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Best proximity points of Kannan type cyclic weak ϕ-contractions in ordered metric spaces

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Abstract

In this manuscript, the existence of the best proximity of Kannan Type cyclic weak ϕ-contraction in ordered metric spaces is investigated. Some results of Rezapour-Derafshpour-Shahzad [22] are generalized.

1 Introduction and Preliminaries

In 1922, Banach [3] stated that every contraction on a complete metric space has a unique fixed point. This theorem is known as Banach contraction mapping principle or Banach fixed point theorem. Banach's theorem preserves its importance in fixed point theory and has applications not only in many branches of mathematics but also in economics. In particular, in micro economics, for the existence of Nash equilibria, fixed point theorems are used (See e.g. [19, 4]).

In 1969, Boyd and Wong [5] gave the definition of Φ-contraction: A self-mapping T on a metric space X is called Φ-contraction if there exists an upper semi-continuous function $\Phi : [0,\infty) \to [0,\infty)$ such that

$$d(Tx,Ty) \leq \Phi(d(x,y)) \quad \text{for all} \;\; x,y \in X.$$

Later, in 1997, Alber and Guerre-Delabriere [1], introduced the definition of weak ϕ-contraction: a self-mapping T on a metric space X is called weak ϕ-contraction if for each $x,y \in X$, there exists a function $\phi : [0,\infty) \to [0,\infty)$ such that

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)).$$

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In addition, Alber and Guerre-Delabriere defined weak ϕ-contraction on Hilbert spaces and proved the existence of fixed points in Hilbert spaces. Rhoades [24] showed that most of the results in [1] are also valid for arbitrary Banach spaces.

Notice that if ϕ is a lower semi-continuous mapping, then $\Phi(u) = u - \phi(u)$ becomes Φ-contraction [5]. The notions Φ-contraction and weak ϕ-contraction have been studied by many authors, (e.g.,[11, 25, 26, 27, 13, 14, 15].)

Next, we give some preliminaries and basic definitions which are used throughout the paper. Let X and Y be nonempty sets and $T : X \to X$ and $S : Y \to Y$. Define the set of all invariant non-empty subsets of X under T as follows:

$$I_T(X) := \{Z \in P_0(X) : T(Z) \subset Z\} \tag{1.1}$$

where $P_0(X)$ denotes the set of all non-empty subsets of X. Moreover, a map $(T \times S) : X \times Y \to X \times Y$ is defined as

$$[T \times S](x, y) = (Tx, Sy). \tag{1.2}$$

For a partially ordered set $(X, \leq)$, we denote the set of all comparable pair in the following way:

$$CP(X) := \{(x, y) \in X \times X : \text{ either } \ x \leq y \ \text{ or } \ y \leq x\}. \tag{1.3}$$

Let $(X, d, \leq)$ be an ordered metric space and $T : X \to X$ be a self-mapping on X. For each x^* and non-empty subset Z of X, we define

$$Z_T(x^*) := \{x \in Z : \lim_{n\to\infty} T^{2n}x = x^*\}. \tag{1.4}$$

Cyclic maps were defined by Kirk-Srinavasan-Veeramani in 2003. They stated the following theorem (see [16], Theorem 1.1).

Theorem 1. *Let A and B be non-empty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is a map satisfying $T(A) \subset B$ and $T(B) \subset A$ and there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$. Then, T has a unique fixed point in $A \cap B$.*

Definition 2. *(See [2]) Let A and B be non-empty closed subsets of a metric space (X, d) and $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A map $T : A \cup B \to A \cup B$ is called a cyclic weak ϕ-contraction if $T(A) \subset B$ and $T(B) \subset A$ and*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B)) \tag{1.5}$$

for all $x \in A$ and $y \in B$ where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

A point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$. Further, if $\alpha \in (0, 1)$ and $\phi(t) = (1 - \alpha)t$, then T is called cyclic contraction (See [9]).

