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Strong convergence theorems for a sequence of nonexpansive mappings with gauge functions

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Abstract

In this paper, we first prove a path convergence theorem for a nonexpansive mapping in a reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function on $[0, \infty)$. Using this result, strong convergence theorems for common fixed points of a countable family of nonexpansive mappings are established.

1 Introduction

Let K be a nonempty, closed and convex subset of a real Banach space E. Let $T : K \to K$ be a nonlinear mapping. We denote by $F(T)$ the fixed points set of T, that is, $F(T) = \{x \in K : x = Tx\}$. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K.$$

One classical way to study convergence of nonexpansive mappings is to use path convergence for approximating the fixed point of mappings [3, 18, 27]. For any $t \in (0, 1)$, we define the mapping $T_t : K \to K$ as follows:

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in K, \tag{1.1}$$

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where $u \in K$ is fixed. Banach's contraction principle ensures that T_t has a unique fixed point x_t in K satisfying

$$x_t = tu + (1 - t)Tx_t. \tag{1.2}$$

Browder [3] first proved that, if E is a real Hilbert space, then $\{x_t\}$ converges strongly to a fixed point of T. Reich [18] showed that Browder's results also valid in a uniformly smooth Banach space. In 2006, Xu [27] proved that Browder's result holds in a reflexive Banach space which has a weakly continuous duality mapping.

On the other hand, Gossez-Lami gave in [9] some geometric properties related to the fixed point theory for nonexpansive mappings. They proved that a space with a weakly continuous duality mapping satisfies Opial's condition [14]. It is also known that all Hilbert spaces and ℓ^p $(1 < p < \infty)$ satisfy the Opial's condition. However, the L^p $(1 < p < \infty)$ spaces do not unless $p = 2$. In this connection, we focus our aim to study a path convergence of (1.2) in a different setting, a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm concerning a gauge function [4]. We note that our class of Banach spaces includes the spaces L^p, ℓ^p $(1 < p < \infty)$ and the Sobolev spaces W^p_m $(1 < p < \infty)$. Moreover, the duality mappings associated with gauge functions also include the generalized and the normalized duality mappings as special cases.

In 1953, Mann [11] introduced the iterative scheme $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\} \subset (0, 1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.3) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [18]). Since 1953, many authors have constructed and proposed the modified version of algorithm (1.3) in order to get strong convergence results (see [5, 6, 10, 13, 16, 24, 26, 29, 30] and the references cited therein). Several applications related to the Mann iterative scheme can be found in [17].

Kim-Xu [10] introduced the following modified Mann's iteration as follows:

$$\begin{cases} x_0 = x \in K, \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

where T is a nonexpansive mapping of K into itself and $u \in K$ is fixed. They proved, in a uniformly smooth Banach space, that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a fixed point of T if the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Recently, Qin et al. [16] introduced the following iteration:

$$\begin{cases} x_0 = x \in K, \\ y_n = \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where W_n is the W-mapping [20] generated by nonexpansive self mappings $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$ and $u \in K$ is fixed. They proved, in a reflexive strictly convex Banach space which has a weakly continuous duality mapping j_φ, that the sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$ if the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Let K be a nonempty, closed and convex subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty} : K \to K$ be a sequence of nonexpansive mappings.

Motivated by the works mentioned above, we consider the following modified Mann-type iteration:

$$\begin{cases} u, x_1 \in K, \\ y_n = \beta_n x_n + (1-\beta_n)T_n x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

In this paper, we first prove a path convergence for a nonexpansive mapping in a real reflexive and strictly convex Banach space which has a Gâteaux differentiable norm and admits the duality mapping associated with a gauge function. Then we discuss strong convergence of the modified Mann-type iteration process (1.6) for a countable family of nonexpansive mappings. Our results improve and extend the recent ones announced by many authors.

2 Preliminaries

A Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1-\delta)$ holds. The *modulus of convexity* of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| :\ \|x\|, \|y\| \le 1,\ \|x-y\| \ge \epsilon\right\}, \quad \forall \epsilon \in [0,2].$$

It is known that a Banach space E is uniformly convex if $\delta_E(0) = 0$ and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \le 2$ and every uniformly convex Banach space is strictly convex and reflexive.

Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for any $x, y \in S(E)$. In this case, E is called *smooth.* The norm of E is said to be *uniformly Gâteaux differentiable* if, for any $y \in S(E)$, the limit is attained uniformly for all $x \in S(E)$.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the *modulus of smoothness* of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 :\ x \in S(E),\ \|y\| \le t\right\}.$$

A Banach space E is said to be *uniformly smooth* if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$ (see [1, 7, 23] for more details).

We recall the following definitions and results which can be found in [1, 4, 7].

Definition 2.1. A continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ is called the *gauge function* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.2. Let E be a normed space and φ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \left\{f^* \in E^* :\ \langle x, f^*\rangle = \|x\|\varphi(\|x\|),\ \|f^*\| = \varphi(\|x\|)\right\}, \quad \forall x \in E,$$

is called the *duality mapping* with gauge function φ.

In particular, if $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the *normalized duality mapping.* If $\varphi(t) = t^{q-1}$ for any $q > 1$, then the duality mapping $J_\varphi = J_q$ is called the *generalized duality mapping.*

It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ and $J_q(x) = \|x\|^{q-2} J(x)$ for any $q > 1$.

Remark 2.3. [1] For the gauge function φ, the function $\Phi : [0,\infty) \to [0,\infty)$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds \tag{2.1}$$

is a continuous convex and strictly increasing function on $[0,\infty)$. Therefore, Φ has a continuous inverse function Φ^{-1}.

Remark 2.4. [1, 7] For any x in a Banach space E, $J_\varphi(x) = \partial\Phi(\|x\|)$, where ∂ denotes the sub-differential.

We know the following subdifferential inequality:

$$\Phi\big(\|x+y\|\big) \le \Phi\big(\|x\|\big) + \big\langle y, j_\varphi(x+y)\big\rangle, \quad \forall j_\varphi(x+y) \in J_\varphi(x+y). \tag{2.2}$$

We also know the following facts (see [1]):
(1) J_φ is a nonempty, closed and convex set in E^* for any $x \in E$.
(2) J_φ is a function when E^* is strictly convex.
(3) If J_φ is single-valued, then

$$J_\varphi(\lambda x) = \frac{sign(\lambda)\varphi(\|\lambda x\|)}{\varphi(\|x\|)} J_\varphi(x), \quad \forall x \in E,\ \lambda \in \mathbb{R},$$

and

$$\langle x-y, J_\varphi(x) - J_\varphi(y)\rangle \ge \big(\varphi(\|x\|) - \varphi(\|y\|)\big)\big(\|x\| - \|y\|\big), \quad \forall x, y \in E.$$

If E is a smooth Banach space, then J_φ is single-valued and also denoted by j_φ.

Remark 2.5. [8] Suppose E has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Then j_φ is uniformly continuous from the norm topology of E to the weak* topology of E^* on each bounded subset of E.

We next give the definition of Banach limit.

Definition 2.6. Let μ be a continuous linear functional on ℓ^∞ and let $(a_0, a_1, \cdots) \in \ell^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. We call μ a Banach limit when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for each $(a_0, a_1, \cdots) \in \ell^\infty$.

For a Banach limit μ, we know that

$$\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$$

for all $a = (a_0, a_1, \cdots) \in \ell^\infty$. Therefore, if $a = (a_0, a_1, \cdots) \in \ell^\infty, b = (b_0, b_1, \cdots) \in \ell^\infty$ and $a_n - b_n \to 0$ as $n \to \infty$, then we have $\mu_n(a_n) = \mu_n(b_n)$ (see [1, 7, 23, 25]).

In the sequel, we need the following crucial lemmas:

Lemma 2.7. [21] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E such that*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \quad \forall n \geq 1,$$

where $\{\beta_n\}$ is a real sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. If $\limsup_{n\to\infty} \big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.8. [28] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(b) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

To deal with a family of mappings, we consider the following condition:

Let K be a subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of K such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy the *AKTT-condition* [2] if, for any bounded subset B of K,

$$\sum_{n=1}^{\infty} \sup\big\{\|T_{n+1}z - T_n z\| : z \in B\big\} < \infty.$$

Lemma 2.9. [2] *Let K be a nonempty and closed subset of a Banach space E and $\{T_n\}$ be a family of mappings of K into itself which satisfies the AKTT-condition. Then, for any $x \in K$, $\{T_n x\}$ converges strongly to a point in K. Moreover, let the mapping T be defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in K.$$

Then, for each bounded subset B of K,

$$\lim_{n\to\infty} \sup\big\{\|Tz - T_n z\| : z \in B\big\} = 0.$$

In the sequel, we write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.9 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Example 2.10. Let $T_1, T_2, \cdots$, be an infinite family of nonexpansive mappings of K into itself and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_i < 1$ for all $i \in \mathbb{N}$. Moreover, let W_n and W be the W-mappings [20] generated by $T_1, T_2, \cdots, T_n$ and $\gamma_1, \gamma_2, \cdots, \gamma_n$, and $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$. Then $\big(\{W_n\}, W\big)$ satisfies the AKTT-condition (see [15, 20]).

