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A characterization of the quaternion group

To Professor Mirela Ştefănescu, at her 70th anniversary

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Abstract

The goal of this note is to give an elementary characterization of the well-known quaternion group Q_8 by using its subgroup lattice.

1 Introduction

One of the most famous finite groups is the quaternion group Q_8. This is usually defined as the subgroup of the general linear group $GL(2,\mathbb{C})$ generated by the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Using matrix multiplication, we have $Q_8 = \{\pm\mathbf{1},\ \pm\mathbf{i},\ \pm\mathbf{j},\ \pm\mathbf{k}\}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. Moreover, $\mathbf{1}$ is the identity of Q_8 and **-1** commutes with all elements of Q_8. Remark that $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have order 4 and that any two of them generate the entire group. In this way, a presentation of Q_8 is

$$Q_8 = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1}\rangle$$

(take, for instance, $\mathbf{i} = a$, $\mathbf{j} = b$ and $\mathbf{k} = ab$). We also observe that the subgroup lattice $L(Q_8)$ consists of Q_8 itself and of the cyclic subgroups $\langle\mathbf{1}\rangle$, $\langle-\mathbf{1}\rangle$, $\langle\mathbf{i}\rangle$, $\langle\mathbf{j}\rangle$, $\langle\mathbf{k}\rangle$. It is well-known that Q_8 is a hamiltonian group, i.e. a non-abelian group all of whose subgroups are normal. More precisely

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Q_8 is the hamiltonian group with the smallest order.

Other basic properties of the subgroups of Q_8 are the following:

- excepting Q_8, they are cyclic;
- $\langle -\mathbf{1} \rangle$ is a *breaking point* in the poset of cyclic subgroups of Q_8, that is any cyclic subgroup of Q_8 either contains $\langle -\mathbf{1} \rangle$ or is contained in $\langle -\mathbf{1} \rangle$;
- $\langle \mathbf{i} \rangle$, $\langle \mathbf{j} \rangle$ and $\langle \mathbf{k} \rangle$ are *irredundant*, that is no one is contained in the union of the other two, and they determine a *covering* of Q_8, that is $Q_8 = \langle \mathbf{i} \rangle \cup \langle \mathbf{j} \rangle \cup \langle \mathbf{k} \rangle$.

These properties can be easily extended to some simple but very nice characterizations of Q_8 (see e.g. [7]), namely

> *Q_8 is the unique non-abelian p-group all of whose proper subgroups are cyclic*,

> *Q_8 is the finite non-cyclic group with the smallest order whose poset of cyclic subgroups has a unique breaking point*

and

> *Q_8 is the unique non-abelian group that can be covered by any three irredundant proper subgroups*,

respectively.

The purpose of this note is to provide a new characterization of Q_8 by using another elementary property of $L(Q_8)$. We recall first a subgroup lattice concept introduced by Schmidt [3] (see also [4]). Given a lattice L, a group G is said to be *L-free* if $L(G)$ has no sublattice isomorphic to L. Interesting results about L-free groups have been obtained for several particular lattices L, as the diamond lattice M_5 and the pentagon lattice N_5 (recall here only that a group is M_5-free if and only if it is locally cyclic, and N_5-free if and only if it is a modular group).

Clearly, for a finite group G the above concept leads to the more general problem of counting the number of sublattices of $L(G)$ that are isomorphic to a certain lattice. Following this direction, our next definition is very natural.

Definition 1.1. Let L be a lattice. A group G is called *almost L-free* if its subgroup lattice $L(G)$ contains a unique sublattice isomorphic to L.

Remark that both the Klein's group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and Q_8 are almost M_5-free (it is well-known that $L(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong M_5$, while for Q_8 the (unique) diamond is determined by the subgroups $\langle -\mathbf{1} \rangle$, $\langle \mathbf{i} \rangle$, $\langle \mathbf{j} \rangle$, $\langle \mathbf{k} \rangle$ and Q_8). Our main theorem proves that these two groups exhaust all finite almost M_5-free groups.

Theorem 1.2. *Let G be a finite almost M_5-free group. Then either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $G \cong Q_8$.*

In particular, we infer the following characterization of Q_8.

Corollary 1.3. *Q_8 is the unique finite non-abelian almost M_5-free group.*

Finally, we observe that there is no finite almost N_5-free group (indeed, if G would be such a group, then the subgroups that form the pentagon of $L(G)$ must be normal; in other words, the normal subgroup lattice of G would not be modular, a contradiction).

Most of our notation is standard and will usually not be repeated here. Basic notions and results on groups can be found in [1] and [5]. For subgroup lattice concepts we refer the reader to [2] and [6].

2 Proof of the main theorem

First of all, we prove our main theorem for p-groups.