Very recently, Rezapour-Derafshpour-Shahzad (see[22],also [23]) stated the following theorem:

Theorem 3. *Let $(X, d, \leq)$ be an ordered metric space, A and B be non-empty subsets of X and $T : A \cup B \to A \cup B$ be decreasing, cyclic weak ϕ-contraction, that is, T satisfies (1.5). Suppose there exists $x_0 \in A$ such that $x_0 \leq T^2x_0 \leq Tx_0$. Define $x_{n+1} = Tx_n$ and $d_n := d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Then $d_n \to d(A, B)$.*

In this manuscript, Theorem 3 and some other results of [22, 13] are generalized.

2 Main Results

In this section we define Kannan type cyclic weak ϕ-contraction and state our main results.

Definition 4. *Let A and B be non-empty subsets of a metric space (X, d) and $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A map $T : A \cup B \to A \cup B$ is said to be Kannan type cyclic weak ϕ-contraction if $T(A) \subset B$ and $T(B) \subset A$ and*

$$d(Tx, Ty) \leq u(x, y) - \phi(u(x, y)) + \phi(d(A, B)) \tag{2.1}$$

for all $x \in A$ and $y \in B$ where $u(x, y) = \frac{1}{2}[d(x, Tx) + d(y, Ty)]$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

Example 5. *Let $X := \mathbb{R}$ with the usual metric. For $A = [0, 1], B = [-1, 0]$, define $T : A \cup B \to A \cup B$ by $Tx = \frac{-3x}{4}$ for all $x \in A \cup B$ and $\phi(t) = \frac{t}{7}$. Then T satisfies (2.1), i.e., T is a Kannan type cyclic weak ϕ-contraction.*

Example 6. *Consider the Euclidean ordered space $X = \mathbb{R}$ with the usual metric. Let $A = B = [0, 1]$ and $T : A \cup B \to A \cup B$ be defined by*

$$Tx = \begin{cases} \frac{1}{4} & \text{if } x = 1, \\ \frac{1}{2} & \text{if } x \in [0, 1). \end{cases}$$

If $\phi : [0, \infty) \to [0, \infty)$ is defined by $\phi(t) = \frac{t}{8}$ then T is a Kannan type cyclic weak ϕ-contraction but not a cyclic weak ϕ-contraction. Indeed, for $x = 7/8$ and $y = 1$ we have

$$\begin{aligned} d(Tx, Ty) &\leq d(x, y) - \phi[d(x, y)] + d(A, B) \\ |T\tfrac{7}{8} - T1| &\leq |\tfrac{7}{8} - 1| - \tfrac{1}{8}|\tfrac{7}{8} - 1| \\ \tfrac{1}{4} &\leq \tfrac{7}{64} \end{aligned}$$

which shows that T is not a cyclic weak ϕ-contraction.

Let us check whether T is a Kannan type cyclic weak ϕ-contraction. Notice that if both x and y are in $[0,1)$, T obviously satisfies (2.1). If $x=y=1$ and T again satisfies (2.1). Now, consider the case $y\in[0,1)$ and $x=1$. Then

$$\begin{array}{rl} d(Tx,Ty) & \leq \frac{1}{2}\left[d(Tx,x)+d(Ty,y)\right]-\phi[\frac{1}{2}\left[d(Tx,x)+d(Ty,y)\right]]+d(A,B)\\ |T1-Ty| & \leq \frac{1}{2}\left[\frac{3}{4}+|y-\frac{1}{2}|\right]-\frac{1}{8}(\frac{1}{2}\left[\frac{3}{4}+|y-\frac{1}{2}|\right])\\ \frac{1}{4} & \leq \frac{21}{64}+\frac{7}{16}|y-\frac{1}{2}| \end{array}$$

which holds for every $y\in[0,1)$. Thus, T is a Kannan type cyclic weak ϕ-contraction.