Example 2.11. Let $T_1, T_2, \cdots$ be an infinite family of nonexpansive mappings of K into itself. For each $n \in \mathbb{N}$, define the mapping $V_n : K \to K$ by

$$V_n x = \sum_{i=1}^{n} \lambda_n^i T_i x, \quad \forall x \in K,$$

where $\{\lambda_n^i\}$ is a family of nonnegative numbers satisfying the following conditions:

(a) $\sum_{i=1}^{n} \lambda_n^i = 1$ for each $n \in \mathbb{N}$;
(b) $\lambda^i := \lim_{n\to\infty} \lambda_n^i > 0$ for each $i \in \mathbb{N}$;
(c) $\sum_{n=1}^{\infty} \sum_{i=1}^{n} |\lambda_{n+1}^i - \lambda_n^i| < \infty$.

Let $V : K \to K$ be the mapping defined by

$$V x = \sum_{i=1}^{\infty} \lambda^i T_i x, \quad \forall x \in K.$$

Then $\big(\{V_n\}, V\big)$ satisfies the AKTT-condition (see [2]).

3 Path convergence theorem

Now, we denote the subset K' of K by

$$K' = \Big\{ x \in K : \ \mu_n \Phi\big(\|x_n - x\|\big) = \inf_{y \in K} \mu_n \Phi\big(\|x_n - y\|\big) \Big\},$$

where Φ is the function defined by (2.1).

Proposition 3.1. [8] *Let K be a nonempty, closed and convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Suppose that $\{x_n\}$ is a bounded sequence of K. Let μ_n be a Banach limit and $z \in K$. Then $z \in K'$ if and only if*

$$\mu_n \big\langle y - z, j_\varphi(x_n - z) \big\rangle \le 0, \quad \forall y \in K.$$

Proposition 3.2. *Let K be a nonempty, closed and convex subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is a bounded sequence in K with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $F(T) \cap K' \neq \emptyset$.*

Proof. Set $g(y) = \mu_n \Phi\big(\|x_n - y\|\big)$ for all $y \in K$. Then g is convex and continuous since Φ is convex and continuous. Further, $g(y_m) \to \infty$ as $\|y_m\| \to \infty$ since $\varphi(\|y_m\|) \to \infty$ as $\|y_m\| \to \infty$. Since E is reflexive, by Theorem 1.3.11 in [23], there exists $z \in K$ such that $g(z) = \inf_{y\in K} g(y)$. Hence K' is nonempty. Further, K' is closed and convex since g is continuous and convex. For any $x \in K'$, we have

$$\begin{aligned} g(Tx) &= \mu_n \Phi\big(\|x_n - Tx\|\big) \\ &\leq \mu_n \Phi\big(\|x_n - Tx_n\| + \|Tx_n - Tx\|\big) \\ &\leq \mu_n \Phi\big(\|x_n - x\|\big) \\ &= g(x). \end{aligned}$$

Therefore, $Tx \in K'$ for all $x \in K'$.

Let $p \in F(T)$. By Day-James's theorem [12], we know that there exists a unique element $v \in K'$ such that

$$\|p - v\| = \inf_{x\in K'} \|p - x\|.$$

Since $p = Tp$ and $Tv \in K'$, we have

$$\|p - Tv\| = \|Tp - Tv\| \leq \|p - v\| \leq \|p - Tv\|.$$

It follows that $v = Tv$ since E is strictly convex. Hence $v \in F(T) \cap K'$. This completes the proof. □

Using Propositions 3.1 and 3.2, we next prove a path convergence theorem, which is important to prove our main theorem.

