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Dirac Operator on a 7-Manifold with Deformed G_2 Structure

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Abstract

In this work, we consider the deforming of a G_2 structure by a vector field on a $7-$manifold. To obtain the metric corresponding to deformed G_2 structure, a new map is defined. By using this new map, the covariant derivatives on associated spinor bundles are compared. Then, the relation between Dirac operators on spinor bundles are investigated under some restrictions.

1 Introduction

There are several deformations of a fixed G_2 structure such as conformal deformations, deformations of a G_2 structure by a vector field and infinitesimal deformations [2, 6]. Conformal deformations were given by Fernández and Gray. They investigated how G_2 structures change after conformally changing the metric [2]. In [4] and [7] relations between Dirac operators on associated spinor bundles were studied.

Other two types of deformations were studied by Karigiannis in [6]. He specially worked on deforming the fundamental 3-form by a vector field and obtained a new metric from the 3-form: Let (M, φ_0, g_0) be a 7-dimensional Riemannian manifold with structure group G_2. If φ_0 is deformed by a vector field w, then the new 3-form

$$\widetilde{\varphi} = \varphi_0 + w \lrcorner * \varphi_0$$

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is always positive-definite. Under this deformation, Karigiannis showed that, for all vector fields u, v the new metric is

$$\widetilde{g}(u,v) = \frac{1}{(1+g_0(w,w))^{\frac{2}{3}}}\left(g_0(u,v) + g_0(u \times w, v \times w)\right),$$

where $\times$ is the cross product associated to the first G_2 structure. He also wrote the new Hodge star $\widetilde{*}$ in terms of the old φ_0, the old $*_0$ and the vector field w corresponding to $\widetilde{\varphi}$ explicitly:

$$\widetilde{*}\alpha = (1+g_0(w,w))^{\frac{2-k}{3}}\left(*_0\alpha + (-1)^{k-1} w \lrcorner (*_0(w \lrcorner \alpha))\right)$$

where α is a k-form [6].

In this paper we consider this type of deformations. For a fixed vector field w, first we define a map

$$\begin{array}{rcl} C_w : \Gamma(TM) & \longrightarrow & \Gamma(TM) \\ u & \longmapsto & C_w(u) = (1+g_0(w,w))^{-1/3}(u + u \times w) \end{array}$$

by use of which we construct the covariant derivative on the spinor bundle under some restrictions. Then we express the Dirac operator $\widetilde{D}$ on the spinor bundle with deformed G_2 structure in terms of the old one as follows:

$$\widetilde{D}\widetilde{\sigma} = b^{-2/3}\Psi_w\left\{D\sigma + \kappa(w)\left(\nabla^S_w\sigma\right) - \left\{\sum_{i=0}^{6}\kappa(e_i)\left(\nabla^S_{(e_i \times w)}\sigma\right)\right\}\right\}.$$

Finally, the relationship between eigenvalues of Dirac operators is investigated.

2 Preliminaries

Let us consider $\mathbb{R}^7$ with the standard basis $\{e_0, ..., e_6\}$ and dual basis $\{e_0^*, ..., e_6^*\}$. Consider the 3-form

$$\varphi = e^{012} + e^{034} + e^{056} + e^{135} - e^{146} - e^{236} - e^{145},$$

where $e^{ijk} = e_i^* \wedge e_j^* \wedge e_k^*$. The group G_2 may be defined as the automorphism group of Octonions. We also have the following characterization:

$$G_2 = \{g \in GL(7,\mathbb{R}) | g^*\varphi = \varphi\}.$$

The group G_2 is a compact, connected, simply connected and simple Lie subgroup of $SO(7)$ of dimension 14. A G_2 structure on a 7-dimensional manifold M is a reduction of the structure group of the frame bundle of M from $SO(7)$

to G_2. Let M be a 7-manifold with a G_2 structure. The classification of such manifolds are done by Fernández and Gray in [2] by decomposing $\nabla\varphi$ into G_2-irreducible components. It turned out that there are 16 such classes. The action of G_2 on the tangent bundle induces an action of G_2 on $\wedge^l(M)$, the space of l-forms on M. This action gives the following orthogonal decompositions of $\wedge^l(M)$:

