}}\frac{\left(-sq^2;q\right)_n}{(-sq^{2-k};q)_n}q^{-nk}\\&=\sum_{k\in\mathbb{Z}}q^{-1}s^{-1}\frac{q^kg(q^k)\left(-\frac{1}{qs};q\right)_k}{\left(-\frac{1}{qs};q\right)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{(q;q)_n}{(-sq^{2-k};q)_n}\frac{\left(-sq^{1-k}\right)^n}{(q;q)_n}.\end{aligned}$$

Using the definition of q-hypergeometric series [4, p. 21]

$${}_1\Phi_1\left(a;c;q;c/a\right)=\frac{(c/a;q)_\infty}{(c;q)_\infty},$$

we find that

$$\begin{aligned}\frac{1}{(1-q)}\int_0^{qs}\frac{G_2(s)}{s}d_qs&=\sum_{k\in\mathbb{Z}}q^{-1}s^{-1}\frac{q^kg(q^k)\left(-\frac{1}{qs};q\right)_k}{\left(-\frac{1}{qs};q\right)_\infty}\,{}_1\Phi_1\left(q;-sq^{2-k};q;-sq^{1-k}\right)\\&=\sum_{k\in\mathbb{Z}}q^{-1}s^{-1}\frac{q^kg(q^k)\left(-\frac{1}{qs};q\right)_k}{\left(-\frac{1}{qs};q\right)_\infty}\frac{(-sq^{1-k};q)_\infty}{(-sq^{2-k};q)_\infty}\\&=\sum_{k\in\mathbb{Z}}q^{-1}s^{-1}\frac{q^kg(q^k)\left(-\frac{1}{s};q\right)_{k-1}}{\left(-\frac{1}{s};q\right)_\infty}\left(1+sq^{1-k}\right)\\&=\sum_{k\in\mathbb{Z}}\frac{g(q^k)\left(-\frac{1}{s};q\right)_k}{\left(-\frac{1}{s};q\right)_\infty}.\end{aligned}$$

Therefore, we obtain the assertion (31)

$$\begin{aligned}\frac{1}{(1-q)}\int_0^{qs}\frac{G_2(s)}{s}d_qs&=s\left(\frac{1}{s\left(-\frac{1}{s};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^k\frac{g(q^k)}{q^k}\left(-\frac{1}{s};q\right)_k\right)\\&=s\mathbb{S}_q\left\{\frac{g(t)}{t};s\right\}.\end{aligned}$$

Proposition 8. *Let $f \in A$, $g \in B$, $S_q\{f(t);s\} = G_1(s)$ and $\mathbb{S}_q\{g(t);s\} = G_2(s)$. Then the following identities hold true:*

$$S_q\left\{\frac{1}{1-q}\int_0^t \frac{f(x)}{x}d_qx;s\right\} = \frac{1}{1-q}\int_0^s \frac{G_1(s)}{s}d_qs, \tag{33}$$

$$\mathbb{S}_q\left\{\frac{1}{1-q}\int_0^t \frac{g(x)}{x}d_qx;s\right\} = \frac{1}{(1-q)q}\int_0^{qs} \frac{G_2(s)}{s}d_qs. \tag{34}$$

Proof. We only give the proof of (33) because the proof of (34) is similar. Using the definition (28) and (30), we have

$$\begin{aligned} S_q\left\{\frac{1}{1-q}\int_0^t \frac{f(x)}{x}d_qx;s\right\} &= sS_q\left\{\frac{f(t)}{t};s\right\} \\ &= \frac{1}{1-q}\int_0^s \frac{G_1(s)}{s}d_qs. \end{aligned}$$

Proposition 9. *Let $f \in A$, $g \in B$, $S_q\{f(t);s\} = G_1(s)$ and $\mathbb{S}_q\{g(t);s\} = G_2(s)$. Then the following identities hold true:*

$$S_q\left\{\frac{1}{1-q}\int_t^\infty \frac{f(x)}{x}d_qx;s\right\} = \frac{1}{1-q}\int_s^\infty \frac{G_1(s)}{s}d_qs \tag{35}$$

$$\mathbb{S}_q\left\{\frac{1}{1-q}\int_t^\infty \frac{g(x)}{x}d_qx;s\right\} = \frac{1}{1-q}\int_s^\infty \frac{G_2(s)}{s}d_qs \tag{36}$$