Theorem 3.3. *Let K be a nonempty, closed and convex subset of a real reflexive and strictly Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $T : K \to K$ be a nonexpansive such that $F(T) \neq \emptyset$. Fix $u \in K$ and let $t \in (0, 1)$. Then the net $\{x_t\}$ defined by* (1.2) *converges strongly as $t \to 0$ to a fixed point p of T which solves the variational inequality:*

$$\big\langle u - p, j_\varphi(w - p)\big\rangle \leq 0, \quad \forall w \in F(T). \tag{3.1}$$

Proof. First, we prove that the solution of variational inequality (3.1) is unique. Suppose that $p, q \in F(T)$ satisfy (3.1). Then we have

$$\big\langle u-p, j_\varphi(q-p)\big\rangle \le 0, \quad \big\langle u-q, j_\varphi(p-q)\big\rangle \le 0.$$

Adding the above inequalities, we obtain

$$\big\langle p-q, j_\varphi(p-q)\big\rangle \le 0,$$

which implies that

$$\|p-q\|\varphi(\|p-q\|) \le 0$$

and so $p=q$.

Next, we prove that $\{x_t\}$ is bounded in K. For any $w \in F(T)$, we see that

$$\begin{aligned} & \|x_t-w\|\varphi(\|x_t-w\|) \\ = {} & \big\langle x_t-w, j_\varphi(x_t-w)\big\rangle \\ = {} & t\big\langle u-w, j_\varphi(x_t-w)\big\rangle + (1-t)\big\langle Tx_t-w, j_\varphi(x_t-w)\big\rangle \\ \le {} & t\big\langle u-w, j_\varphi(x_t-w)\big\rangle + (1-t)\|x_t-w\|\varphi\big(\|x_t-w\|\big), \end{aligned}$$

which implies

$$\begin{aligned} \|x_t-w\|\varphi(\|x_t-w\|) &\le \big\langle u-w, j_\varphi(x_t-w)\big\rangle \\ &\le \|u-w\|\varphi\big(\|x_t-w\|\big). \end{aligned} \tag{3.2}$$

Hence $\|x_t-w\| \le \|u-w\|$ and, consequently, $\{x_t\}$ is bounded. So is $\{Tx_t\}$. We see that

$$\|x_t-Tx_t\| = t\|u-Tx_t\| \to 0 \quad (t \to 0).$$

Since E is reflexive, $\{x_t\}$ has a weakly convergent subsequence $\{x_{t_n}\}$. Thus $\{x_{t_n}\}$ is bounded. Putting $x_n := x_{t_n}$, in particular, we also have

$$\|x_n-Tx_n\| \to 0 \quad (n \to \infty).$$

By Proposition 3.2, since $\{x_n\}$ is bounded, there exists $p \in F(T)$ such that

$$\mu_n\Phi\big(\|x_n-p\|\big) = \inf_{y\in K}\mu_n\Phi\big(\|x_n-y\|\big).$$

It follows from Proposition 3.1 that

$$\mu_n\big\langle y-p, j_\varphi(x_n-p)\big\rangle \le 0, \quad \forall y \in K.$$

Since $u \in K$, in particular, we have

$$\mu_n\big\langle u-p, j_\varphi(x_n-p)\big\rangle \le 0. \tag{3.3}$$

Observe that

$$\Phi(\|y\|) = \int_0^{\|y\|} \varphi(s)ds \leq \|y\|\varphi(\|y\|).$$

It follows from (3.2) and (3.3) that

$$\mu_n \Phi\big(\|x_n - p\|\big) \leq \mu_n \big\langle u - p, j_\varphi(x_n - p)\big\rangle \leq 0$$

and hence

$$\mu_n \Phi\big(\|x_n - p\|\big) = 0. \tag{3.4}$$

Since Φ is continuous, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. From (3.4), we have

$$\mu_j \Phi\big(\|x_{n_j} - p\|\big) = \Phi\big(\|q - p\|\big) = 0$$

and so $p = q$. Therefore, the sequence $\{x_n\}$ converges strongly to a fixed point p of T.

Next, we prove that $p \in F(T)$ is a solution to the variational inequality (3.1). For any $w \in F(T)$, we see that

