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Geodesic Flow on the Quotient Space of the Action of $\langle z+2, -\frac{1}{z}\rangle$ on the Upper Half Plane

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Abstract

Let G be the group generated by $z \mapsto z+2$ and $z \mapsto -\frac{1}{z}$, $z \in \mathbb{C}$. This group acts on the upper half plane and the associated quotient surface is topologically a sphere with two cusps. Assigning a "geometric" code to an oriented geodesic not going to cusps, with alphabets in $\mathbb{Z} \setminus \{0\}$, enables us to conjugate the geodesic flow on this surface to a special flow over the symbolic space of these geometric codes. We will show that for $k \geq 1$, a subsystem with codes from $\mathbb{Z} \setminus \{0, \pm 1, \pm 2, \cdots, \pm k\}$ is a TBS: topologically Bernouli scheme. For similar codes for geodesic flow on modular surface, this was true for $k \geq 3$. We also give bounds for the entropy of these subsystems.

Introduction

Let $\mathcal{H} = \{z = x + iy : y > 0\}$ be the upper half plane endowed with the hyperbolic metric $ds = \frac{|dz|}{y}$. Then the geodesics are the vertical lines or the semi-circles perpendicular to real axis. Let G be a finitely generated Fuchsian group of the first kind with generators $T(z) = z + 2$ and $S(z) = -\frac{1}{z}$. The group G $(= \langle T, S \rangle)$ acts on $\mathcal{H}$ discontinuously with a Dirichlet fundamental domain

$$F = \{z \in \mathcal{H} : |z| \geq 1, |\mathrm{Re} z| \leq 1\} \tag{1}$$

whose boundary consists of a semicircle and two vertical lines $x = -1$ and $x = 1$ (see Figure 1). The associated quotient space $G\backslash\mathcal{H}$ is a finite area

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Riemann surface with one elliptic point, as its only singular point, and two cusps. This surface is topologically a sphere with two punctures and we denote it by M^{c2} (a sphere with two cusps). If G' is another group giving a Riemann surface with the same signature, then these two surfaces are quasi-conformally equivalent [3]. One of our goals is to study the dynamics of geodesic flow on M^{c2} which will not go to the cusps in either directions. We may do this by considering oriented geodesics. A geodesic will not go to the cusp if it is the projection of an oriented geodesic $\gamma = (w, u)$ in $\mathcal{H}$ intersecting the real line at irrationals u and w where we let u represent the repelling fixed point and w the attracting fixed point of γ. Two geodesics γ and γ' project to the same geodesic on M^{c2} if there exists $g \in G$ such that $\gamma = g\gamma' g^{-1}$. Lifting the geodesics to TM^{c2}, the unit tangent bundle of M^{c2}, gives the geodesic flow as an invariant set on M^{c2}.

When the generators are $z \mapsto z+1$ and $z \mapsto -\frac{1}{z}$, correspondingly a surface called *modular surface* arises. Modular surface has two singular points and a cusp and we show it by M^{c1}. Extensive studies has been done for the dynamics of geodesic flow on M^{c1} [2], [5], [6], [8] and [13]. Here we do similar studies for such dynamics on M^{c2}. Our second goal is to have a comparison of some of the main results in M^{c2} with similar ones in M^{c1}.

Results on geodesic flow on M^{c1} are based on the codes which are assigned to the geodesics. Several techniques for developing codes have been introduced [4], [5] and [8] and one of the most natural codes is geometric codes appearing in [15]. In this note, we introduce geometric codes for geodesic flow on M^{c2}. Basically, these codes are bi-infinite sequences of nonzero integers which tell how a geodesic enters F infinitely many times in past and future.

In fact, these codes together with the length of geodesic between two successive return of geodesic to F reveals the dynamics of geodesic flow. For then we can construct a special flow, conjugate to our flow, whose base space is the symbolic space obtained by these codes and its height function is the aforementioned length.

Our technique for acquiring the codes is different from the one given in [8]. We first introduce parameter space and then we will obtain the codes from that space. This will considerably lessen the task of computations. For instance in Theorem 2.3, rather easily will be shown that all geodesics whose entries of their geometric codes are in $\mathbb{Z} \setminus \{0, \pm 1\}$ are topologically Bernouli scheme which is a sort of 1-step Markov chain. For geodesic flow on M^{c1}, this is true when entries are from $\mathbb{Z} \setminus \{n : |n| \leq 3\}$.

Last section is devoted to give an upper and a lower bound for the topological entropy of subsystems with codes in $\mathbb{Z} \setminus \{n : |n| \leq k,\, k \geq 2\}$.

1 Geometric and arithmetic codes for geodesics on M^{c2}

We apply Morse method to have the geometric codes of the geodesic flow on M^{c2}. This requires a cross section and we will show how F defines our cross section. Recall that a cross section is a set that almost all flow lines intersect infinitely many times in past and future. Let F be as in (1). Label the circular side of F by s and the sides $x = -1$ and $x = 1$ by t^{-1} and t respectively (see Figure 1). The left and right parts of the semicircle are identified by S and the sides t^{-1} and t are identified by T and T^{-1}.