$$\wedge^2(M) = \wedge^2_7 \oplus \wedge^2_{14},$$

$$\wedge^3(M) = \wedge^3_1 \oplus \wedge^3_7 \oplus \wedge^3_{27},$$

where

$$\wedge^2_7 = \{\beta \in \wedge^2(M) \mid *(\varphi \wedge \beta) = -2\beta\},$$

$$\wedge^2_{14} = \{\beta \in \wedge^2(M) \mid *\varphi \wedge \beta = 0\},$$

$$\wedge^1_3 = \{t\varphi \mid t \in \mathbb{R}\},$$

$$\wedge^3_7 = \{*(\beta \wedge \varphi) \mid \beta \in \wedge^1(M)\} = \{w \lrcorner * \varphi \mid w \in \Gamma(T(M))\},$$

$$\wedge^3_{27} = \{\gamma \in \gamma^3(M) \mid \gamma \wedge \varphi = 0, \gamma \wedge *\varphi = 0\}$$

and $\wedge^l_k$ denotes a k-dimensional G_2-irreducible subspace of $\wedge^l(M)$ and $\Gamma(T(M))$ is the set of smooth vector fields on M.

It is known that $Spin(7)$ is the double cover of $SO(7)$ and a 7-dimensional manifold is called a spin one if the structure group $SO(7)$ of M can be lifted to Spin(7). In addition a 7-dimensional manifold has a G_2 structure if and only if it is a spin manifold [7].

Let M be a *spin* manifold of dimension n. By a spinor bundle for M we mean a vector bundle S associated to a representation of $Spin(n)$ by Clifford multiplication,

$$S = P_{Spin(n)} \times_\kappa \Delta_n,$$

where $\Delta_n \cong \mathbb{C}^{2^n}$ and $\kappa : Spin(n) \to End(\Delta_n)$ is the restriction of the representation of the Clifford algebra Cl_n to $Spin(n)$.

Let $\Gamma(S)$ be the set of sections of the spinor bundle S. It is known that the Levi-Civita covariant derivative ∇ on M determines a covariant derivative

$$\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S)$$

on the spinor bundle S. The covariant derivative ∇^S is given locally by the formula

$$\nabla^S_v \sigma = d\sigma(v) + \frac{1}{4}\sum_{i,j} g_0(\nabla_v e_i, e_j)\kappa(e_i)\kappa(e_j)\sigma,$$

where $v \in \Gamma(TM)$, $\varepsilon = \{e_0, ..., e_{n-1}\}$ is a local section of $P_{SO(n)}M$ and $\sigma \in \Gamma(S)$. Then it can be defined a first order differential operator $D : \Gamma(S) \to \Gamma(S)$ called the Dirac operator of S by setting

$$D\sigma = \sum_{j=0}^{n-1} \kappa(e_j)\nabla_{e_j}\sigma$$

[3], [7] .

3 Covariant Derivation of the Deformed Spinor Bundle

Let (M, g_0) be a 7-dimensional Riemannian manifold with G_2 structure φ_0. For an arbitrary vector field w, we define a map C_w as follows:

$$C_w(u) := \frac{1}{(1 + g_0(w,w))^{\frac{1}{3}}}(u + u \times w),$$

where $\times$ is the cross product associated to φ_0 and u is a vector field. Note that the map C_w is one-to-one and C^∞-linear. The inverse of this map is

$$C_w^{-1}(u) = b^{-\frac{2}{3}}\{u - u \times w + g_0(w,u)w\},$$

where $b = 1 + g_0(w,w)$. From the equation $g_0(x \times y, z) = -g_0(z \times y, x)$ for all $x, y, z \in \Gamma(T(M))$, the new metric $\widetilde{g}$ we mentioned in our introduction can also be written in the following form

$$\widetilde{g}(x,y) = g_0(C_w(x), C_w(y)).$$

Let $\widetilde{M}$ denote this Riemannian manifold with the new metric $\widetilde{g}$. It is known that if M has a G_2 structure, then M is a spin manifold [7]. Then for a deformation $\widetilde{g}(x,y) := g_0(C_w(x), C_w(y))$, the spin structure on (M, g_0) induces a spin structure on $(M, \widetilde{g})$. To each orthonormal frame field $\varepsilon = \{e_0, e_1, \cdots, e_6\}$ on M, we can associate an orthonormal frame $\psi_w(\varepsilon) = \{\widetilde{e}_0, \cdots, \widetilde{e}_6\}$ on $\widetilde{M}$, where $\widetilde{e}_j = C_w^{-1}(e_j)$ for each j. This gives us an $SO(7)$-equivariant map $\psi_w : P_{SO}(M) \to P_{SO}(\widetilde{M})$. The map ψ_w lifts to a $Spin(7)-$equivariant map

