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Approximate multipliers and approximate double centralizers: A fixed point approach

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Abstract

In the present paper, the Hyers-Ulam stability and also the superstability of double centralizers and multipliers on Banach algebras are established by using a fixed point method. With this method, the condition of without order on Banach algebras is no longer necessary.

1 Introduction

The concept of the stability and the superstability for Banach algebra has been a main stream in the theory of Banach algebras in the last decades. A functional equation is called *stable* if any approximately solution to the functional equation is near to a true solution of that functional equation, and is *superstable* if every approximately solution is an exact solution of it.

In 1940, Ulam [21] proposed the following question concerning the stability of group homomorphisms: *under what condition does there exist an additive mapping near an approximately additive mapping?* Hyers [13] answered the problem of Ulam for the case where X and Y are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mapping was given by Th. M. Rassias [19]. Since then, the stability problems of various functional equation have been extensively investigated by a number of authors (for instances, [6], [7], [10], and [14]).

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In 2003, Cădariu and Radu [3] applied the fixed point method to the investigation of the Jensen functional equation (see [2, 4, 8, 9] for more applications of this method). They presented a short and a simple proof (different from the "*direct method*", initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation [18], for the Cauchy functional equation [4] and for the quadratic functional equation [3].

Let $\mathcal{A}$ be a non-unital Banach algebra. Then $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbb{C}$ is a unital Banach algebra such that $\mathcal{A}$ is a closed subalgebra of $\mathcal{A}^{\#}$. In fact $\mathcal{A}^{\#}$ is the smallest unitization of $\mathcal{A}$. Also there are other unitizations for Banach algebras. For instance, the multiplier of $\mathcal{A}$, $\mathcal{M}(\mathcal{A})$ is one of them. However, $\mathcal{M}(\mathcal{A})$ is very much bigger than $\mathcal{A}^{\#}$.

The concept of the multipliers of Banach algebras were defined by Helgason in [11]. Later, Wang in [22] studied the multipliers on commutative Banach algebras. For some non-unital Banach algebras, their multipliers are computed. If X is a locally compact Housdorff space, then $\mathcal{M}(C_0(X)) = C_b(X)$, where $C_0(X)$ is Banach algebra (C^*-algebra) of continuous functions on X which vanish at infinity, and $C_b(X)$ is Banach algebra of all bounded continuous complex-valued functions on X. For Hilbert space $\mathcal{H}$, the multiplier of the compact operators on $\mathcal{H}$ is the bounded operators on $\mathcal{H}$.

Let $\mathcal{A}$ be an algebra. Recall that $A_l(\mathcal{A}) := \{a \in \mathcal{A} : a\mathcal{A} = \{0\}\}$ is the left annihilator ideal and $A_r(\mathcal{A}) := \{a \in \mathcal{A} : \mathcal{A}a = \{0\}\}$ is the right annihilator ideal on $\mathcal{A}$. We say a Banach algebra $\mathcal{A}$ is (*strongly*) *without order* if $A_l(\mathcal{A}) = A_r(\mathcal{A}) = \{0\}$. Obviously, a Banach algebra is strongly without order when $\mathcal{A}$ is unital or approximately unital.

Miura, Hirasawa and Takasaki in [16, Theorem 1.3] investigated the stability of multipliers on Banach algebras, and showed that every approximately multiplier on a Banach algebra can be approximated by a multiplier. They also proved the superstability multipliers with the condition of without order on Banach algebras. On the other hand, the notion of double centralizer was introduced by Hochschild [12] and Johnson [15] independently. The stability and the superstability of double centralizers of a Banach algebra $\mathcal{A}$ which is (strongly) without order is investigated in [17].

In this paper, we remove the condition of without order on Banach algebras. In other words, we show that the hypothesis on Banach algebras being without order in [16, 17] can be eliminated, and establish the stability and the superstability of double centralizers and multipliers on a Banach algebra by a method of the fixed point.