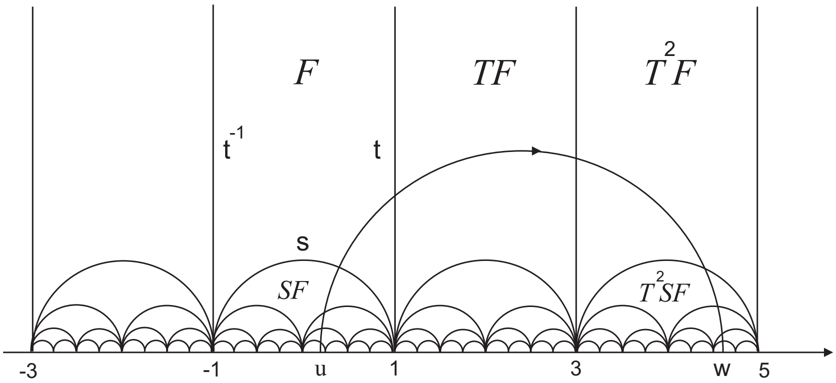

Figure 1: F is the fundamental domain for the action of $G = \langle z+2, -\frac{1}{z}\rangle$ on the upper half plane.

We consider the oriented geodesics which enter F via side s and call them *reduced geodesics*. In Lemma 1.1, we will show that any geodesic on $\mathcal{H}$ is G-equivalent to a reduced geodesic, that means for given $\gamma \in \mathcal{H}$, there exists $g \in G$ and a reduced geodesic γ' such that $\gamma = g\gamma' g^{-1}$. If $\gamma = (w, u)$ is a reduced geodesic with repelling and attracting endpoints w and u respectively, then $|w| > 1$ and $|u| < 1$. By Morse method we start from an initial point of a reduced geodesic on s and move in the direction of the geodesic and count the number of times that the geodesic hits sides t or t^{-1}. A bi-infinite sequence of non-zero integers will be assigned to γ called the *geometric code* of γ where entries $n_i > 0$ (respectively $n_i < 0$) in geometric code shows the number of times that γ has hit the side t (respectively t^{-1}) between two successive hits to s. Denote the geometric code of γ by $[\gamma] = [..., n_{-1}, n_0, n_1, ...]$. This is similar to geometric code given for geodesics in M^{c1} described in [8, 9].

Our method to compute the geometric code of γ is to consider the parameter space for the reduced geodesics in $\mathcal{H}$. Consider the wu coordinate in the plane.

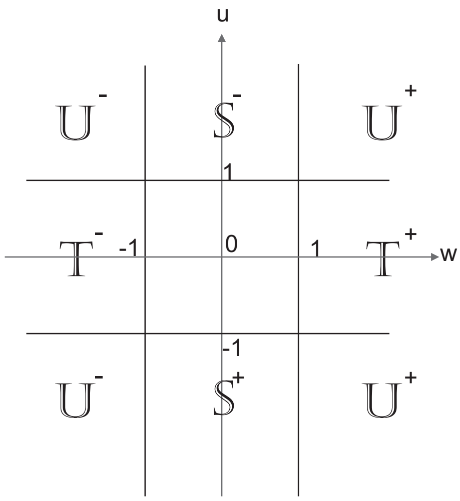

Figure 2: The parameter space for reduced geodesics is $\mathbb{T}^+ \cup \mathbb{T}^-$.

The lines $w = \pm 1$ and $u = \pm 1$ partition the plane to 9 regions named $\mathbb{T}^+$, $\mathbb{T}^-$, $\mathbb{S}^+$, $\mathbb{S}^-$, $\mathbb{U}^+$ and $\mathbb{U}^-$ (see Figure 2). Let $\mathbb{T} = \mathbb{T}^+ \cup \mathbb{T}^-$, $\mathbb{S} = \mathbb{S}^+ \cup \mathbb{S}^-$ and $\mathbb{U} = \mathbb{U}^+ \cup \mathbb{U}^-$. Also let $\mathbb{T}_n \subset \mathbb{T}$ be the square whose opposite vertices are $(2n-1, -1)$ and $(2n+1, 1)$, $n \in \mathbb{Z} \setminus \{0\}$ and let $\mathbb{S}_n := T^{-n}(\mathbb{T}_n) \subset \mathbb{S}$. The action of transformations $T(z) = z + 2$ and $S(z) = -\frac{1}{z}$ induces a map T_R on $\mathbb{R}^2$ which is defined as

$$T_R(w,u) = \begin{cases} T^{-1}(w,\, u) = (w-2,\, u-2) & \text{on } \mathbb{T}^+ \cup \mathbb{U}^+ \\ S(w,\, u) = (\frac{-1}{w},\, \frac{-1}{u}) & \text{on } \mathbb{S} \\ T(w,\, u) = (w+2,\, u+2) & \text{on } \mathbb{T}^- \cup \mathbb{U}^-. \end{cases} \quad (2)$$

We show T_R by T or S when the domain is understood from context. The geodesics whose one of their endpoints is a rational number will go to the cusp and depending on cusps two cases happen. 1) After some iterations of T_R, this rational endpoint sits on zero which then the geometric code associated to the direction of this endpoint is finite. 2) It eventually lands on 1 or -1 where there remains forever. That is, the code will have at least a tail of 1's or -1's. So as geodesics do not go to cusps, geometric codes are bi-infinite sequences with no tails of 1's or -1's.