$$\psi_w : P_{Spin(7)}(M) \to P_{Spin(7)}(\widetilde{M})$$

between principal $Spin(7)$ bundles. Let $\kappa : Spin(7) \to Aut(\Delta_7)$ be the spinor representation. An isomorphism between associated spinor bundles may explicitly be given by

$$\Psi_w : S = P_{Spin(7)}(M) \times_\kappa \Delta_7 \to \widetilde{S} = P_{Spin(7)}(\widetilde{M}) \times_\kappa \Delta_7,$$

$$\Psi_w([s,\rho]) = [\psi_w(s),\rho],$$

and the relation between spinor representations is

$$\widetilde{\kappa}\left(\widetilde{e}_i\right)\left(\Psi_w(\sigma)\right) = \Psi_w\left(\kappa(e_i)\sigma\right),$$

where $\sigma = [s,\rho]$ is a spinor field.

Let ∇ denote the Levi-Civita covariant derivative of g_0 and $\widetilde{\nabla}$ that of $\widetilde{g}$. If we take $\nabla C_w = 0$, applying the Kozsul formula, we get

$$\begin{aligned} 2g_0\left(C_w\left(\widetilde{\nabla}_x y\right), C_w(z)\right) &= g_0\left(\left(\nabla_y C_w\right)(z) - \left(\nabla_z C_w\right)(y), C_w(x)\right) \\ &+ g_0\left(\left(\nabla_x C_w\right)(z) - \left(\nabla_z C_w\right)(x), C_w(y)\right) \\ &+ g_0\left(\left(\nabla_y C_w\right)(x), C_w(z)\right) \\ &+ g_0\left(C_w\left(\nabla_x y\right), C_w(z)\right) + g_0\left(\nabla_x\left(C_w(y)\right), C_w(z)\right). \end{aligned}$$

Since $\nabla C_w = 0$, we have

$$\widetilde{\nabla} = \nabla.$$

Note that if the G_2 structure φ_0 is parallel, then for all vector fields x, y, z we have

$$\nabla_x(y\times z) = (\nabla_x y)\times z + y\times(\nabla_x z),$$

implying

$$0 = \nabla_x(C_w)(y) = x[b^{-1/3}](y + y\times w) + b^{-1/3} y\times(\nabla_x w).$$

Now if we take the inner product with y in the above relation, we get $x[b^{-1/3}] = 0$. Thus we obtain $\nabla_x w = 0$. On the other hand, if $\nabla w = 0$, it can easily be seen that $\nabla C_w = 0$. Thus when φ_0 is parallel, the condition $\nabla C_w = 0$ is equivalent to the condition $\nabla w = 0$. The existence of such non-trivial parallel vector fields on 7-dimensional manifolds with holonomy a subgroup of G_2 is due to R.Bryant and S.Salamon [1]. Now we find out how the G_2 structure changes. We have $\widetilde{\nabla}_u\widetilde{\varphi}(x,y,z) = \nabla_u\widetilde{\varphi}(x,y,z) = (*\varphi_0)(\nabla_u w, x, y, z)$ for all $u, x, y, z \in \Gamma(TM)$, when $\nabla\varphi_0 = 0$. Since $\nabla w = 0$, we get $\widetilde{\nabla}\widetilde{\varphi} = 0$ and thus the G_2 structure remains parallel.