2 Stability of double centralizers

Before proceeding to the main results, we will state the following theorem which is useful to our purpose (an extension of the result was given in [20]).

Theorem 2.1. (*The fixed point alternative [5]*) *Let* (Ω, d) *be a complete generalized metric space and* $\mathfrak{J} : \Omega \to \Omega$ *be a mapping with Lipschitz constant* $L < 1$. *Then, for each element* $x \in \Omega$, *either* $d(\mathfrak{J}^n x, \mathfrak{J}^{n+1}x) = \infty$ *for all* $n \geq 0$, *or there exists a natural number* n_0 *such that:*

(i) $d(\mathfrak{J}^n x, \mathfrak{J}^{n+1}x) < \infty$ *for all* $n \geq n_0$*;*

(ii) *the sequence* $\{\mathfrak{J}^n x\}$ *is convergent to a fixed point* y^* *of* $\mathfrak{J}$*;*

(iii) y^* *is the unique fixed point of* $\mathfrak{J}$ *in the set*

$$\Lambda = \{y \in \Omega : d(\mathfrak{J}^{n_0}x, y) < \infty\};$$

(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathfrak{J}y)$ *for all* $y \in \Lambda$.

Throughout this paper, we assume that A is a complex Banach algebra and denote $\overbrace{A \times A \times ... \times A}^{n-times}$ by A^n. A linear mapping $L : A \longrightarrow A$ is said to be *left centralizer* on A if $L(ab) = L(a)b$ for all $a, b \in A$. Similarly, a linear mapping $R : A \longrightarrow A$ satisfying $R(ab) = aR(b)$ for all $a, b \in A$ is called *right centralizer* on A. A *double centralizer* on A is a pair (L, R), where L is a left centralizer, R is a right centralizer and $aL(b) = R(a)b$ for all $a, b \in A$. For example, (L_c, R_c) is a double centralizer, where $L_c(a) := ca$ and $R_c(a) := ac$. The set $D(A)$ of all double centralizers equipped with the multiplication $(L_1, R_1) \cdot (L_2, R_2) = (L_1L_2, R_1R_2)$ is an algebra.

A mapping $T : A \longrightarrow A$ is said to be a *multiplier* if $aT(b) = T(a)b$ for all $a, b \in A$. Clearly, if $A_l(A) = \{0\}$ ($A_r(A) = \{0\}$, respectively) then T is a left (right) centralizer. For all $a, b \in A$, we put $a^0 - b^0 = 0, a^0 b = b$. We establish the Hyers-Ulam stability of double centralizers as follows:

Theorem 2.2. *Let* $f_i : A \to A$ *be mappings with* $f_i(0) = 0$ $(i = 0, 1)$, *and let* $\varphi : A^6 \to [0, \infty)$ *be a function such that*

$$\|f_i(\mu x + y + zw) - \mu f_i(x) - f_i(y) - [(1-i)(f_i(z)w)^{1-i} + i(zf_i(w))^i]$$

$$-sf_0(t) + f_1(s)t\| \leq \varphi(x, y, z, w, t, s) \tag{1}$$

for all $\mu \in \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *and all* $x, y, z, w, s, t \in A$, $i = 0, 1$. *If there exists a constant* $K \in (0, 1)$ *such that*

$$\varphi(2x, 2y, 2z, 2w, 2s, 2t) \leq 2K\varphi(x, y, z, w, s, t) \tag{2}$$

for all $x, y, z, w, s, t \in A$, then there exists a unique double centralizer (L, R) on A satisfying

$$\|f_0(x) - L(x)\| \leq \frac{1}{2(1-K)}\varphi(x, x, 0, 0, 0, 0) \tag{3}$$

and

$$\|f_1(x) - R(x)\| \leq \frac{1}{2(1-K)}\varphi(x, x, 0, 0, 0, 0) \tag{4}$$

for all $x \in A$.