Lemma 1.1. *Each geodesic in $\mathcal{H}$ is G-equivalent to a reduced geodesic.*

Proof. Suppose γ is a geodesic with endpoints $(w, u) \in \mathbb{R}^2$, $w \neq u$. Let $D = \mathbb{T} \cup \mathbb{S} \cup \mathbb{U}$. It suffices to show that by finite applications of S, T and T^{-1}, the point (w, u) will map to $\mathbb{T} \cup \mathbb{S}$. Evidently there is $k \in \mathbb{Z}$ such that $(T^k w, T^k u) \in (-1, 1) \times \mathbb{R}$. Therefore, we only have to care about points (w, u) such that $(w, u) \in E := ((-1, 1) \times \mathbb{R}) \setminus D$.

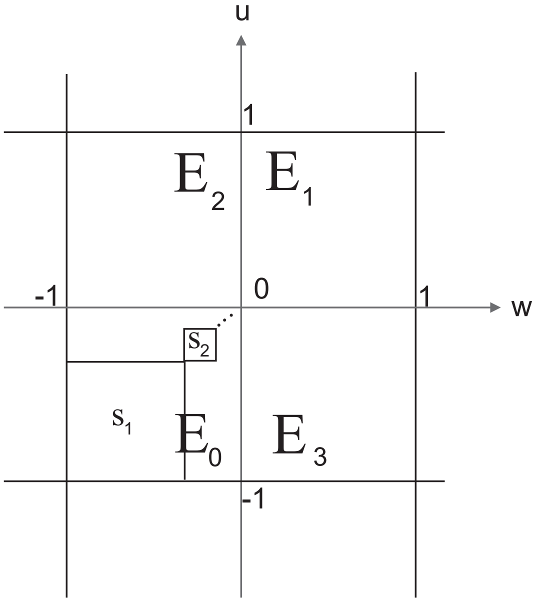

Figure 3: Any (w, u) eventually maps to $\mathbb{T}$ under T_R.

Partition E to $E_0 \cup E_1 \cup E_2 \cup E_3$ as in Figure 3. We have $S(E_0) \subset \mathbb{U}^+ \cap \{(w, u) : u > 0\}$, $S(E_1) \subset \mathbb{U}^- \cap \{(w, u) : u < 0\}$. Note that if $(w, u) \in E_2$ or E_3, then there is $\ell \in \mathbb{Z} \setminus \{0\}$ such that $T^\ell S(w, u) \in \mathbb{S}$ and so the associated geodesic will be reduced. We only give a proof for those (w, u) in $(-1, 0) \times (-1, 0) = E_0$. The proof for E_1 is similar. Consider the infinite chain of squares shown in E_0 in the Figure 3 which is the image under S of the union of all squares with two vertices at $(2n-1, 2n-1)$ and $(2n+1, 2n+1)$ for $n \in \mathbb{N}$, that is $S(\cup_{n\geq 1}[2n-1, 2n+1] \times [2n-1, 2n+1])$. Let s_k be the image of the square $[2k-1, 2k+1] \times [2k-1, 2k+1]$ in E_0 under S. Call this chain of squares E_0^1 $(= s_1 \cup s_2 \cup ...)$. Now $S(E_0 \setminus E_0^1)$ is a subset of $(1, \infty) \times (1, \infty)$ outside of the squares with vertices at $(2n-1, 2n-1)$ and $(2n+1, 2n+1)$ and map to $\mathbb{T}$ by some finite applications of T and so the associated geodesics are reduced.

We have $T^{-k}S(s_k) = E$, $k \in \mathbb{N}$. Hence a copy of E_0^1, say $E_{0,\,k}^1$ appears in s_k where by the same procedure any geodesic in $s_k \backslash E_{0,\,k}^1$ can be reduced. Applying the same reasoning to the subsequence chains of squares in sub-squares, all geodesics will be reduced. □

Every irrational number x where $|x| > 1$ has a unique *E-expansion* (E for even) of nonzero integers $(n_0, n_1, ...)$ as follows:

Set $x = x_0$. For $k \in \mathbb{Z} \setminus \{0\}$, let $n_i = k$ if $2k - 1 \leq x_i < 2k + 1$ where $x_{i+1} = -\frac{1}{x_i - 2n_i}$. Let $\gamma = (w, u) \in \mathbb{T}$ be a given geodesic whose E-expansion of w and u are

$$w = 2n_0 - \cfrac{1}{2n_1 - \cfrac{1}{\ddots}}, \qquad u = \cfrac{1}{2n_{-1} - \cfrac{1}{2n_{-2} - \cfrac{1}{\ddots}}}. \tag{3}$$

So to a reduced geodesic $\gamma = (w, u)$, with respect to the E-expansion of its endpoints, a bi-infinite sequence of nonzero integers is assigned. This sequence is called the *arithmetic code* of γ and is denoted by

$$(\gamma) = (..., n_{-1}, n_0, n_1, ...).$$

It is easy to see that the geometric and arithmetic codes of a reduced geodesic on M^{c2} coincide. So if the reduced geodesic $\gamma = (w, u)$ has the geometric code $[\gamma] = [..., n_{-1}, n_0, n_1, ...]$, then w and u satisfy (3).