Proof. We only give the proof of (35) since the proof of (36) is similar. Indeed, to prove (35) , we start by using the definition (9) of q-improper integral

$$\begin{aligned} \frac{1}{1-q}\int_t^\infty \frac{f(x)}{x}d_qx &= \frac{1}{(1-q)}(1-q)\,t\sum_{k=1}^\infty q^{-k}\frac{f\left(q^{-k}t\right)}{q^{-k}t} \\ &= \sum_{k=1}^\infty f\left(q^{-k}t\right). \end{aligned} \tag{37}$$

Applying S_q-transform to both sides of (37) and using the Proposition 1, we have

$$\begin{aligned} S_q\left\{\frac{1}{1-q}\int_t^\infty \frac{f(x)}{x}d_qx;s\right\} &= \sum_{k=1}^\infty S_q\left\{f(t);q^{-k}s\right\} \\ &= \sum_{k=1}^\infty G_1\left(q^{-k}s\right). \end{aligned}$$

Now, we obtain

$$S_q\left\{\frac{1}{1-q}\int_t^{\infty}\frac{f(x)}{x}d_qx;s\right\}=\sum_{k=1}^{\infty}G_1\left(q^{-k}s\right)$$
$$=\frac{1}{1-q}(1-q)\sum_{k=1}^{\infty}q^{-k}s\frac{G_1\left(q^{-k}s\right)}{q^{-k}s}. \tag{38}$$

The summation on the right-hand side of (38) is the q-improper integral of $\frac{G_1(s)}{s}$. Therefore, we find that

$$S_q\left\{\frac{1}{1-q}\int_t^{\infty}\frac{f(x)}{x}d_qx;s\right\}=\frac{1}{1-q}\int_s^{\infty}\frac{G_1(s)}{s}d_qs.$$

3 q-Sumudu Transforms of Power Functions $x^{\alpha-1}$

This section envisages the evaluation of the q-images of power function $x^{\alpha-1}$ under the q-Sumudu transforms introduced in the previous section.

Theorem 8. *Let* $f(x)=x^{\alpha-1}$ $(\alpha>0)$. *Then*

$$S_q\left\{x^{\alpha-1};s\right\}=s^{\alpha-1}(1-q)^{\alpha-1}\Gamma_q(\alpha), \tag{39}$$

and

$$\mathbb{S}_q\left\{x^{\alpha-1};s\right\}=s^{\alpha-1}(1-q)^{\alpha-1}\frac{\Gamma_q(\alpha)}{K(s;\alpha)}. \tag{40}$$

Proof. Setting $f(x)=x^{\alpha-1}$ in the definition (20) of S_q-transform, we have

$$S_q\left\{x^{\alpha-1};s\right\}=\frac{1}{(1-q)}\int_0^1 E_q(qx)(sx)^{\alpha-1}d_qx.$$

Changing the variable $x=(1-q)t$, we find that

$$S_q\left\{x^{\alpha-1};s\right\}=s^{\alpha-1}(1-q)^{\alpha-1}\int_0^{1/1-q}t^{\alpha-1}E_q(q(1-q)t)\,d_qt.$$

Using the definition (12) of q-gamma function, we get the desired result. Similarly, setting $f(x)=x^{\alpha-1}$ in the definition (22) of $\mathbb{S}_q$-transform, we have

$$\mathbb{S}_q\left\{x^{\alpha-1};s\right\}=\frac{1}{1-q}\int_0^{\infty/s}e_q(-x)(sx)^{\alpha-1}d_qx.$$

Changing the variable $x = (1-q)\,t$, we find that

$$\mathbb{S}_q\left\{x^{\alpha-1};s\right\} = s^{\alpha-1}\left(1-q\right)^{\alpha-1}\int_0^{\infty/s(1-q)} t^{\alpha-1}e_q\left(-\left(1-q\right)t\right)d_qt.$$

Using the definition (13) of q-gamma function, we get the desired result.