Now assume $\nabla C_w = 0$. We want to find the relation between covariant derivations on spinor bundles S and $\widetilde{S}$ under this restriction. Thus we calculate $\widetilde{w}_{ij}(v) = \widetilde{g}(\widetilde{\nabla}_v\widetilde{e}_i, \widetilde{e}_j)$. First note that $\nabla C_w = 0$ gives

$$\nabla_x(y\times w) - (\nabla_x y)\times w = \frac{2}{3} b^{-2/3} g_0(\nabla_x w, w) C_w(y)$$

and

$$(\nabla_x w) \times w = -\frac{2}{3} b^{-1} g_0(\nabla_x w, w) w.$$

Then we compute equations below:

$$\nabla_v \widetilde{e}_i = \nabla_v (C_w^{-1}(e_i)) = b^{-2/3}\{\nabla_v e_i - \nabla_v (e_i \times w) + \nabla_v \left(g_0(w, e_i) w\right)\} + b^{2/3} v[b^{-2/3}] C_w^{-1}(e_i),$$

and

$$\begin{aligned} C_w\left(\nabla_v (C_w^{-1}(e_i))\right) &= b^{-2/3} C_w\left(\nabla_v e_i\right) - b^{-2/3} C_w\left(\nabla_v (e_i \times w)\right) \\ &\quad + b^{-2/3} g_0(w, e_i) C_w\left(\nabla_v w\right) + b^{-1} v[g_0(w, e_i)] w + b^{2/3} v[b^{-2/3}] e_i \\ &= b^{-1} \nabla_v e_i - \tfrac{2}{3} b^{-5/3} g_0(\nabla_v w, w) C_w(e_i) - b^{-1} (\nabla_v (e_i \times w)) \times w \\ &\quad + b^{-1} g_0(w, e_i) \nabla_v w - \tfrac{2}{3} b^{-2} g_0(\nabla_v w, w) g_0(w, e_i) w \\ &\quad + b^{-1} g_0(\nabla_v w, e_i) w + b^{-1} g_0(\nabla_v e_i, w) w + b^{2/3} v[b^{-2/3}] e_i. \end{aligned}$$

Hence,

$$\begin{aligned} \widetilde{w}_{ij}(v) &= \widetilde{g}(\widetilde{\nabla}_v \widetilde{e}_i, \widetilde{e}_j) \\ &= g_0\left(C_w\left(\nabla_v (C_w^{-1}(e_i))\right), e_j\right) \\ &= g_0\left(\nabla_v e_i, e_j\right) - \tfrac{2}{3} b^{-2} g_0(\nabla_v w, w) g_0\left(e_i, e_j\right) \\ &\quad + \tfrac{2}{3} b^{-2} g_0(\nabla_v w, w) g_0(w, w) g_0\left(e_i, e_j\right) + b^{2/3} v[b^{-2/3}] g_0(e_i, e_j) \\ &\quad - \tfrac{4}{3} b^{-2} g_0(\nabla_v w, w) g_0(e_i \times w, e_j) - \tfrac{4}{3} b^{-2} g_0\left(\nabla_v w, w\right) g_0(w, e_i) g_0(w, e_j) \\ &\quad + b^{-1} g_0(w, e_i) g_0\left(\nabla_v w, e_j\right) + b^{-1} g_0(\nabla_v w, e_i) g_0\left(w, e_j\right). \end{aligned}$$

The local tangent frame field $\varepsilon = \{e_0, \cdots, e_6\}$ on the open set U_α, determines a local frame field $S = \{\sigma_0, \cdots, \sigma_6\}$ for spinor bundle S. Similarly, $\widetilde{\varepsilon} = \{\widetilde{e}_0, \cdots, \widetilde{e}_6\}$ determines a local frame field $\widetilde{S} = \{\widetilde{\sigma}_0, \cdots, \widetilde{\sigma}_6\}$ for the spinor bundle $\widetilde{S}$, where $\widetilde{\sigma}_i = \Psi_w(\sigma_i)$ for each $0 \le i \le 6$. From the local expression for the covariant derivation on the spinor bundle, we get