The block $[n_0, n_1, ..., n_k]$ is an *admissible* block if there is a geodesic γ which has $n_0, n_1, ..., n_k$ as a part of its geometric code. Infinite blocks are likewise defined.

It is a natural question to ask if a given bi-infinite sequence of nonzero integers is realized by a geodesic. We will show in Corollary 1.3 that this is the case when the sequence has not a tail of just 1 or -1 in either directions.

Now we show how the parameter space evolves geometric codes. Start with a reduced geodesic $\gamma = (w, u) \in \mathbb{T}_{n_0}$. The left (respectively right) edge of $\mathbb{S}_{n_0}$ $(= T^{-n_0}(\mathbb{T}_{n_0}))$ will map to a short interval on the left (respectively right) edge of $\mathbb{T}_1$ (respectively $\mathbb{T}_{-1}$) by S. By identifying $-\infty$ and ∞, $S(\mathbb{S}_{n_0})$ will be a long horizontal rectangle intersecting any $\mathbb{T}_n$, $n \in \mathbb{Z} \setminus \{0\}$. Let $ST^{-n_0}(w, u) \in \mathbb{T}_{n_1}$ and set $\mathbb{T}_{n_0,n_1} = ST^{-n_0}(\mathbb{T}_{n_0}) \cap \mathbb{T}_{n_1}$. Inductively, let $\mathbb{T}_{n_0,\,n_1,\,...,\,n_k} := ST^{-n_{k-1}}(\mathbb{T}_{n_0,\,n_1,\,...,\,n_{k-1}}) \cap \mathbb{T}_{n_k}$ containing the reduced geodesic $ST^{-n_{k-1}}ST^{-n_{k-2}}...ST^{-n_0}(w, u)$.

Note that $\mathbb{T}_{n_0,\,n_1,\,...,\,n_k}$ contains all geodesics having n_i as the ith entry in their geometric code, $0 \leq i \leq k$.

Likewise, $S(\mathbb{T}_{n_0})$ is a long vertical rectangle in $\mathbb{S}$ containing $S(w, u) \in \mathbb{S}_{n_{-1}} \subset \mathbb{S}$. Then $T^{n_1}\mathbb{S}_{n_{-1}}(w, u)$ is in a vertical rectangle $T^{n_{-1}}S(\mathbb{T}_{n_0})$ denoted by $\mathbb{T}^{n_{-1},\,n_0}$. Inductively, $\mathbb{T}^{n_{-k},\,...,\,n_{-1},\,n_0}$ will be constructed which contains all geodesics with ith entry n_i, $-k \leq i \leq 0$. Carrying out the same process, the geometric code of γ will be obtained.

Let $\mathcal{A}$ be a set of countable alphabets. Consider the space $\Sigma \subseteq \Sigma_{\mathcal{A}} = \{x = (x_i)_{i=-\infty}^{\infty}, x_i \in \mathcal{A}\}$ and the shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(x_i) = x_{i+1}$. The symbolic dynamical system (Σ, σ) is called two-sided *countable topological Bernoulli scheme* (TBS) if $\Sigma = \Sigma_{\mathcal{A}}$.

Set $\beta_2 = \{B = [n, m] : n, m \in \mathcal{A}\}$ to be the set of all blocks of length two. Let $\tau : \beta_2 \to \{0, 1\}$ be the transition map, that is a map which assigns 1 to $[n, m] \in \beta$ if $[n, m]$ is an admissible block and zero otherwise. Then the subsystem $X_\tau = \{x \in \{\mathcal{A}\} : \tau([n_i, n_{i+1}]) = 1, \forall i \in \mathbb{Z}\}$ is a 1-*step Markov chain*. Obviously, a TBS is a 1-step Markov chain.

Let $\mathcal{B} \subseteq \mathbb{Z} \setminus \{0, \pm 1\}$ and $\Sigma_{\mathcal{B}}$ be the space of geometric codes whose alphabets are from $\mathcal{B}$. The easy proof of the following theorem shows how using the parameter space to detect the properties associated to geometric codes is effective.

