An. Şt. Univ. Ovidius Constanţa Vol. 20(3), 2012, 5–20

ON VARIABLE EXPONENT AMALGAM SPACES

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Abstract

We derive some of the basic properties of weighted variable exponent Lebesgue spaces $L_w^{p(.)}(\mathbb{R}^n)$ and investigate embeddings of these spaces under some conditions. Also a new family of Wiener amalgam spaces $W(L_w^{p(.)}, L_v^q)$ is defined, where the local component is a weighted variable exponent Lebesgue space $L_w^{p(.)}(\mathbb{R}^n)$ and the global component is a weighted Lebesgue space $L_v^q(\mathbb{R}^n)$. We investigate the properties of the spaces $W(L_w^{p(.)}, L_v^q)$. We also present new Hölder-type inequalities and embeddings for these spaces.

1 Introduction

A number of authors worked on amalgam spaces or some special cases of these spaces. The first appearance of amalgam spaces can be traced to N.Wiener [26]. But the first systematic study of these spaces was undertaken by F. Holland [18], [19]. The *amalgam* of L^p and l^q on the real line is the space $(L^p, l^q)(\mathbb{R})$ (or shortly (L^p, l^q)) consisting of functions f which are locally in L^p and have l^q behavior at infinity in the sense that the norms over $[n, n+1]$ form an l^q -sequence. For $1 \leq p, q \leq \infty$ the norm

$$\|f\|_{p,q} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty$$

Key Words: Variable exponent Lebesgue spaces, Amalgam spaces, embedding, Fourier transform
2010 Mathematics Subject Classification: Primary 46E30; Secondary 43A25.
Received: April, 2011.
Revised: April, 2011.
Accepted: January, 2012.

makes (L^p, l^q) into a Banach space. If $p = q$ then (L^p, l^q) reduces to L^p. A generalization of Wiener's definition was given by H.G. Feichtinger in [10], describing certain Banach spaces of functions (or measures, distributions) on locally compact groups by global behaviour of certain local properties of their elements. C. Heil [17] gave a good summary of results concerning amalgam spaces with global components being weighted $L^q(\mathbb{R})$ spaces. For a historical background of amalgams see [16]. The variable exponent Lebesgue spaces (or generalized Lebesgue spaces) $L^{p(\cdot)}$ appeared in literature for the first time already in a 1931 article by W. Orlicz [22]. The major study of this spaces was initiated by O. Kovacik and J. Rakosnik [20], where basic properties such as Banach space, reflexivity, separability, uniform convexity, Hölder inequalities and embeddings of type $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ were obtained in higher dimension Euclidean spaces. Also there are recent many interesting and important papers appeared in variable exponent Lebesgue spaces (see, [4], [5], [6] [8], [9]). The spaces $L^{p(\cdot)}$ and classical Lebesgue spaces L^p have many common properties, but a crucial difference between this spaces is that $L^{p(\cdot)}$ is not invariant under translation in general (Ex. 2.9 in [20] and Lemma 2.3 in [6]). Moreover , the Young theorem $\|f * g\|_{p(.)} \leq \|f\|_{p(.)} \|g\|_1$ is not valid for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. But the Young theorem was proved in a special form and derived more general statement in [25]. Aydın and Gürkanlı [3] defined the weighted variable Wiener amalgam spaces $W(L^{p(\cdot)}, L^q_w)$ where the local component is a variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the global component is a weighted Lebesgue space $L^q_w(\mathbb{R}^n)$. They proved new Hölder-type inequalities and embeddings for these spaces. They also showed that under some conditions the Hardy-Littlewood maximal function does not map the space $W(L^{p(\cdot)}, L^q_w)$ into itself.

Let $0 < \mu(\Omega) < \infty$. It is known that $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ if and only if $p(x) \leq q(x)$ for a.e. $x \in \Omega$ by Theorem 2.8 in [20]. This paper is concerned with embeddings properties of $L^{p(\cdot)}_w(\mathbb{R}^n)$ with respect to variable exponents and weight functions. We will discuss the continuous embedding $L^{p_2(\cdot)}_{w_2}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}_{w_1}(\mathbb{R}^n)$ under different conditions. We investigate the properties of the spaces $W(L^{p(\cdot)}_w, L^q_v)$. We also present new Hölder-type inequalities and embeddings for these spaces.

2 Definition and Preliminary Results

In this paper all sets and functions are Lebesgue measurable. The Lebesgue measure and the characteristic function of a set $A \subset \mathbb{R}^n$ will be denoted by $\mu(A)$ and χ_A, respectively. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two normed linear spaces and $X \subset Y$. $X \hookrightarrow Y$ means that X is a subspace of Y and the identity operator I from X into Y is continuous. This implies that there exists a constant $C > 0$ such that

$$\|u\|_Y \leq C\,\|u\|_X$$

for all $u \in X$.