$$\begin{aligned}
\nabla_v^{\widetilde{S}}\widetilde{\sigma}_\alpha &= \tfrac{1}{4}\sum_{i,j}\widetilde{w}_{ij}(v)\widetilde{\kappa}(\widetilde{e}_i)\widetilde{\kappa}(\widetilde{e}_j)\widetilde{\sigma}_\alpha \\
&= \tfrac{1}{4}\Psi_w\left\{\sum_{i,j}\widetilde{w}_{ij}(v)\kappa(e_i)\kappa(e_j)\sigma_\alpha\right\} \\
&= \tfrac{1}{4}\Psi_w\left\{\sum_{i,j}g_0(\nabla_v e_i, e_j)\kappa(e_i)\kappa(e_j)\sigma_\alpha + \tfrac{14}{3}b^{-2}g_0(\nabla_v w, w)\left(3+g_0(w,w)\right)\sigma_\alpha\right. \\
&\quad + b^{-1}\left(w\cdot\nabla_v w + \nabla_v w\cdot w\right)\sigma_\alpha - \tfrac{4}{3}b^{-2}g_0(\nabla_v w, w)\left(w\cdot w\right)\sigma_\alpha \\
&\quad \left. + \tfrac{4}{3}b^{-2}g_0(\nabla_v w, w)\kappa\left(\sum_{k=0}^{6}\left(e_k\times w\right)\cdot e_k\right)\sigma_\alpha\right\} \\
&= \Psi_w\left\{\nabla_v^S\sigma_\alpha + b^{-2}g_0(\nabla_v w, w)\left(3+g_0(w,w)\right)\sigma_\alpha\right. \\
&\quad \left. + \tfrac{1}{3}b^{-2}g_0(\nabla_v w, w)\kappa\left(\sum_{k=0}^{6}\left(e_k\times w\right)\cdot e_k\right)\sigma_\alpha\right\},
\end{aligned}$$

since $v\cdot w + w\cdot v = -2g_0(v,w)1$ for all vector fields v, w. Thus we obtain the following lemma:

Lemma 1. *Let ∇^S and $\nabla^{\widetilde{S}}$ denote the covariant derivatives on spinor bundles S and $\widetilde{S}$ respectively. If $\nabla C_w = 0$, then*

$$\begin{aligned}
\nabla_v^{\widetilde{S}}\widetilde{\sigma} &= \Psi_w\left\{\nabla_v^S\sigma + b^{-2}g_0(\nabla_v w, w)\left(3+g_0(w,w)\right)\sigma\right. \\
&\quad \left. + \tfrac{1}{3}b^{-2}g_0(\nabla_v w, w)\kappa\left(\sum_{k=0}^{6}\left(e_k\times w\right)\cdot e_k\right)\sigma\right\},
\end{aligned}$$

where $\Psi_w(\sigma) = \widetilde{\sigma} \in \Gamma(\widetilde{S})$ and $v \in \Gamma(T(M))$.

In particular, assume $\nabla\varphi_0 = 0$. In this case, the condition $\nabla C_w = 0$ is equivalent to the condition $\nabla w = 0$ and thus, the covariant derivative on the spinor bundle $\widetilde{S}$ is $\nabla_v^{\widetilde{S}}\widetilde{\sigma} = \Psi_w(\nabla_v^S\sigma)$.

4 The Dirac Operator on the Deformed Spinor Bundle

Let $\nabla C_w = 0$. We can calculate the Dirac operator on the spinor bundle $\widetilde{S}$: For any spinor $\Psi_w(\sigma) = \widetilde{\sigma}$,

$$
\begin{aligned}
\widetilde{D}\widetilde{\sigma} &= \sum_{i=0}^{6} \widetilde{\kappa}(\widetilde{e}_i) \left(\nabla^{\widetilde{S}}_{\widetilde{e}_i} \widetilde{\sigma}\right) \\
&= \sum_{i=0}^{6} \widetilde{\kappa}(\widetilde{e}_i) \circ \Psi_w \left\{ \nabla^{S}_{\widetilde{e}_i} \sigma + b^{-2} g_0(\nabla_{\widetilde{e}_i} w, w)\left(3 + g_0(w,w)\right)\sigma \right. \\
&\quad \left. + \tfrac{1}{3} b^{-2} g_0(\nabla_{\widetilde{e}_i} w, w)\kappa \left(\sum_{k=0}^{6} (e_k \times w)\cdot e_k \right) \sigma \right\} \\
&= \sum_{i=0}^{6} \Psi_w \circ \kappa(e_i) \left\{ \nabla^{S}_{\widetilde{e}_i} \sigma + b^{-2} g_0(\nabla_{\widetilde{e}_i} w, w)\left(3 + g_0(w,w)\right)\sigma \right. \\
&\quad \left. + \tfrac{1}{3} b^{-2} g_0(\nabla_{\widetilde{e}_i} w, w)\kappa \left(\sum_{k=0}^{6} (e_k \times w)\cdot e_k \right) \sigma \right\} \\
&= \Psi_w \left\{ b^{-2/3} D\sigma + b^{-2/3}\kappa(w)\left(\nabla^S_w \sigma\right) - b^{-2/3} \left\{ \sum_{i=0}^{6} \kappa(e_i) \left(\nabla^S_{(e_i \times w)} \sigma \right) \right\} \right\} \\
&\quad + \tfrac{1}{2} b^{-1/3} \Psi_w \left\{ \kappa\left(C_w(u)\right) \left\{ b^{-2}\left(3 + g_0(w,w)\right)\sigma + \kappa\left(\widetilde{w}\right)\sigma \right\} \right\} \\
&\quad + b^{-2/3} \Psi_w \left\{ g_0\left(\nabla_w w, w\right) \kappa(w) \left\{ b^{-2}\left(3 + g_0(w,w)\right)\sigma + \kappa\left(\widetilde{w}\right)\sigma \right\} \right\},
\end{aligned}
$$