Theorem 1.2. *The space $\Sigma_{\mathcal{B}}$ is a TBS.*

Proof. Consider the region $\mathbb{T}_{n_0}$, $n_0 \in \mathcal{B}$. Since $ST^{-n_0}(\mathbb{T}_{n_0}) \cap \mathbb{T}_{n_1}$ is nonempty for all $n_1 \in \mathcal{B}$ it follows that $[n_0, n_1]$ is an admissible block. □

A similar theorem can be stated for geodesic flow over modular surface M^{c1} and then we must choose $\mathcal{B} = \mathbb{Z} \setminus \{n : |n| \geq 3\}$. That is because a bi-infinite sequence $\cdots, n_{-1}, n_0, n_1, \cdots$ is realized as a geodesic code when $|\frac{1}{n_i} + \frac{1}{n_{i+1}}| < \frac{1}{2}$ [8, Theorem 1.5]. When this sufficiency condition holds we say that geometric code satisfies *Katok's criterion.*

A geodesic goes to a cusp in future or past if and only if its geometric code in that direction is finite or has a tail of all 1 or all -1. Now the following is immediate.

Corollary 1.3. *Let $\cdots, n_{-1}, n_0, n_1, \cdots$ be a bi-infinite sequence in $\mathbb{Z} \setminus \{0\}$ whose tails are neither all 1 nor all -1. Then this sequence represents a geometric code of an oriented geodesic on M^{c2} not going to cusps in positive or negative times.*

2 Entropy

When codes are from a set of infinite symbols, there are not much routines available to compute the entropy of the geodesic flow. Savchenko [12] is the first who defines the entropy for a class of systems called *topological Markov chains* (TMC): a system which can be represented by a countable directed connected graph. The first application on practical problems appears in [11] for a subclass of TMC called the *local perturbation of a* TBS. In a local perturbation, one considers the graph of TBS which is a complete graph and deletes some finite edges. Then this was extended to a larger class in [1] with a different and simpler technique. We use the method in [1] to give bounds for topological entropy on our subsystems which are all TBS. Clearly the same results will be obtained if one uses the method and formulas in [11]. The subsystems we have chosen are those whose alphabets are in $\mathbb{Z}\backslash\{0, \pm 1, \pm 2, ..., \pm k\}$ and those with alphabets in $\mathbb{N}\backslash\{1, 2, ..., k\}$, $k \geq 2$. Recall that geometric codes for modular surface M^{c1} has also $\mathbb{Z} \setminus \{0\}$ as its alphabets. Hence similar subsystems can be defined there as well. For M^{c1}, bounds for entropies of subsystems with alphabets in $\mathbb{Z}\backslash\{0, \pm 1, \pm 2\}$, $\mathbb{N}\backslash\{1, 2\}$ and $\mathbb{Z}\backslash\{0, \pm 1\}$ which satisfy Katok's criterion has been reported in [9, 5, 1]. Hence our results is for a class of subsystems not just some individual examples. See also Remark 2.4

First we recall a general theorem for the quantity of topological entropy of the action of the geometrically finite Fuchsian groups on $\mathcal{H}$ which implies the geodesic flows on both M^{c1} and M^{c2} have entropies equal to 1.

Theorem 2.1. *[5, Theorem 12]. The topological entropy of geodesic flow on a quotient of $\mathcal{H}$ by a geometrically finite Fuchsian group of the first kind is equal to 1.*

Our computations are done via special flows conjugated to our subsystems. To define this special flow, let (Σ, σ) be a subsystem of geodesic codes and let $\ell(x)$ be the length of geodesic between two successive hits of s. Set $Y(G) = \{(x, t) : x \in \Sigma, 0 \leq t \leq \ell(x)\}$ with the points $(x, \ell(x))$ and $(\sigma(x), 0)$ identified. Then for $0 \leq s, s+t \leq \ell(x)$, set $T^s_{\ell, \Sigma}(x, t) = (x, s+t)$. Define the family $T_{\ell, \Sigma} = \{T^s_{\ell, \Sigma}\}_{s \in \mathbb{R}}$ to be the *special flow* constructed over the *base space* Σ and *height function* ℓ. Geodesic flow on M^{c2} is conjugate to this family of special flow and the same can be formulated for any subsystem.

For $k \in \mathbb{N} \setminus \{1\}$, let $\mathcal{A}_k = \{n : |n| \geq k\}$ and $\mathcal{A}^+_k = \{n : n \geq k\}$. Denote by $h(T_{\ell, \Sigma_{\mathcal{A}_k}})$ (respectively $h(T_{\ell, \Sigma_{\mathcal{A}^+_k}})$) the topological entropy of $T_{\ell, \Sigma_{\mathcal{A}_k}}$ (respectively $T_{\ell, \Sigma_{\mathcal{A}^+_k}}$).

Theorem 2.2. *For $k \in \mathbb{N} \setminus \{1\}$, let $\Sigma_{\mathcal{A}_k}$, $\Sigma_{\mathcal{A}^+_k}$, $T_{\ell, \Sigma_{\mathcal{A}_k}}$ and $T_{\ell, \Sigma_{\mathcal{A}^+_k}}$ be as before and let $\zeta(\cdot)$ be the Riemann zeta function. Then $x_l < h(T_{\ell, \Sigma_{\mathcal{A}_k}}) < x_u$ where for $\alpha \in \{l, u\}$, x_α is the unique solution of*

$$2c_\alpha^{-2x}\left(\zeta(2x) - \sum_{n=1}^{k-1} \frac{1}{n^{2x}} + \frac{1}{k^{2x}}\right) = 1. \tag{4}$$

Here $c_u = 2 - \frac{1}{k[2k]} = 2 - \frac{1}{k(k+\sqrt{k^2-1})}$ and $c_l = 2 + \frac{1}{k[2k]}$.