$$\begin{aligned}
\|x_n - w\|\varphi\big(\|x_n - w\|\big) &= \big\langle x_n - w, j_\varphi(x_n - w)\big\rangle \\
&= t_n\big\langle u - p, j_\varphi(x_n - w)\big\rangle + t_n\big\langle p - x_n, j_\varphi(x_n - w)\big\rangle \\
&\quad + t_n\big\langle x_n - w, j_\varphi(x_n - w)\big\rangle \\
&\quad + (1 - t_n)\big\langle Tx_n - w, j_\varphi(x_n - w)\big\rangle \\
&\leq t_n\big\langle u - p, j_\varphi(x_n - w)\big\rangle + t_n\|x_n - p\|\varphi\big(\|x_n - w\|\big) \\
&\quad + t_n\|x_n - w\|\varphi(\|x_n - w\|) \\
&\quad + (1 - t_n)\|x_n - w\|\varphi\big(\|x_n - w\|\big) \\
&= t_n\big\langle u - p, j_\varphi(x_n - w)\big\rangle + t_n\|x_n - p\|\varphi\big(\|x_n - w\|\big) \\
&\quad + \|x_n - w\|\varphi\big(\|x_n - w\|\big).
\end{aligned}$$

This implies that

$$\big\langle u - p, j_\varphi(w - x_n)\big\rangle \leq \|x_n - p\|\varphi\big(\|x_n - w\|\big). \tag{3.5}$$

Since j_φ is norm-weak* uniformly continuous on bounded subsets of E, we have

$$\big\langle u - p, j_\varphi(w - x_n)\big\rangle \to \big\langle u - p, j_\varphi(w - p)\big\rangle \quad (n \to \infty).$$

Thus, taking the limit as $n \to \infty$ in both sides of (3.5), we get

$$\big\langle u - p, j_\varphi(w - p)\big\rangle \leq 0, \quad \forall w \in F(T).$$

Finally, we prove that $x_t \to p$ as $t \to 0$. To this end, let $\{x_{s_n}\}$ be another subsequence of $\{x_t\}$ such that $x_{s_n} \to p'$ as $n \to \infty$. We have to show that $p = p'$. For any $w \in F(T)$, we have

$$\begin{aligned}\big\langle Tx_t - x_t, j_\varphi(x_t - w)\big\rangle &= \big\langle Tx_t - w, j_\varphi(x_t - w)\big\rangle + \big\langle w - x_t, j_\varphi(x_t - w)\big\rangle \\ &\le \|x_t - w\|\varphi\big(\|x_t - w\|\big) + \big\langle w - x_t, j_\varphi(x_t - w)\big\rangle \\ &= \big\langle x_t - w, j_\varphi(x_t - w)\big\rangle + \big\langle w - x_t, j_\varphi(x_t - w)\big\rangle \\ &= 0.\end{aligned}$$

On the other hand, since

$$x_t - Tx_t = \frac{t}{1-t}(u - x_t),$$

we have

$$\big\langle x_t - u, j_\varphi(x_t - w)\big\rangle \le 0, \quad \forall w \in F(T).$$

In particular, we have

$$\big\langle x_{t_n} - u, j_\varphi(x_{t_n} - p')\big\rangle \le 0$$

and

$$\big\langle x_{s_n} - u, j_\varphi(x_{s_n} - p)\big\rangle \le 0$$

or, equivalently,

$$\|x_{t_n} - p'\|\varphi\big(\|x_{t_n} - p'\|\big) + \big\langle p' - u, j_\varphi(x_{t_n} - p')\big\rangle \le 0$$

and

$$\|x_{s_n} - p\|\varphi\big(\|x_{s_n} - p\|\big) + \big\langle p - u, j_\varphi(x_{s_n} - p)\big\rangle \le 0.$$

Taking the limit as $n \to \infty$, since φ is continuous and j_φ is norm-to-weak* uniformly continuous on bounded subsets of E, we obtain

$$\|p - p'\|\varphi\big(\|p - p'\|\big) + \big\langle p' - u, j_\varphi(p - p')\big\rangle \le 0$$

and

$$\|p' - p\|\varphi\big(\|p' - p\|\big) + \big\langle p - u, j_\varphi(p' - p)\big\rangle \le 0.$$

Summing the above inequalities, we also have

$$2\|p - p'\|\varphi\big(\|p - p'\|\big) + \big\langle p' - p, j_\varphi(p - p')\big\rangle \le 0.$$

This implies that

$$\big\langle p - p', j_\varphi(p - p')\big\rangle \le 0$$

and hence $p = p'$. Therefore, $\{x_t\}$ converges strongly to a fixed point of T. This completes the proof. □

4 Strong convergence theorems

In this section, using Theorem 3.3, we prove a strong convergence theorem in a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function on $[0,\infty)$.

Theorem 4.1. *Let K be a nonempty closed and convex subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\{T_n\}_{n=1}^\infty : K \to K$ be a sequence of nonexpansive mappings such that $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Let $u \in K$ be fixed. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $(0,1)$ such that*

(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^\infty \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

If $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition, then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.6) converge strongly to $p \in F$ which also solves the variational inequality (3.1).