Proof. We consider the set $X = \{h : A \longrightarrow A | h(0) = 0\}$ and introduce the generalized metric on X as follows:

$$d(h_1, h_2) := inf\{C \in (0, \infty) : \|h_1(x) - h_2(x)\| \leq C\varphi(x, x, 0, 0, 0, 0), \quad \forall x \in A\},$$

if there exist such constant C, and $d(h_1, h_2) = \infty$, otherwise. Similar to the proof of [1, Theorem 2.2], we can show that d is a generalized metric on X and the metric space (X, d) is complete. We define a mapping $T : X \longrightarrow X$ via

$$Th(x) = \frac{1}{2}h(2x) \tag{5}$$

for all $x \in A$. First, we show that T is strictly contractive on X. Given $h_1, h_2 \in X$, let $C \in (0, \infty)$ be an arbitrary constant with $d(h_1, h_2) \leq C$, i.e.,

$$\|h_1(x) - h_2(x)\| \leq C\varphi(x, x, 0, 0, 0, 0) \tag{6}$$

for all $x \in A$. If we substitute x in the inequality (6) by $2x$ and make use of (2) and (5), then we have

$$\begin{aligned}\|Th_1(x) - Th_2(x)\| &= \frac{1}{2}\|h_1(2x) - h_2(2x)\| \\ &\leq \frac{1}{2}C\varphi(2x, 2x, 0, 0, 0, 0) \\ &\leq CK\varphi(x, x, 0, 0, 0, 0)\end{aligned}$$

for all $x \in A$. Then $d(Th_1, Th_2) \leq CK$. Hence we conclude that

$$d(Th_1, Th_2) \leq Kd(h_1, h_2)$$

for all $h_1, h_2 \in X$. Hence, T is a strictly contractive mapping on X with a Lipschitz constant K. Now, we prove that $d(Tf_0, f_0) < \infty$. Putting $i = 0, \mu = 1, x = y, z = w = t = s = 0$ in (1), we obtain

$$\|f_0(2x) - 2f_0(x)\| \leq \varphi(x, x, 0, 0, 0, 0)$$

for all $x \in A$. Hence

$$\|\frac{1}{2}f_0(2x) - f_0(x)\| \leq \frac{1}{2}\varphi(x, x, 0, 0, 0, 0) \tag{7}$$

for all $x \in A$. It follows from (7) that $d(Tf_0, f_0) \leq \frac{1}{2}$. By Theorem 2.1, there exists a unique mapping $L : A \to A$ such that L is a fixed point of T and that $T^n f_0 \to L$, i.e.,

$$\lim_{n\to\infty} \frac{f_0(2^n x)}{2^n} = L(x) \tag{8}$$

for all $x \in A$, and so

$$d(f_0, L) \leq \frac{1}{1-K} d(Tf_0, f_0) \leq \frac{1}{2(1-K)}.$$

In fact, the inequality (3) is true for all $x \in A$. It follows from (2) that

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n w, 2^n s, 2^n t)}{2^n} = 0. \tag{9}$$

Now, replace $2^n x$ and $2^n y$ by x and y respectively, and put $i = 0$, $z = w = t = s = 0$ in (1). If we divide both sides of the resulting inequality by 2^n, and letting n tend to infinity, then the equalities (8) and (9) imply that

$$L(\mu x + y) = \mu L(x) + L(y)$$

for all $x, y \in A$ and all $\mu \in \mathbb{T}$. Now assume that $\mu \in \mathbb{C}$ and $\mu = \mu_1 + i\mu_2$, where μ_j $(j = 1, 2)$ are real numbers. Let $\mu_1 = \alpha_1 + \beta_1$ such that α_1 is the integer part of μ_1 and $0 \leq \beta_1 < 1$. Easily, we can write $\beta_1 = \frac{\beta_{1,1}+\beta_{1,2}}{2}$, where $\beta_{1,1}, \beta_{1,2} \in \mathbb{T}$. We have

$$L(\mu_1 x) = L(\alpha_1 x + \beta_1 x) = \alpha_1 L(x) + \frac{\beta_{1,1} + \beta_{1,2}}{2} L(x) = \mu_1 L(x).$$