Also, $x_l < h(T_{\ell,\Sigma_{\mathcal{A}_k^+}}) < x_u$ *where for* $\alpha \in \{l, u\}$*,* x_α *is the unique solution of*

$$c_\alpha^{-2x}\left(\zeta(2x) - \sum_{n=1}^{k-1}\frac{1}{n^{2x}} + \frac{2}{k^{2x}}\right) = 1. \tag{5}$$

Here again $c_u = 2 - \frac{1}{k[2k]}$*, but* $c_l = 2$*.*

Example 2.3. If $k = 2$ or 3, then the bounds for the entropy of geodesic flow on M^{c2} are $0.7491 < h(T_{\ell,\Sigma_{\mathcal{A}_2}}) < 0.9330$ and $0.7218 < h(T_{\ell,\Sigma_{\mathcal{A}_3}}) < 0.7994$ respectively. Also, for positive geodesic flow on M^{c2}, we have $0.7137 < h(T_{\ell,\Sigma_{\mathcal{A}_2^+}}) < 0.8041$ and $0.6736 < h(T_{\ell,\Sigma_{\mathcal{A}_3^+}}) < 0.7077$. We used the computer algebra software Maple to perform our computations.

In fact our bounds for entropy stems out from the bounds for c_α.

The reported bounds for entropies for M^{c1} are those satisfying Katok's criterion. If this criterion is satisfied, the entropy is greater than 0.8417 when codes are in $\mathbb{Z}\backslash\{0, \pm1, \pm2\}$ [8]; it is between 0.7771 and 0.8161 when codes are in $\mathbb{N}\backslash\{1, 2\}$ [5] and it is greater than 0.8665 when codes are in $\mathbb{Z}\backslash\{0, \pm1\}$ [1].

Remark 2.4. Formulas similar to (4) and (5) can be derived for other subsystems; in particular, for the subsystems whose codes are in $\mathbb{Z}\setminus\mathcal{A}$ where $\mathcal{A}\subset\mathbb{Z}$ is finite and contains zero. Note that in practice the main task is to be able to evaluate c_α and this can be done, as we will do later, when the height function depends only on its zero coordinate.

To prove Theorem 2.2, we need to determine explicitly the height function of the special flow, that is $\ell(x)$.

Theorem 2.5. *Let* $x = [\gamma]$ *be the geometric code of* γ *with repelling and attracting points* $w = w(x)$ *and* $u = u(x)$ *respectively. Then* $\ell(x) = 2\ln(w(x)) + \ln(g(x)) - \ln(g(\sigma x))$ *where* $g(x) = \frac{(w(x)-u(x))\sqrt{w(x)^2-1}}{w(x)^2\sqrt{1-u(x)^2}}$*.*

Proof. With almost no change, the lines of proof is similar to the proof of [5, Theorem 4]. Just let z_1 and z_1' be the intersection of $\gamma = (w, u)$ with $|z| = 1$ and $|z - 2n_1| = 1$ respectively and $z_2 = ST^{-n_1}z_1'$. Then the same computations in [5] imply that the distance between z_1 and z_1' is equal to $2\ln(w(x)) + \ln(g(x)) - \ln(g(\sigma x))$ where $g(x) = \frac{(w(x)-u(x))\sqrt{w(x)^2-1}}{w(x)^2\sqrt{1-u(x)^2}}$. □

Let $\Sigma' \subseteq \Sigma$ and let $f_1, f_2 : \Sigma' \to \Sigma'$. Then f_1 and f_2 are called *cohomologous*, if there exists a function $h : \Sigma' \to \mathbb{R}$ such that $f_1(x) = f_2(x) + h(x) - h(\sigma(x))$. When f_1 and f_2 are height functions, then the special flows $T_{f_1,\Sigma'}$

and $T_{f_2,\Sigma'}$ are conjugate and therefore have the same topological entropy [10]. By Theorem 2.5, $\ell(x)$ is cohomologous to $f(x) = 2\ln(w(x))$.

When the height function depends only on its zero coordinate, an estimate for entropy of special flow can be obtained which we will briefly explain here. For any subsystem $\Sigma_{\mathcal{B}}$ of Σ denote the positive continued functions like $f(x)$ depending on the zero coordinate and satisfying the condition $\sum_{k=1}^{\infty} f(\sigma^k(x)) = \sum_{k=1}^{\infty} f(\sigma^{-k}(x)) = \infty$ by $\mathcal{F}_0(\Sigma_{\mathcal{B}})$.