Theorem 7. *Let $(X,d,\leq)$ be an ordered metric space, A and B be non-empty subsets of X. Suppose $T: A\cup B\to A\cup B$ is decreasing, Kannan type cyclic weak ϕ-contraction, that is, T satisfies (2.1). Assume that there exists $x_0\in A$ such that $x_0\leq T^2x_0\leq Tx_0$. Define $x_{n+1}=Tx_n$ and $d_n:=d(x_{n+1},x_n)$ for all $n\in\mathbb{N}$. Then $d_n\to d(A,B)$.*

Proof. By the assumption, one can easily observe that

$$x_0\leq x_2\leq\cdots\leq x_{2n}\leq x_{2n+1}\leq\cdots\leq x_3\leq x_1,\quad \text{for all } n\in\mathbb{N}.$$

Due to (2.1), we have

$$\begin{array}{rl} d_{n+1} & = d(x_{n+2},x_{n+1})=d(Tx_{n+1},Tx_n)\\ & \leq \frac{1}{2}[d(x_{n+1},Tx_{n+1})+d(x_n,Tx_n)]-\phi(\frac{1}{2}[d(x_{n+1},Tx_{n+1})+\\ & +d(x_n,Tx_n)])+\phi(d(A,B))\\ & \leq \frac{1}{2}[d_{n+1}+d_n]-\phi(\frac{1}{2}[d_{n+1}+d_n])+\phi(d(A,B)). \end{array} \tag{2.2}$$

Notice that $d(A,B)\leq\frac{1}{2}[d(x_{n+1},x_{n+2})+d(x_n,x_{n+1})]$. Since ϕ is a strictly increasing map, we have

$$\phi(d(A,B))\leq\phi\left(\frac{1}{2}[d(x_{n+1},x_{n+2})+d(x_n,x_{n+1})]\right). \tag{2.3}$$

Thus, regarding $0\leq d_{n+1}$, the expression (2.2) turns into

$$0\leq d_{n+1}\leq\frac{1}{2}[d_{n+1}+d_n]-\phi(\frac{1}{2}[d_{n+1}+d_n])+\phi(d(A,B)) \tag{2.4}$$

which implies $0\leq\frac{1}{2}d_{n+1}\leq\frac{1}{2}d_n$. Hence, the sequence $\{d_n\}$ is non-increasing and bounded below. If $d_{n_0}=0$ for some $n_0\in\mathbb{N}$, then clearly $d_n\to 0$, $d(A,B)=0$ and so, $d_n\to d(A,B)$. Suppose $d_n\neq 0$ (that is, $d_n>0$) for all $n\in\mathbb{N}$ and $d_n\to L$ for some $L\geq d(A,B)$. From (2.4), we obtain that

$$\phi(\frac{1}{2}[d_{n+1}+d_n])\leq\frac{1}{2}[d_{n+1}+d_n]+\phi(d(A,B))-d_{n+1}. \tag{2.5}$$

If we combine (2.3) and (2.5) we get

$$\phi(d(A,B)) \leq \phi(\frac{1}{2}[d_{n+1}+d_n]) \leq \frac{1}{2}[d_{n+1}+d_n] - d_{n+1} + \phi(d(A,B)) \quad (2.6)$$

which yields that $\phi(L) = \phi(d(A,B))$. Since ϕ is strictly increasing, then $L = d(A,B)$ which completes the proof. □

The following result is a consequence of Theorem 7 and Theorem 2.2 of [22]. Regarding analogy, we omit the proof.

Corollary 8. *Let $(X,d,\leq)$ be an ordered metric space, A and B be non-empty subsets of X and $T : A\cup B \to A\cup B$ be decreasing map satisfying*

$$d(Tx,Ty) \leq m(x,y) - \phi(m(x,y)) + \phi(d(A,B)) \quad (2.7)$$

for all $x \in A$ and $y \in B$ where $m(x,y) = \max\{d(x,y), \frac{1}{2}[d(x,Tx)+d(y,Ty)]\}$. Suppose that there exists $x_0 \in A$ such that $x_0 \leq T^2x_0 \leq Tx_0$. Define $x_{n+1} = Tx_n$ and $d_n := d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$. Then $d_n \to d(A,B)$.

Our next results are concerned with orbital continuous maps and topological spaces with C-condition.