Similarly, we have $L(\mu_2 x) = \mu_2 L(x)$. Thus L is $\mathbb{C}$-linear. We may also show from (1) that $L(xy) = L(x)y$, and so it is a left centralizer of A. According to the above argument, one can show that there exists a unique mapping $R : A \to A$ which is a fixed point of T such that

$$\lim_{n\to\infty} \frac{f_1(2^n x)}{2^n} = R(x) \tag{10}$$

for all $x \in A$. Indeed, R belongs to the set $\{h \in X, d(Tf_1, h) < \infty\}$. Also, it follows from (2) that

$$\lim_{n\to\infty} \frac{\varphi(0, 0, 0, 0, 2^n s, 2^n t)}{2^n} = 0 \tag{11}$$

for all $s, t \in A$. If we put $x = y = z = w = 0$ and substitute s and t by $2^n s$ and $2^n t$ in (1) respectively and we divide the both sides of the obtained inequality by 4^n, then we get

$$\|s\frac{f_0(2^n t)}{2^n} - \frac{f_1(2^n s)}{2^n}t\| \leq \frac{\varphi(0,0,0,0,2^n s, 2^n t)}{4^n}.$$

Passing to the limit as $n \to \infty$ and from (11), we conclude that $sL(t) = R(s)t$, for all $s, t \in A$. □

Corollary 2.3. *Let $r \in (0,1), \theta$ be a non-negative real number and let $f_i : A \to A$ be mappings with $f_i(0) = 0$ $(i = 0, 1)$ such that*

$$\|f_i(\mu x + y + zw) - \mu f_i(x) - f_i(y) - [(1-i)(f_i(z)w)^{1-i} + i(zf_i(w))^i]$$

$$-sf_0(t) + f_1(s)t\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r + \|s\|^r + \|t\|^r)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z, w, r, s \in A$. Then there exists a unique double centralizer (L, R) on A satisfying

$$\|f_0(x) - L(x)\| \leq \frac{\theta}{2 - 2^r}\|x\|^r$$

and

$$\|f_1(x) - R(x)\| \leq \frac{\theta}{2 - 2^r}\|x\|^r$$

for all $x, y \in A$.

Proof. The result follows immediately from Theorem 2.2 by taking

$$\varphi(x, y, z, w, s, t) = \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r + \|s\|^r + \|t\|^r)$$

for all $x, y, z, w, s, t \in A$ and by letting $K = 2^{r-1}$. □

In the following corollary, we show that if f_1, f_2 are additive mappings, then the superstability for the inequality (1) is valid.

Corollary 2.4. *Suppose that additive mappings $f_0, f_1 : A \to A$ satisfy (1) and a function $\varphi : A^6 \to [0, \infty)$ satisfies (2). Then (f_0, f_1) is a double centralizer.*

Proof. Since f_i is additive, $f_i(0) = 0$ for $i = 0, 1$. On the other hand, we have $f_i(2^n x) = 2^n f_i(x)$ for all $x \in A$ and $i = 0, 1$. By Theorem 2.2, we have $(f_0, f_1) = (L, R)$ is a double centralizer. □

Corollary 2.5. *Let p_j, θ be positive real numbers $(1 \leq j \leq 6)$ with $\sum_{j=1}^{6} p_j \neq 1$, and let $f_i : A \to A$ be mappings with $f_i(0) = 0$ $(i = 0, 1)$ such that*

$$\|f_i(\mu x + y + zw) - \mu f_i(x) - f_i(y) - [(1-i)(f_i(z)w)^{1-i} + i(zf_i(w))^i]$$

$$-sf_0(t) + f_1(s)t\| \leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3}\|w\|^{p_4}\|s\|^{p_5}\|t\|^{p_6}) \qquad (12)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z, w, r, s \in A$. Then (f_0, f_1) is a double centralizer.