Let H be a directed graph with vertex set $V = \mathcal{A}$ and the edge set $E = \{(v, w) : v, w \in \mathcal{A}\}$. A path τ with length n in H from v_0 to v_n is a sequence $\tau = (v_0, ..., v_n)$ of vertices in $V(H)$. The path $\tau = (v_0, ..., v_n)$ is called a *simple v-cycle* if $v_0 = v_n = v$ and $v_i \neq v$ for $1 \leq i \leq n-1$. Let $C(H; v)$ be the set of all simple v-cycles in the graph H. Let $f \in \mathcal{F}_0(\Sigma_{\mathcal{B}})$ and $F_{f,V}(x) = \sum_{v \in V} x^{f(v)}$ be a series for $x \geq 0$ and set

$$\phi_{H,f,w}(x) = \sum_{\tau \in C(H;w)} x^{f^*(\tau)}, \qquad x \geq 0 \tag{6}$$

be the *generating function* with respect to the special flow $T_{f,\Sigma}$ where $f^*(\tau) = \sum_{i=0}^{n} f(v_i)$, $\tau = (v_0, ..., v_n)$.

Remark 2.6. Let $(\Sigma_{\mathcal{B}}, \sigma)$ be a 1-step topological Markov chain and $f \in \mathcal{F}_0(\Sigma_{\mathcal{B}})$. Then by [1, Remark 1], $h(T_{f,\Sigma}) = -\ln(\hat{x}_f)$ where $\hat{x}_f$ is either the unique solution of $\phi_{H,f,v}(x) = 1$ or $\hat{x}_f = r(\phi_{H,f,v})$. Here H is the graph associated to $\Sigma_{\mathcal{B}}$ and $v \in V(H)$.

Lemma 2.7. *Let $c > 1$ and $f(x) = 2\ln(cn_0)$, $|n_0| \geq k \geq 2$. Let $\alpha \in \{\mathcal{A}_k, \mathcal{A}_k^+\}$. Then $h(T_{f,\Sigma_\alpha}) = -\hat{x}_f^\alpha$ where $\hat{x}_f^\alpha$ is the unique solution of $\phi_{H_\alpha,f,v_k}(x) = 1$.*

Proof. Let $\alpha = \mathcal{A}_k$. Since $f(x) = 2\ln(cn_0)$ and $|n_0| \geq k$ so $f \in \mathcal{F}_0(\Sigma_{\mathcal{A}_k})$. Let $H_k := H_{\mathcal{A}_k}$ be a complete graph with vertex set $V(H_k) = \mathcal{A}_k$ and edge set $E(H_k)$. Set $V_0 = \{v_k\}$ and $V_1 = \mathcal{A}_k - \{v_k\}$. Define a new complete graph P_k with vertex set $\{V_0, V_1\}$ and edge set $E(P_k)$. Set $\alpha_i(x) = \sum_{v \in V_i} x^{f(v)}$ and $\alpha_{ij}(x) = \alpha_i(x)$ if $(V_i, V_j) \in E(P_k)$ and zero otherwise for $i = 1, 2$. Then we may apply [1, Lemma 1] for $m = 1$ to have a series

$$A_1(x) = \alpha_{10}(x) + \alpha_{11}(x)A_1(x)$$

and a matrix $M(x) = \begin{pmatrix} \alpha_{11}(x) - 1 & \alpha_{12}(x) \\ \alpha_{21}(x) & \alpha_{22}(x) - 1 \end{pmatrix}$. Now the generating function for the flow $T_{f,\Sigma_{\mathcal{A}_k}}$ is $\phi_{H_k,f,v_k}(x) = \alpha_{00}(x) + \alpha_{10}(x)A_1(x)$. This implies $r(\phi_{H_k,f,v_k})$, the radius of convergent of $\phi_{H_k,f,v_k}(x)$, is equal to $r(A_1) \leq r(F_{f,V(H_k)})$. Since $f \in \mathcal{F}_0(\Sigma_{\mathcal{A}_k})$ then by Remark 2.6, $h(T_{f,\Sigma_{\mathcal{A}_k}}) = -\ln(\hat{x}_f)$ where $\hat{x}_f$ $(= \hat{x}_f^\alpha))$ is either the unique solution of $\phi_{H_k,f,v_k}(x) = 1$ or $\hat{x}_f =$

$r(\phi_{H_k,f,v_k}) = r(A_1)$. We want to show that for our case $\hat{x}_f$ is the unique solution of $\phi_{H_k,f,v_k}(x) = 1$.

For $0 \le x < r(F_{f,V(H_k)})$ set

$$\tilde{x}_0 = \begin{cases} r(F_{f,V(H_k)}), & \text{if } M(x) \text{ is invertible} \\ \inf\{x : 0 \le x < r(F_{f,V(H_k)}), \det M(x) = 0\}, & \text{otherwise.} \end{cases} \tag{7}$$

From [1, Theorem 2] we have $r(\phi_{H_k,f,v_k}) = \tilde{x}_0$ and if $\tilde{x}_0 < r(F_{f,V(H_k)})$, then $\lim_{x\to\tilde{x}_0^-} \phi_{H_k,f,v_k}(x) = \infty$ which means $\phi_{H_k,f,v_k}(x) = 1$ has a solution in $0 < x < r(F_{f,V(H_k)})$. We will show that this is indeed the case. We achieve this if $\det M(x) = 0$ in $0 < x < r(F_{f,V(H_k)})$. But for $T_{f,\Sigma_{\mathcal{A}_k}}$,

$$\det M(x) = 1 - F_{f,V(H_k)}(x) = 1 - \sum_{v\in\mathcal{A}_k} x^{f(v)} = 1 - 2\sum_{n=k}^{\infty} x^{2\ln cn}.$$