Definition 9. *(See [22]) A topological space X is said to have condition (C) if for each convergent monotone sequence $\{x_n\}$ (that is, $x_n \to x$, $x \in X$), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every element of $\{x_{n_k}\}$ is comparable with the limit x. Further, X is called regular if every bounded monotone sequence in X is convergent.*

Definition 10. *(See [6, 7],see also [12]) A mapping T on metric space (X,d) is said to be orbitally continuous if $\lim_{i\to\infty} T^{n_i}(x) = z$ implies $\lim_{i\to\infty} T(T^{n_i}(x)) = Tz$.*

Remark 11. *It is clear that orbital continuity of T implies orbital continuity of T^m for any $m \in \mathbb{N}$.*

Theorem 12. *Let $(X,d,\leq)$ be a regular ordered metric space, A and B be non-empty subsets of X where A is closed. Assume $T : A\cup B \to A\cup B$ is a decreasing map satisfying Kannan type cyclic weak ϕ-contraction, that is, T satisfies (2.1). Suppose that there exists $x_0 \in A$ such that $x_0 \leq T^2x_0 \leq Tx_0$. Define $x_{n+1} = Tx_n$ and $d_n := d(x_{n+1},x_n)$ for all $n \in \mathbb{N}$. If T is orbitally continuous or X has the property (C), then there exists $x \in A$ such that $d(x,Tx) = d(A,B)$.*

Proof. By the assumption, one can easily see that

$$x_0 \leq x_2 \leq \cdots \leq x_{2n} \leq x_1, \quad \text{for all } n \in \mathbb{N}.$$

Regarding that X is regular and A is closed, the sequence $\{x_{2n}\}$ converges to a point, say $x \in A$. Observe that

$$d(A,B) \leq d(x_{2n}, Tx) = d(Tx_{2n-1}, Tx) \leq d(Tx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Tx), \tag{2.8}$$

for all $n \in \mathbb{N}$. If T is orbitally continuous, then $d(Tx_{2n}, Tx) \to 0$. Regarding Theorem 7, we have $d(Tx_{2n-1}, Tx_{2n}) \to d(A,B)$ and hence

$$d(x, Tx) = d(A,B)$$

due to (2.8). Suppose now that X has the property (C). Taking the fact that the sequence $\{x_{2n}\}$ is bounded and increasing into account, one can find a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ such that

$$x_{2n_1} \leq x_{2n_2} \leq \cdots \leq x_{2n_k} \leq \cdots \leq x.$$

Hence,

$$\begin{aligned} d(A,B) &\leq d(x_{2n_k}, Tx) = d(x_{2n_k-1}, Tx) \\ &\leq d(x_{2n_k-1}, Tx_{2n_k}) + d(Tx_{2n_k}, Tx) \\ &\leq d(x_{2n_k-1}, Tx_{2n_k}) + d(x_{2n_k}, Tx) \end{aligned} \tag{2.9}$$

which implies that $d(x, Tx) = d(A,B)$. □

The following corollary results from Theorem 7 and Theorem 2.3 of [22]. The proof is omitted because of analogy.

Corollary 13. *Let $(X, d, \leq)$ be a regular ordered metric space, A and B be non-empty subsets of X where A is closed. Assume that $T : A \cup B \to A \cup B$ is a decreasing map satisfying*

$$d(Tx, Ty) \leq m(x,y) - \phi(m(x,y)) + \phi(d(A,B)) \tag{2.10}$$

for all $x \in A$ and $y \in B$ where $m(x,y) = \max\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)]\}$. Suppose that there exists $x_0 \in A$ such that $x_0 \leq T^2x_0 \leq Tx_0$. Define $x_{n+1} = Tx_n$ and $d_n := d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. If T is orbitally continuous or X has the property (C), then there exists $x \in A$ such that $d(x, Tx) = d(A,B)$.