where $u = grad(g_0(w,w))$ and $\widetilde{w} = \sum_{k=0}^{6} (e_k \times w).e_k$. Hence we obtain the theorem below:

Theorem 2. *If $\nabla C_w = 0$, then the Dirac operator $\widetilde{D}$ on the spinor bundle $\widetilde{S}$ is*

$$
\begin{aligned}
\widetilde{D}\widetilde{\sigma} &= \Psi_w \left\{ b^{-2/3} D\sigma + b^{-2/3}\kappa(w)\left(\nabla^S_w \sigma\right) - b^{-2/3} \left\{ \sum_{i=0}^{6} \kappa(e_i) \left(\nabla^S_{(e_i \times w)} \sigma \right) \right\} \right\} \\
&\quad + \tfrac{1}{2} b^{-1/3} \Psi_w \left\{ \kappa\left(C_w(u)\right) \left\{ b^{-2}\left(3 + g_0(w,w)\right)\sigma + \kappa\left(\widetilde{w}\right)\sigma \right\} \right\} \\
&\quad + b^{-2/3} \Psi_w \left\{ g_0\left(\nabla_w w, w\right) \kappa(w) \left\{ b^{-2}\left(3 + g_0(w,w)\right)\sigma + \kappa\left(\widetilde{w}\right)\sigma \right\} \right\},
\end{aligned}
$$

where $u = grad(g_0(w,w))$ *and* $\widetilde{w} = \sum_{k=0}^{6}(e_k \times w).e_k$.

In addition assume $\nabla\varphi_0 = 0$. In this case, since $\nabla C_w = 0$ if and only if $\nabla w = 0$, the Dirac operator is

$$\widetilde{D}\widetilde{\sigma} = b^{-2/3}\Psi_w \left\{ D\sigma + \kappa(w)\left(\nabla^S_w \sigma\right) - \left\{ \sum_{i=0}^{6} \kappa(e_i)\left(\nabla^S_{(e_i \times w)}\sigma\right)\right\}\right\}.$$

Let λ be an eigenvalue of D associated with the spinor σ. Then we get

$$\widetilde{D}\widetilde{\sigma} = b^{-2/3}\Psi_w \left\{ \lambda\sigma + \kappa(w)\left(\nabla^S_w \sigma\right) - \left\{ \sum_{i=0}^{6} \kappa(e_i)\left(\nabla^S_{(e_i \times w)}\sigma\right)\right\}\right\}.$$

Hence $\Psi_w(\sigma) = \widetilde{\sigma}$ may not be an eigenspinor. Thus, there is not any obvious relation between eigenvalues of Dirac operators D and $\widetilde{D}$. Nevertheless, we can deduce the following:

Let $\nabla\varphi_0 = 0$ and $\nabla w = 0$. If λ is an eigenvalue of $\widetilde{D}$ associated with the spinor $\widetilde{\sigma} = \Psi_w(\sigma)$, from the equation

$$\widetilde{D}\Psi_w\sigma = b^{-2/3}\Psi_w \left\{ D\sigma + \kappa(w)\left(\nabla^S_w \sigma\right) - \left\{ \sum_{i=0}^{6} \kappa(e_i)\left(\nabla^S_{(e_i \times w)}\sigma\right)\right\}\right\},$$

we get

$$\lambda\sigma = b^{-2/3} \left\{ D\sigma + \kappa(w)\left(\nabla^S_w \sigma\right) - \left\{ \sum_{i=0}^{6} \kappa(e_i)\left(\nabla^S_{(e_i \times w)}\sigma\right)\right\}\right\}.$$