4 The Convolution Theorem

In 1949, Hahn [5, (9.5), p. 372] gave the convolution theorem for the first kind of q-Laplace transform as follow:

$$L_q\left\{(f*g)(x);s\right\} = L_q\left\{f(x);s\right\}\ L_q\left\{g(x);s\right\}$$

where

$$(f*g)(x) = \frac{t}{1-q}\int_0^1 f(tx)\,g\left[t\left(1-qx\right)\right]d_qx.$$

This convolution theorem mentioned above is valid only for L_q-transform i.e. first kind of q-Laplace transform. Therefore, this section, establishes a convolution theorem for S_q-transform.

Theorem 9. *Suppose that $f, g \in A$ have the following series expansion*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \ \text{and}\ g(x) = \sum_{r=0}^{\infty} b_r x^r.$$

Then, the convolution of f and g is

$$S_q\left\{(f*g)(x);s\right\} = sS_q\left\{f(x);s\right\}\ S_q\left\{g(x);s\right\}. \tag{41}$$

Proof. Using the power series expansion of f and g, and relation (7), we can write

$$f(tx) = \sum_{k=0}^{\infty} a_k t^k x^k,$$

and

$$g\left[t\left(1-qx\right)\right] = \sum_{r=0}^{\infty} b_r t^r \left(1-qx\right)_q^r.$$

Thus, we get

$$\begin{aligned}(f*g)(x) &= \frac{t}{1-q}\int_0^1 f(tx)\,g\left[t(1-qx)\right]d_qx \\ &= \frac{t}{1-q}\int_0^1 \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k t^k x^k b_r t^r (1-qx)_q^r\, d_qx \\ &= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k b_r \frac{t^{k+r+1}}{1-q}\int_0^1 x^k (1-qx)_q^r\, d_qx.\end{aligned}$$

Using the definition (14) of q-beta integral, we obtain that

$$(f*g)(x) = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k b_r \frac{t^{k+r+1}}{1-q} B_q(k+1; r+1).$$

Now, on using the well-known relation, namely

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)},$$

we have

$$(f*g)(x) = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k b_r \frac{t^{k+r+1}}{1-q}\frac{\Gamma_q(k+1)\Gamma_q(r+1)}{\Gamma_q(k+r+2)}. \tag{42}$$

Applying S_q-transform to both side of (42), we obtain

$$S_q\{(f*g)(x); s\} = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k b_r \frac{1}{1-q}\frac{\Gamma_q(k+1)\Gamma_q(r+1)}{\Gamma_q(k+r+2)} S_q\left\{t^{k+r+1}; s\right\}.$$

By the result (39) of Theorem 8, we have

$$\begin{aligned}S_q\{(f*g)(x); s\} &= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} a_k b_r \Gamma_q(k+1)\Gamma_q(r+1)\, s^{k+r+1}(1-q)^{k+r} \\ &= s\left(\sum_{k=0}^{\infty} a_k \Gamma_q(k+1)\, s^k (1-q)^k\right)\left(\sum_{r=0}^{\infty} b_r \Gamma_q(r+1)\, s^r (1-q)^r\right) \\ &= s\left(\sum_{k=0}^{\infty} a_k S_q\left\{x^k; s\right\}\right)\left(\sum_{r=0}^{\infty} b_r S_q\left\{x^r; s\right\}\right) \\ &= sS_q\{f(x); s\}\, S_q\{g(x); s\}.\end{aligned}$$

5 Conclusions and Further Directions

This paper infer that the q-Sumudu transforms are of exemplary nature. Furthermore, the q-Sumudu transforms have very interesting properties which makes it easy to visualize. We conclude with the remark that, the concept of q-Sumudu transforms can find certain applications to solve q-difference and q-integral equations, and can lead to plenty applications on the subject. More importantly, using the results of Theorem 8, one can deduce a number of q-images under q-Sumudu transforms involving various q-Special functions.

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