Corollary 14. *Let $(X, d, \leq)$ be an ordered metric space, A and B be non-empty subsets of X. Define $T : A \cup B \to A \cup B$ such that $T(A) \subset B$, $T(B) \subset A$*

and $(A \times B) \cap CP(X) \in I_T(X)$*. Assume that there exists* $x_0 \in A$ *such that* $(x_0, Tx_0) \in CP(X)$ *and*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Tx)+d(Ty,y)] - \phi(\frac{1}{2}[d(x,Tx)+d(Ty,y)]) + \phi(d(A,B)), \tag{2.11}$$

for all $x \in A$ *and* $y \in B$ *with* $(x,y) \in CP(X)$*, where* $\phi : [0,\infty) \to [0,\infty)$ *is a strictly increasing map. Define* $x_{n+1} = Tx_n$ *and* $d_n := d(x_{n+1}, x_n)$ *for all* $n \in \mathbb{N}$*. If* T *is orbitally continuous or* X *has the property* (C)*, then* $d_n \to d(A,B)$*.*

Proof. By the assumption, it is easy to see that,

$$\begin{aligned} d(T^{2n+1}x_0, T^{2n}x_0) &\leq \tfrac{1}{2}[d(T^{2n}x_0, T^{2n+1}x_0) + d(T^{2n}x_0, T^{2n-1}x_0)] \\ &-\phi(\tfrac{1}{2}[d(T^{2n}x_0, T^{2n+1}x_0) + d(T^{2n}x_0, T^{2n-1}x_0)]) + \phi(d(A,B)), \end{aligned} \tag{2.12}$$

for all $n \in \mathbb{N}$. Clearly, $d(A,B) \leq \frac{1}{2}[d(T^{2n}x_0, T^{2n+1}x_0) + d(T^{2n}x_0, T^{2n-1}x_0)]$. Since ϕ is a strictly increasing map, then

$$\phi(d(A,B)) \leq \phi\left(\frac{1}{2}[d(T^{2n}x_0, T^{2n+1}x_0) + d(T^{2n}x_0, T^{2n-1}x_0)]\right).$$

Regarding $0 \leq d_{n+1}$, the expression (2.12) becomes

$$0 \leq d_{n+1} \leq \frac{1}{2}[d_{n+1} + d_n] - \phi(\frac{1}{2}[d_{n+1} + d_n]) + \phi(d(A,B)) \tag{2.13}$$

which implies that $0 \leq \frac{1}{2}d_{n+1} \leq \frac{1}{2}d_n$. Hence, $\{d_n\}$ is decreasing and bounded below. If $d_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then clearly $d_n \to d(A,B) = 0$. Consider the other case $d_n > 0$ for all $n \in \mathbb{N}$. Let $d_n \to L$ for some $L \geq d(A,B)$. By the assumptions, we have

$$\phi(d(A,B)) \leq \phi(\frac{1}{2}[d_{n+1} + d_n]) \leq \frac{1}{2}[d_{n+1} + d_n] - d_{n+1} + \phi(d(A,B)) \tag{2.14}$$

which yields $\phi(d_n) \to \phi(d(A,B))$, and therefore $\phi(L) = \phi(d(A,B))$. Since ϕ is strictly increasing, $L = d(A,B)$. □

The following result is a consequence of Theorem 14 and Theorem 2.3 of [22]. Regarding analogy, we omit the proof.

Corollary 15. *Let* $(X, d, \leq)$ *be an ordered metric space,* A *and* B *be non-empty subsets of* X*. Define* $T : A \cup B \to A \cup B$ *such that* $T(A) \subset B$*,* $T(B) \subset A$ *and* $(A \times B) \cap CP(X) \in I_T(X)$*. Assume that there exists* $x_0 \in A$ *such that* $(x_0, Tx_0) \in CP(X)$ *and*

$$d(Tx,Ty) \leq m(x,y) - \phi(m(x,y)) + \phi(d(A,B)) \tag{2.15}$$

for all $x \in A$ *and* $y \in B$ *with* $(x,y) \in CP(X)$ *where*

$$m(x,y) = \max\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)]\}$$

and $\phi : [0,\infty) \to [0,\infty)$ *is a strictly increasing map. Define* $x_{n+1} = Tx_n$ *and* $d_n := d(x_{n+1}, x_n)$ *for all* $n \in \mathbb{N}$. *If* T *is orbitally continuous or* X *has the property (C), then* $d_n \to d(A,B)$.