Lemma 2.1. *Let G be a finite almost M_5-free p-group for some prime p. Then $p = 2$ and we have either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $G \cong Q_8$.*

Proof. Let M be a minimal normal subgroup of G.

If there is $N \in L(G)$ with $\mid N \mid = p$ and $N \neq M$, then $MN \in L(G)$ and $MN \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Obviously, $\mathbb{Z}_p \times \mathbb{Z}_p$ has more than one diamond for $p \geq 3$. So, we have $p = 2$ and we easily infer that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

If M is the unique minimal subgroup of G, then by (4.4) of [5], II, G is a generalized quaternion 2-group, that is there exists an integer $n \geq 3$ such that $G \cong Q_{2^n}$. If $n \geq 4$, then G contains a subgroup $H \cong Q_{2^{n-1}}$ and therefore $G/\Phi(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong H/\Phi(H)$. This shows that G has more than one diamond, a contradiction. Hence $n = 3$ and $G \cong Q_8$, as desired. ∎

We are now able to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We will proceed by induction on $\mid G \mid$. Let H be the top of the unique diamond of G. We distinguish the following two cases.

Case 1. $H = G$.

We infer that every proper subgroup of G is M_5-free and therefore cyclic. Assume that G is not a p-group. Then the Sylow subgroups of G are cyclic. If all these subgroups would be normal, then G would be the direct product of its cyclic Sylow subgroups and hence it would be cyclic, a contradiction. It follows that there is a prime q such that G has more than one Sylow q-subgroup. Let $S, T \in Syl_q(G)$ with $S \neq T$. Since S and T are cyclic, $S \wedge T$ is normal in $S \vee T$ and the quotient $S \vee T/S \wedge T$ is not cyclic (because it contains two different Sylow q-subgroups). Hence $S \vee T = G$ and $G/S \wedge T$ is almost M_5-free. If $S \wedge T \neq 1$, then the inductive hypothesis would imply that $G/S \wedge T$ would be a 2-group (isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or to Q_8), contradicting the fact that it has two different Sylow q-subgroups. Thus $S \wedge T = 1$. This shows that $Syl_q(G) \cup \{1, G\}$ is a sublattice of $L(G)$. Since G is almost M_5-free, one obtains $\mid Syl_q(G) \mid = 3$. By Sylow's theorem we infer that $q = 2$ and $\mid G : N_G(S) \mid = \mid Syl_q(G) \mid = 3$. In this way, we can choose a 3-element $x \in G \setminus N_G(S)$. It follows that $X = \langle x \rangle$ operates transitively on $Syl_q(G)$. Then for every $Q \in Syl_q(G)$, we have $Q \vee X \geq Q \vee Q^x = G$ and consequently $Q \vee X = G$. On the other hand, we obviously have $Q \wedge X = 1$ because Q and X are of coprime orders. So $\{1, S, T, X, G\}$ is a second sublattice of $L(G)$ isomorphic to M_5, contradicting our hypothesis. Hence G is a p-group and the conclusion follows from Lemma 2.1.

Case 2. $H \neq G$.

By the inductive hypothesis we have either $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $H \cong Q_8$. We also infer that H is the unique Sylow 2-subgroup of G. Let p be an odd prime dividing $\mid G \mid$ and K be a subgroup of order p of G. Then HK is an almost M_5-free subgroup of G, which is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or to Q_8. This shows that $HK = G$. Denote by n_p the number of Sylow p-subgroups of G. If $n_p = 1$, then either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$ or $G \cong Q_8 \times \mathbb{Z}_p$. It is clear that the subgroup lattices of these two direct products contain more than one diamond, contradicting our assumption. If $n_p \neq 1$, then $n_p \geq p+1 \geq 4$ and so we can choose two distinct Sylow p-subgroups K_1 and K_2. For $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ one obtains that $L_1 = \{1, H, K_1, K_2, G\}$ forms a diamond of $L(G)$, which is different from $L(H)$, a contradiction. For $H \cong Q_8$ the same thing can be said by applying a similar argument to the quotient G/H_0, where H_0 is the (unique) subgroup of order 2 of G. This completes the proof. ∎

We end our note by indicating three open problems concerning this topic.

Problem 2.2. Describe the (almost) L-free groups, where L is a lattice different from M_5 and N_5.

Problem 2.3 Determine explicitly the number of sublattices isomorphic to a given lattice that are contained in the subgroup lattices of some important classes of finite groups.

Problem 2.4. Extend the concepts of L-free group and almost L-free group to other remarkable posets of subgroups of a group (e.g. what can be said about a group whose normal subgroup lattice/poset of cyclic subgroups contains a certain number of sublattices isomorphic to a given lattice ?).

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