Proof. First, we see that the sequences $\{x_n\}$ and $\{y_n\}$ is bounded. In fact, for any $w \in F$, we have

$$\|y_n - w\| \leq \beta_n \|x_n - w\| + (1-\beta_n)\|T_n x_n - w\| \leq \|x_n - w\|$$

and so

$$\begin{aligned} \|x_{n+1} - w\| &\leq \alpha_n \|u - w\| + (1-\alpha_n)\|y_n - w\| \\ &\leq \alpha_n \|u - w\| + (1-\alpha_n)\|x_n - w\| \\ &\leq \max\Big\{\|x_n - w\|, \|u - w\|\Big\}. \end{aligned}$$

Hence the sequence $\{x_n\}$ is bounded by induction and so is $\{y_n\}$.

Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Putting $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, we get

$$x_{n+1} = (1-\beta_n) l_n + \beta_n x_n, \quad \forall n \geq 1.$$

Thus we have

$$\begin{aligned} & l_{n+1} - l_n \\ = \; & \frac{\alpha_{n+1}u + (1-\alpha_{n+1})y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\alpha_n)y_n - \beta_n x_n}{1-\beta_n} \\ = \; & \frac{\alpha_{n+1}(u - y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(u-y_n)}{1-\beta_n} + T_{n+1}x_{n+1} - T_n x_n, \end{aligned}$$

which implies

$$\begin{aligned} & \|l_{n+1} - l_n\| \\ \leq \; & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u - y_n\| + \|x_{n+1} - x_n\| + \|T_{n+1}x_n - T_n x_n\| \\ \leq \; & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u - y_n\| + \|x_{n+1} - x_n\| + \sup_{z\in\{x_n\}}\|T_{n+1}z - T_n z\|. \end{aligned}$$

Since $\{T_n\}$ satisfies the AKTT-condition, it follows from the conditions (a) and (c) that

$$\limsup_{n\to\infty}\big(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

By Lemma 2.7, we also obtain

$$\lim_{n\to\infty}\|l_n - x_n\| = 0.$$

Since

$$x_{n+1} - x_n = (1-\beta_n)(l_n - x_n),$$

we have

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|l_n - x_n\| \to 0 \quad (n\to\infty). \tag{4.1}$$

On the other hand, we see that

$$\|x_{n+1} - y_n\| = \alpha_n\|u - y_n\| \to 0 \quad (n\to\infty). \tag{4.2}$$

Combining (4.1) and (4.2) we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{4.3}$$

Noting that

$$\begin{aligned} \|x_n - T_n x_n\| & \leq \|x_n - y_n\| + \|y_n - T_n x_n\| \\ & = \|x_n - y_n\| + \beta_n\|x_n - T_n x_n\|, \end{aligned}$$

from (4.3) and the condition (c), we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{4.4}$$

Further, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\leq \|x_n - T_n x_n\| + \sup_{z\in\{x_n\}} \|T_n z - Tz\|. \end{aligned}$$

Thus, by Lemma 2.9 and (4.4), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.5}$$

Since T is nonexpansive, by Theorem 3.3, we know that the net $\{x_t\}$ generated by (1.2) converges strongly to a fixed point $p \in F(T) = F$ which also solves the variational inequality (3.1).

Next, we prove that

$$\limsup_{n\to\infty} \big\langle u - p, j_\varphi(x_n - p)\big\rangle \leq 0.$$

Observe that

$$\begin{aligned} & \|x_t - x_n\|\varphi\big(\|x_t - x_n\|\big) \\ = \ & t\big\langle u - x_n, j_\varphi(x_t - x_n)\big\rangle + (1-t)\big\langle Tx_t - x_n, j_\varphi(x_t - x_n)\big\rangle \\ = \ & t\big\langle p - x_t, j_\varphi(x_t - x_n)\big\rangle + t\big\langle u - p, j_\varphi(x_t - x_n)\big\rangle \\ & + t\big\langle x_t - x_n, j_\varphi(x_t - x_n)\big\rangle + (1-t)\big\langle Tx_t - Tx_n, j_\varphi(x_t - x_n)\big\rangle \\ & + (1-t)\big\langle Tx_n - x_n, j_\varphi(x_t - x_n)\big\rangle \\ \leq \ & t\|p - x_t\|\varphi(\|x_t - x_n\|) + t\big\langle u - p, j_\varphi(x_t - x_n)\big\rangle \\ & + \|x_t - x_n\|\varphi(\|x_t - x_n\|) + \|Tx_n - x_n\|\varphi(\|x_t - x_n\|). \end{aligned}$$