Theorem 16. *Let* $(X,d,\leq)$ *be an ordered metric space,* A *and* B *be non-empty subsets of* X. *Define* $T : A \cup B \to A \cup B$ *such that* $T(A) = B$, $T(B) \subset A$ *and* $(A \times B) \cap CP(X) \in I_T(X)$. *Assume that there exist* $x, y \in A$ *such that* $(x,z),(z,y) \in CP(X)$. *Suppose also that there exist* $x_0, u \in A$ *such that* $x_0 \in A_T(u)$, $(x_0, Tx_0) \in CP(X)$ *and*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Tx) + d(Ty,y)] - \phi(\frac{1}{2}[d(x,Tx) + d(Ty,y)]) + \phi(d(A,B)) \tag{2.16}$$

for all $x \in A$ *and* $y \in B$ *with* $(x,y) \in CP(X)$ *where* $\phi : [0,\infty) \to [0,\infty)$ *is a strictly increasing map. Moreover,* $y \in A$, $(x,y) \in CP(X)$ *and* $x \in A_T(u)$ *implies that* $y \in A_T(u)$. *Then,* $A_T(u) = A$ *and the following holds:*

$$B_T(Tu) = B \text{ and } d(u,Tu) = d(A,B) \Leftrightarrow T \text{ is orbitally continuous.} \tag{2.17}$$

Proof. First, we prove that $A_T(u) = A$. Take $x \in A$. If $(x,x_0) \in CP(X)$, then $x \in A_T(u)$ and so $A_T(u) = A$. If $(x,x_0) \notin CP(X)$, then by assumption, there exists $z \in A$ such that $(x,z),(x_0,z) \in CP(X)$. Therefore, $x \in A_T(u)$. Hence, in any case, $A_T(u) = A$.

Now we show (2.17). Assume that T is orbitally continuous. Fix $y \in B$. Choose $\tilde{x} \in A$ such that $T\tilde{x} = y$. Since $A_T(u) = A$ and $T^{2n}\tilde{x} \to u$, then $T^{2n+1}\tilde{x} \to Tu$. That is, $T^{2n}y \to Tu$ and thus $B_T(Tu) = B$. If $d(A,B) = d(u,Tu)$ then the proof is done. Assume that $d(A,B) \neq d(u,Tu)$. Since $d(x_0,Tx_0) \in CP(X)$ and by(2.16), the sequence $\{d(T^{2n+1}x_0, T^{2n}x_0)\}$ is decreasing. Hence, Theorem 14 implies that $d(T^{2n+1}x_0, T^{2n}x_0) \downarrow d(A,B)$. Choose $n \in \mathbb{N}$ in a way that

$$d(A,B) \leq d(T^{2n+1}x_0, T^{2n}x_0) < d(u,Tu). \tag{2.18}$$

Substitute $r = T^{2n+1}x_0$ and $s = T^{2n}x_0$. Since $(r,s) \in CP(X)$ then $(Tr,Ts) \in CP(X)$ and thus $\{d(T^{2n}r, T^{2n}s)\}$ is a decreasing sequence.

Hence $d(T^{2n}r, T^{2n}s) \downarrow d(u,Tu)$ and so $d(u,Tu) \leq d(r,s) = d(T^{2n+1}x_0, T^{2n}x_0) < d(u,Tu)$ which contradicts (2.18). Thus, $d(A,B) = d(u,Tu)$.