Proof. Putting $x = y = z = w = s = t = 0$ in (12), we get $f_i(0) = 0$ for $i = 0, 1$. Now, if we put $x = y$, $z = w = s = t = 0$ and $\mu = 1$ in (12), then we have $f_i(2x) = 2f_i(x)$ for all $x \in A$. It is easy to see by induction that $f_i(2^n x) = 2^n f_i(x)$, and so $f_i(x) = \frac{f_i(2^n x)}{2^n}$ for all $x \in A$ and $n \in \mathbb{N}$. It follows from the proof of Theorem 2.2 that (f_0, f_1) is a double centralizer on A. $\square$

3 Stability of multipliers

In this section, we investigate the Hyers-Ulam stability and the superstability of multipliers.

Theorem 3.1. *Let $f : A \to A$ be a mapping with $f(0) = 0$ and let $\phi : A^4 \to [0, \infty)$ be a function such that*

$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y) - f(z)w + zf(w)\| \leq \phi(x, y, z, w) \qquad (13)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z, w \in A$. If there exists a constant $K \in (0, 1)$ such that

$$\phi(2x, 2y, 2z, 2w) \leq 2K\phi(x, y, z, w) \qquad (14)$$

for all $x, y, z, w \in A$, then there exists a unique multiplier T on A satisfying

$$\|f(x) - T(x)\| \leq \frac{1}{2(1-K)}\phi(x, x, 0, 0) \qquad (15)$$

for all $x \in A$.

Proof. It follows from $\phi(2x, 2y, 2z, 2w) \leq 2K\phi(x, y, z, w)$ that

$$lim_{n\to\infty}\frac{\phi(2^n x, 2^n y, 2^n z, 2^n w)}{2^n} = 0 \qquad (16)$$

for all $x, y, z, w \in A$. Putting $\mu = 1, x = y$ and $z = w = 0$ in (13), we obtain

$$\|f(2x) - 2f(x)\| \leq \phi(x, x, 0, 0)$$

for all $x \in A$. So

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2}\phi(x, x, 0, 0) \tag{17}$$

for all $x \in A$. Consider the set $X := \{h : A \to A \mid h(0) = 0\}$ and introduce the generalized metric on X:

$$d(h_1, h_2) := inf\{C \in \mathbb{R}^+ : \|h_1(x) - h_2(x)\| \leq C\phi(x, x, 0, 0) \text{ for all } x \in A\},$$

if there exist such constant C, and $d(h_1, h_2) = \infty$, otherwise. It is easy to show that (X, d) is complete. We define a mapping $\Phi : X \to X$ by

$$\Phi h(x) = \frac{1}{2}h(2x)$$

for all $x \in A$. By the same reasoning as in the proof of Theorem 2.2, Φ is strictly contractive on X. It follows from (17) that

$$d(\Phi f, f) \leq \frac{1}{2}.$$

By Theorem 2.1, Φ has a unique fixed point in the set $X_1 := \{h \in X : d(f, h) < \infty\}$. Let T be the fixed point of Φ. Then T is the unique mapping with

$$T(2x) = 2T(x)$$

for all $x \in A$ such that there exists $C \in (0, \infty)$ such that

$$\|T(x) - f(x)\| \leq K\phi(x, x, 0, 0)$$

for all $x \in A$. On the other hand, we have $lim_{n\to\infty} d(\Phi^n(f), h) = 0$. Thus

$$lim_{n\to\infty}\frac{1}{2^n}f(2^n x) = T(x) \tag{18}$$

for all $x \in A$. Hence

$$d(f, T) \leq \frac{1}{1-K}d(f, \Phi f) \leq \frac{1}{2(1-K)}. \tag{19}$$

This implies the inequality (15). It follows from (13), (16) and (18) that

$$\begin{aligned}\|T(x+y) - T(x) - T(y)\| &= lim_{n\to\infty}\frac{1}{2^n}\|f(2^n(x+y)) + f(2^n(x)) - f(2^n y)\| \\ &\leq lim_{n\to\infty}\frac{1}{2^n}\phi(2^n x, 2^n y, 0, 0) = 0\end{aligned}$$