Now by using the equation

$$\sum_{i=0}^{6} \kappa(e_i)\left(\nabla^S_{e_i \times w}\sigma\right) = -\sum_{i=0}^{6} \kappa(e_i \times w)\nabla^S_{e_i}\sigma,$$

we have the identity

$$\lambda\sigma b^{2/3} - \kappa(w)(\nabla^S_w \sigma) = b^{1/3}\sum_{i=0}^{6} \kappa(C_w(e_i))\nabla^S_{e_i}\sigma.$$

Hence we express the following lemma:

Lemma 3. *Let* $\nabla\varphi_0 = 0$ *and* $\nabla w = 0$*. If there exist a scalar* λ *and a spinor* $\sigma \in \Gamma(S)$ *satisfying the equation*

$$\sum_{i=0}^{6} b^{1/3}\kappa\left(C_w(e_i)\right)\nabla^S_{e_i}\sigma = \lambda b^{2/3}\sigma - \kappa(w)\left(\nabla^S_w\sigma\right),$$

then λ *is an eigenvalue of* $\widetilde{D}$ *associated with the spinor* $\widetilde{\sigma} = \Psi_w(\sigma)$*.*

It is known that if the G_2 structure φ_0 is parallel, then there are nonzero parallel spinors in $\Gamma(S)$ [5]. Let $\sigma \in \Gamma(S)$ be a nonzero parallel spinor. In this case since Ψ_w is an isomorphism, the spinor $\widetilde{\sigma} = \Psi_w(\sigma) \in \Gamma(\widetilde{S})$ is nonzero too. We showed that for all vector fields v, $\nabla^{\widetilde{S}}_v\widetilde{\sigma} = \Psi_w(\nabla^S_v\sigma)$ when $\nabla w = 0$. So $\nabla^S_v\sigma = 0$ if and only if $\nabla^{\widetilde{S}}_v\widetilde{\sigma} = 0$, i.e. there is a one-to-one correspondence between nonzero parallel spinors on S and nonzero parallel spinors on $\widetilde{S}$.

Recall that a spinor $\sigma \in \Gamma(S)$ is called harmonic if $D\sigma = 0$, that is, if $\sigma \in Ker(D)$. The existence of nonzero harmonic spinors is determined by the sign of the scalar curvature of the metric. Let (M, g) be compact. If the scalar curvature s of g is zero, then every harmonic spinor on M is parallel. If the scalar curvature is positive, then there are no nonzero harmonic spinors [5].

Now $\nabla\varphi_0 = 0$ implies that g_0 is Ricci-flat [5]. Hence the scalar curvature s_0 of g_0 is 0. Similarly the scalar curvature $\widetilde{s}$ of $\widetilde{g}$ is 0 since $\widetilde{\nabla}\widetilde{\varphi} = 0$. Thus there exist nonzero harmonic spinors in $\Gamma(S)$ and $\Gamma(\widetilde{S})$. Moreover every harmonic spinor is parallel. It is easy to see that each parallel spinor is harmonic.

Let $\widetilde{\sigma} = \Psi_w(\sigma) \in \Gamma(\widetilde{S})$ be a nonzero harmonic spinor, i.e. $\widetilde{D}\widetilde{\sigma} = 0$. Then we have $\widetilde{\nabla}\widetilde{\sigma} = 0$, which is possible if and only if $\nabla\sigma = 0$. Since $D\sigma = \sum_{j=0}^{6}\kappa(e_j)\nabla_{e_j}\sigma$, we get $D\sigma = 0$. Thus $\widetilde{D}\widetilde{\sigma} = 0$ if and only if $D\sigma = 0$. Therefore we obtain

$$\Psi_w(KerD) = Ker(\widetilde{D})$$

which shows that if we apply the deformation $\widetilde{\varphi} = \varphi_0 + w\lrcorner * \varphi_0$ to a 3-form on a manifold with parallel G_2 structure, then kernels of Dirac operators D and $\widetilde{D}$ are isomorphic.

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