To prove the inverse inclusion of (2.17), assume that $B_T(Tu) = B$ and $d(A,B) = d(u,Tu)$. Let $x \in A \cup B$ and $T^{n(i)}x \to w$ for some $w \in A \cup B$. To complete the proof, it is sufficient to show that T is orbitally continuous, that is, $T^{n(i)+1}x \to Tw$. Define the subsets $S_A = A \cap \{T^{n(i)}x\}$ and $S_B = B \cap \{T^{n(i)}x\}$. Observe that S_A and S_B are subsequences of $\{T^{n_i}x\}$, that is, $S_A = A \cap \{T^{n(i)}x\} = \{T^{n_1(i)}x\}$ and $S_B = B \cap \{T^{n(i)}x\} = \{T^{n_2(i)}x\}$. We consider the following two cases:

The first case, $d(u,Tu) = d(A,B) = 0$, in other words $Tu = u$. On account of Theorem 14 and using the fact that $\{T^{n_1(i)}x\}$ is a subsequence of $\{T^{n(i)}x\}$ and the assumption $T^{n(i)}x \to w$, we get $T^{n_1(i)}x \to w$. On the other hand, $\{T^{n_1(i)}x\}$ is also a subsequence of $\{T^{2n}x\}$ and thus, $T^{n_1(i)}x \to u$. Hence, we conclude that $u = w$, and further $w = u = Tu = Tw$. Then we have

$$\begin{aligned} d(T(T^{n(i)})x, Tw) &\leq \tfrac{1}{2}[d(T^{n(i)}x, T(T^{n(i)})x) + d(Tw,w)] \\ &\quad -\phi(\tfrac{1}{2}[d(T^{n(i)}, T(T^{n(i)})) + d(Tw,w)]) + \phi(d(A,B)) \\ &\leq \tfrac{1}{2}[[d(T^{n(i)}x, T(T^{n(i)})x) + d(Tw,w)] \\ &\leq \tfrac{1}{2}[d(T^{n(i)}x, T(T^{n(i)})x)] \\ &\leq \tfrac{1}{2}[d(T^{n(i)}x, Tw) + d(T^{n(i)}x, w)] \quad (\text{ since } Tw = w) \end{aligned} \tag{2.19}$$

which implies that $\frac{1}{2}d(T(T^{n(i)})x, Tw) \leq \frac{1}{2}d(T^{n(i)}x, w)$, or equivalently, $T(T^{n(i)})x \to Tw$. Hence, T is orbitally continuous.

Consider the second case, $d(u,Tu) = d(A,B) > 0$. We assert that both S_A or S_B are finite. Indeed, if both S_A and S_B are infinite, then as in the first case, $T^{n_1(i)+1}x \to u$ and $T^{n_2(i)+1}x \to Tu$. Since both $\{T^{n_1(i)}x\}$ and $\{T^{n_2(i)}x\}$ are subsets of $\{T^{n(i)}x\}$ and $T^{n(i)}x \to w$, then both subsequences converge to w and thus $w = Tu = u$. So $d(A,B) = d(u,Tu) = 0$ which is a contradiction. Therefore, either S_A or S_B is finite. Assume that $S_B = \{b_1, b_2, \cdots, b_m\}$ is finite. As in the first case, we get that $T^{n_1(i)x} \to u$, $T^{n_1(i)+1}x \to Tu$ and $w = u$. Then, $\{T^{n(i)+1}x\} = \{T^{n_1(i)}x\} \cup \{Tb_1, \cdots, Tb_m\}$ and $T^{n(i)+1} \to Tw$. Suppose now that $S_A = \{a_1, a_2, \cdots, a_j\}$ is finite and recall that $\{T^{n_2(i)}x\}$ is a subsequence of $\{T^{n(i)}\}$. Then, $T^{n_2(i)}x \to w$. Moreover, $\{T^{n_2(i)}x\}$ is also a subsequence of $\{T^{2n+1}x\} = T^{2n}(Tx)$ and $Tx \in B = B_T(Tu)$, so $T^{n_2(i)}x \to Tu$. Hence, $w = Tu$. Due to the fact that $\{T^{n_2(i)}x\}$ is a subset of $\{T^{2n+2}x\} = \{T^{2n}(T^2x)\}$ and $T^2x \in A = A_T(u)$, we have

$$T^{n_2(i)+1} \to u. \tag{2.20}$$

We claim that $Tw = u$. Since $(u,u) \in CP(X)$, by triangle inequality we have $d(u, T^2u) \leq d(u, T^{2n}u) + d(T^{2n}u, T^2u)$. By assumptions of the theorem, we have $d(T^{2n}u, T^2u) \leq d(T^{2n-2}u, u)$ which implies that