for all $x, y \in A$. So
$$T(x+y) = T(x) + T(y)$$
for all $x, y \in A$. Thus T is Cauchy additive. Putting $y = x, z = w = 0$ in (13), we have
$$\|2\mu f(x) - f(2\mu x)\| \leq \phi(x, x, 0, 0)$$
for all $x \in A$. Hence
$$\begin{aligned}\|T(2\mu x) - 2\mu T(x)\| &= lim_{n\to\infty} \frac{1}{2^n}\|f(2\mu 2^n x) - 2\mu f(2^n x)\| \\ &\leq lim_{n\to\infty} \frac{1}{2^n}\phi(2^n x, 2^n x, 0, 0) = 0\end{aligned}$$
for all $\mu \in \mathbb{T}$ and $x \in A$. So $T(2\mu x) = 2\mu T(x)$ for all $\mu \in \mathbb{T}$ and $x \in A$. Since T is a additive map, $T(\mu x) = \mu T(x)$ for all $\mu \in \mathbb{T}$ and $x \in A$. The proof of Theorem 2.2 shows that T is $\mathbb{C}$-linear. If we substitute z and w by $2^n z$ and $2^n w$ in (13) respectively, and put $x = y = 0$ and we divide the both sides of the obtained inequality by 4^n, we get
$$\|z\frac{f(2^n w)}{2^n} - \frac{f(2^n z)}{2^n}w\| \leq \frac{\phi(0, 0, 2^n z, 2^n w)}{4^n}.$$

Passing to the limit as $n \to \infty$ and using (16), we conclude that $zT(w) = T(x)w$ for all $z, w \in A$. □

Corollary 3.2. *Let $r \in (0,1), \theta$ be non-negative real number and let $f : A \to A$ be a mapping with $f(0) = 0$ such that*
$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y) - f(z)w - zf(w)\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$$
for all $\mu \in \mathbb{T}$ and all $x, y, z, w \in A$. Then there exists a unique multiplier T on A satisfying
$$\|f(x) - T(x)\| \leq \frac{\theta}{2 - 2^r}\|x\|^r$$
for all $x \in A$.

Proof. We can deduce the desired result from Theorem 3.1 if we take
$$\phi(x, y, z, w) = \theta(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$$
for all $x, y, z, w \in A$. □

In analogy with corollaries 2.4 and 2.5, we have the following results which show that under what conditions the multipliers on Banach algebras are superstable.

Corollary 3.3. *Suppose that an additive mapping $f : A \to A$ satisfies (13) and a function $\phi : A^4 \to [0, \infty)$ satisfies (14). Then f is a multiplier on A.*

Proof. Since f is additive, $f(0) = 0$. On the other hand, we have $f(2^n x) = 2^n f(x)$ for all $x \in A$. By Theorem 3.1, f is a multiplier on A. □

Corollary 3.4. *Let p_j $(1 \leq j \leq 4), \theta$ be positive real numbers with $\sum_{j=1}^{4} p_j \neq 1$, and let $f : A \to A$ be a mapping such that*

$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y) - f(z)w - zf(w)\|$$

$$\leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3}\|w\|^{p_4}) \tag{20}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z, w \in A$. Then f is a multiplier on A.

Proof. If we put $x = y = z = w = 0$ in (20), we have $f(0) = 0$. Again, by letting $x = y$, $z = w = 0$ and $\mu = 1$ in (20), we get $f(2x) = 2f(x)$ for all $x \in A$. Similar to the proof of Corollary 2.5, one can obtain $f(x) = \frac{f(2^n x)}{2^n}$ for all $x \in A$ and $n \in \mathbb{N}$. Now, the proof of Theorem 3.1 shows that f is a multiplier on A. □

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