So $1 = 2c^{2\ln x}\sum_{n=k}^{\infty} n^{2\ln x}$. By setting $\ln\frac{1}{x} = s$, we have

$$\frac{c^{2s}}{2} = \sum_{n=k}^{\infty}\frac{1}{n^{2s}}. \tag{8}$$

But the series is convergent for $s > \frac{1}{2}$ and decreases strictly on $\frac{1}{2} < s < \infty$ from ∞ to zero. Since $c > 1$, $\frac{c^{2s}}{2}$ is greater than $\frac{1}{2}$ on $s = \frac{1}{2}$ and increases to infinity on $\frac{1}{2} < s < \infty$. So (8) has a unique solution on $\frac{1}{2} < s < \infty$ or $M(x)$ has a unique solution on $0 < x < \frac{1}{\sqrt{e}}$.

Now let $\alpha = \mathcal{A}_k^+$ and let H_k be the associated complete graph. Then $\det M(x) = 1 - F_{f,V(H_k)}(x) = 1 - \sum_{v\in\mathcal{A}_k} x^{f(v)} = 1 - \sum_{n=k}^{\infty} x^{2\ln cn}$. Therefore in this situation, (8) turns to $c^{2s} = \sum_{n=k}^{\infty}\frac{1}{n^{2s}}$ and by a similar discussion has a unique solution on $0 < x < \frac{1}{\sqrt{e}}$. □

Proof of Theorem 2.2. Recall that if two height functions f_1 and f_2 on $\Sigma' \subseteq \Sigma$ are cohomologus, then they have the same topological entropy. So applying Theorem 2.5, it suffices to let the height function to be $f(x) = 2\ln(w(x))$ for $x = (..., n_0, n_1, ...)$. But

$$c_u|n_0| = |2n_0| - \frac{1}{[2k]} \le |w(x)| = |2n_0 - \cfrac{1}{2n_1 - \cfrac{1}{2n_2 - \cfrac{1}{\ddots}}}| \le |2n_0| + \frac{1}{[2k]} = c_l|n_0|, \tag{9}$$

where $c_l = 2 + \frac{1}{k[2k]}$ and $c_u = 2 - \frac{1}{k[2k]}$. Let $f_\alpha(x) = 2\ln c_\alpha|n_0|$ where $\alpha \in \{l, u\}$. Then by Abramov formula, $h(T_{f_l,\Sigma_{\mathcal{A}_k}}) \le h(T_{f,\Sigma_{\mathcal{A}_k}}) \le h(T_{f_u,\Sigma_{\mathcal{A}_k}})$. We give the proof for the lower bound and for the upper bound, it follows similarly.

Since $(\Sigma_{\mathcal{A}_k}, \sigma)$ is a TBS,

$$\phi_{H_k,f,v_k}(x) = \frac{x^{f(v)}}{1 - x^{f(v)} - F_{f,V(H_k)}(x)},$$

when $1-x^{f(v)}-F_{f,V(H_k)}(x) > 0$ [11] and H_k is the complete graph introduced in the proof of Lemma 2.7. See also [1, Remark 2].

By the above lemma, $\hat{x}_l$ is the unique solution of $\phi_{H_k,f_l,v_k}(x) = 1$ or equivalently it is the unique solution of

$$\sum_{n\in\mathcal{A}_k} x^{2\ln(c_l n)} + 2x^{2\ln(c_l k)} = 1, \quad 0 < x < 1. \tag{10}$$

But

$$\sum_{n\in\mathcal{A}_k} x^{2\ln(c_l n)} = 2\sum_{n=k}^{\infty} x^{2\ln(c_l n)} = 2c_l^{2\ln x}\left(\zeta(-2\ln x) - \sum_{n=1}^{k-1} n^{2\ln x}\right).$$

Where $\zeta(.)$ is the Riemann zeta function. Since by Remark 2.6, the entropy equals $-2\ln\hat{x}_l$ where $\hat{x}_l$ is the solution of (10), so by letting $x_l = -\ln\hat{x}_l$, we have x_l is the solution of

$$2c_l^{-2x}\left(\zeta(2x) - \sum_{n=1}^{k-1}\frac{1}{n^{2x}} + \frac{1}{k^{2x}}\right) = 1.$$

The proof for the $h(T_{\ell,\Sigma_{\mathcal{A}_k^+}})$ is similar with a minor change. We only need to use the relation

$$c_u|n_0| = |2n_0| - \frac{1}{\lceil 2k\rceil} \leq |w(x)| = |2n_0 - \cfrac{1}{2n_1 - \cfrac{1}{2n_2 - \cfrac{1}{\ddots}}}| = 2|n_0| = c_l|n_0|,$$

instead of (9). □

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