Therefore, it follows that

$$\big\langle u - p, j_\varphi(x_n - x_t)\big\rangle \leq \frac{\|Tx_n - x_n\|\varphi(\|x_t - x_n\|)}{t} + \|x_t - p\|\varphi(\|x_t - x_n\|). \tag{4.6}$$

Using (4.5) and taking the limit as $n \to \infty$ first and then, as $t \to 0$, the inequality (4.6) becomes

$$\limsup_{t\to 0}\limsup_{n\to\infty} \big\langle u - p, j_\varphi(x_n - x_t)\big\rangle \leq 0. \tag{4.7}$$

Since j_φ is norm-weak* uniformly continuous on bounded sets,

$$\big\langle u-p, j_\varphi(x_n-x_t)\big\rangle \to \big\langle u-p, j_\varphi(x_n-p)\big\rangle \quad (t\to 0).$$

We see that

$$\big\langle u-p, j_\varphi(x_n-p)\big\rangle = \big\langle u-p, j_\varphi(x_n-x_t)\big\rangle + \big\langle u-p, j_\varphi(x_n-p)-j_\varphi(x_n-x_t)\big\rangle.$$

By the uniform continuity of j_φ, we can interchange the two limits above and deduce that

$$\limsup_{n\to\infty} \big\langle u-p, j_\varphi(x_n-p)\big\rangle \le 0. \tag{4.8}$$

Finally, we prove that $x_n \to p$ as $n\to\infty$. Observe that

$$\begin{aligned}\Phi\big(\|y_n-p\|\big) &= \Phi\big(\|\beta_n(x_n-p)+(1-\beta_n)(T_nx_n-p)\|\big)\\ &\le \beta_n\Phi\big(\|x_n-p\|\big)+(1-\beta_n)\Phi\big(\|T_nx_n-p\|\big)\\ &\le \Phi\big(\|x_n-p\|\big).\end{aligned}$$

Form (2.2), it follows that

$$\begin{aligned}\Phi\big(\|x_{n+1}-p\|\big) &= \Phi\big(\|\alpha_n(u-p)+(1-\alpha_n)(y_n-p)\|\big)\\ &\le \Phi\big((1-\alpha_n)\|y_n-p\|\big)+\alpha_n\big\langle u-p, j_\varphi(x_{n+1}-p)\big\rangle\\ &\le (1-\alpha_n)\Phi\big(\|x_n-p\|\big)+\alpha_n\big\langle u-p, j_\varphi(x_{n+1}-p)\big\rangle.\end{aligned}$$

Applying Lemma 2.8, we have $\Phi\big(\|x_n-p\|\big)\to 0$ as $n\to\infty$ by the condition (b) and (4.8). Hence $x_n\to p$ as $n\to\infty$ since Φ is continuous. Moreover, the sequence $\{y_n\}$ also strongly converges to p. This completes the proof. □

Remark 4.2. From Examples 2.10 and 2.11, the ordered pair $\big(\{T_n\},T\big)$ in Theorem 4.1 can be replaced by $\big(\{W_n\},W\big)$ and $\big(\{V_n\},V\big)$.

Remark 4.3. Theorem 4.1 mainly improves and extends the results of Kim-Xu [10] in the following aspects:

(1) we relax the restrictions imposed on the parameters in Theorem 1 of [10];

(2) we extend Theorem 1 of [10] from a single nonexpansive mapping to an infinite family of nonexpansive mappings;

(3) we extend Theorem 1 of [10] from a uniformly smooth Banach space to a much more general setting.

Remark 4.4. If $f: K\to K$ is a contraction and we replace u by $f(x_n)$ in the recursion formula (1.6), we can obtain the so-called viscosity iteration method (see [22]).

Remark 4.5. Theorem 3.3 and Theorem 4.1 can be applied to the spaces L^p, ℓ^p $(1 < p < \infty)$, the Sobolev spaces W_m^p $(1 < p < \infty)$ and Hilbert spaces. Moreover, our results hold for a Banach space which has the generalized duality mapping j_q $(q > 1)$ and the normalized the duality mapping j.

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