$$d(u, T^2u) \leq d(u, T^{2n}u) + d(T^{2n-2}u, u).$$

Taking into account that $u \in A = A_T(u)$, we have $T^{2n}u \to u$ and $T^{2n-2}u \to u$. Hence, $u = T^2u$. Since $w = Tu$, then $Tw = T^2u = u$. So, the equation (2.20) becomes $T^{n_2(i)+1} \to u = Tw$. Thus, $T^{n(i)+1}x \to Tw$ where $\{T^{n(i)+1}x\} = \{T^{n_2(i)}x\} \cup \{Ta_1, \cdots, a_j\}$ which completes the proof. □

Notice that the condition

$$d(Tx, Ty) \le \frac{1}{2}[d(Tx, x) + d(Ty, y)] - \phi(\frac{1}{2}[d(Tx, x) + d(Ty, y)]) + d(A, B),$$

for all $x \in A$, $y \in B$ with $(x, y) \in CP(X)$, does not imply

$$y \in A, (x, y) \in CP(X), x \in A_T(u) \Rightarrow y \in A_T(u).$$

The following example is given to support our assertion:

Example 17. *Let $X := \mathbb{R}^2$ with the usual metric and the following order:*

$$(a, b) \le (c, d) \Leftrightarrow a \le c \ \text{ and } \ c \le d.$$

For $A = \{x_1 = (5, 2), x_2 = (1, 2)\}, B = \{y_1 = (3, 0), y_2 = (0, 4)\}$, define $T : A \cup B \to A \cup B$ by $Tx_1 = y_2, Tx_2 = y_1, Ty_1 = x_2, Ty_2 = x_1$. Here, $x_2 \le x_1$ and $y_1 \le x_1$ and the others are not comparable. Let $\phi(t) = \frac{t}{3}$. Observe that $d(Tx_1, Ty_1) = d(x_2, y_2) = d(A, B) = \sqrt{5}$ and $\frac{1}{2}[d(Tx_1, x_1) + d(Ty_1, y_1)] = \frac{1}{2}[\sqrt{8} + \sqrt{29}]$. Then we have

$$d(Tx_1, Ty_1) \le \frac{1}{2}[d(Tx_1, x_1) + d(Ty_1, y_1)] - \phi(\frac{1}{2}[d(Tx_1, x_1) + d(Ty_1, y_1)]) + d(A, B),$$

while $T^{2n}x_1 \to x_1$ and $T^{2n}x_2 \to x_2$.

Last we give a consequence of Theorem 7 and Theorem 2.4 of [22].

Corollary 18. *Let $(X, d, \le)$ be an ordered metric space, A and B be non-empty subsets of X. Define $T : A \cup B \to A \cup B$ such that $T(A) = B$, $T(B) \subset A$ and $(A \times B) \cap CP(X) \in I_T(X)$. Assume that there exist $x, y \in A$ such that $(x, z), (z, y) \in CP(X)$. Suppose that there exist $x_0, u \in A$ such that $x_0 \in A_T(u)$, $(x_0, Tx_0) \in CP(X)$ and*

$$d(Tx, Ty) \le m(x, y) - \phi(m(x, y)) + \phi(d(A, B)) \tag{2.21}$$

for all $x \in A$ and $y \in B$ with $(x, y) \in CP(X)$ where

$$m(x, y) = \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\}$$

and $\phi : [0, \infty) \to [0, \infty)$ *is a strictly increasing map. Moreover,* $y \in A$, $(x, y) \in CP(X)$ *and* $x \in A_T(u)$ *implies that* $y \in A_T(u)$. *Then,* $A_T(u) = A$ *and the following hold:*

$$B_T(Tu) = B \quad \text{and} \quad d(u, Tu) = d(A, B) \Leftrightarrow T \text{ is orbitally continuous.} \tag{2.22}$$

Regarding analogy, we omit the proof.

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