The space $L^1_{loc}(\mathbb{R}^n)$ consists of all (classes of) measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$. It is a topological vector space with the family of seminorms $f \to \|f\chi_K\|_{L^1}$. A Banach function space (shortly BF-space) on $\mathbb{R}^n$ is a Banach space $(B, \|.\|_B)$ of measurable functions which is continosly embedded into $L^1_{loc}(\mathbb{R}^n)$, i.e. for any compact subset $K \subset \mathbb{R}^n$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_{L^1} \leq C_K\,\|f\|_B$ for all $f \in B$. A BF-space $(B, \|.\|_B)$ is called solid if $g \in L^1_{loc}(\mathbb{R}^n)$, $f \in B$ and $|g(x)| \leq |f(x)|$ almost everywhere (shortly a.e.) implies that $g \in B$ and $\|g\|_B \leq \|f\|_B$. A BF- space $(B, \|.\|_B)$ is solid iff it is a $L^\infty(\mathbb{R}^n)$-module. We denote by $C_c(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ the space of all continuos, complex-valued functions with compact support and the space of infinitely differentiable functions with compact support in $\mathbb{R}^n$ respectively. The character operator M_t is defined by $M_t f(y) = \langle y, t\rangle\, f(y)$, $y \in \mathbb{R}^n$,$t \in \mathbb{R}^n$. $(B, \|.\|_B)$ is strongly character invariant if $M_t B \subseteq B$ and $\|M_t f\|_B = \|f\|_B$ for all $f \in B$ and $t \in \mathbb{R}^n$.

We denote the family of all measurable functions $p : \mathbb{R}^n \to [1, \infty)$ (called the variable exponent on $\mathbb{R}^n$) by the symbol $\mathcal{P}(\mathbb{R}^n)$. For $p \in \mathcal{P}(\mathbb{R}^n)$ put

$$p_* = \underset{x\in\mathbb{R}^n}{\text{ess inf}} p(x), \qquad p^* = \underset{x\in\mathbb{R}^n}{\text{ess sup}} p(x).$$

For every measurable functions f on $\mathbb{R}^n$ we define the function

$$\varrho_p(f) = \int\limits_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx.$$

The function ϱ_p is a convex modular; that is, $\varrho_p(f) \geq 0$, $\varrho_p(f) = 0$ if and only if $f = 0$, $\varrho_p(-f) = \varrho_p(f)$ and ϱ_p is convex. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all $\mu-$measurable functions f on $\mathbb{R}^n$ such that $\varrho_p(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_p(\frac{f}{\lambda}) \leq 1\right\}.$$

If $p^* < \infty$, then $f \in L^{p(\cdot)}(\mathbb{R}^n)$ iff $\varrho_p(f) < \infty$. If $p(x) = p$ is a constant function, then the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$. The space $L^{p(\cdot)}(\mathbb{R}^n)$ is a particular case of the so-called Orlicz-Musielak space [20]. The function p always denotes a variable exponent and we assume that $p^* < \infty$.

Definition 2.1. Let w be a measurable, positive a.e. and locally $\mu-$integrable function on $\mathbb{R}^n$. Such functions are called weight functions. By a Beurling weight on $\mathbb{R}^n$ we mean a measurable and locally bounded function w on $\mathbb{R}^n$ satisfying $1 \leq w(x)$ and $w(x+y) \leq w(x)w(y)$ for all $x, y \in \mathbb{R}^n$. Let $1 \leq p < \infty$ be given. By the classical weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ we denote the set of all $\mu-$measurable functions f for which the norm

$$\|f\|_{p,w} = \|fw\|_p = \left(\int_{\mathbb{R}^n} |f(x)w(x)|^p \, dx \right)^{1/p} < \infty.$$

We say that $w_1 \prec w_2$ if and only if there exists a $C > 0$ such that $w_1(x) \leq Cw_2(x)$ for all $x \in \mathbb{R}^n$. Two weight functions are called equivalent and written $w_1 \approx w_2$, if $w_1 \prec w_2$ and $w_2 \prec w_1$ [13], [15].

Lemma 2.2. (a) A Beurling weight function w is also weight function in general.

(b) For each $p \in \mathcal{P}(\mathbb{R}^n)$, both $w^{p(.)}$ and $w^{-p(.)}$ are locally integrable.

Proof. **(a)** Let any compact subset $K \subset \mathbb{R}^n$ be given. Since w is locally bounded function, then we write

$$\sup_{x \in K} w(x) < \infty.$$

Hence

$$\int_K w(x)dx \leq \left(\sup_{x \in K} w(x) \right) \mu(K) < \infty.$$

(b) Since $w(x) \geq 1$, then

$$\int_K w(x)^{p(x)} dx \leq \int_K w(x)^{p^*} dx \leq \left(\sup_{x \in K} w(x)^{p^*} \right) \mu(K) < \infty.$$

Also $w(x) \neq 0$ and $w(x)^{-1} \leq 1$

$$\int_K w(x)^{-p(x)} dx \leq \int_K w(x)^{-p_*} dx \leq \left(\sup_{x \in K} w(x)^{-p_*} \right) \mu(K) < \infty.$$

□

Let w be a Beurling weight function on $\mathbb{R}^n$ and $p \in \mathcal{P}(\mathbb{R}^n)$. The weighted variable exponent Lebesgue space $L^{p(.)}_w(\mathbb{R}^n)$ is defined as the set of all measurable functions f, for which

$$\|f\|_{p(.),w} = \|fw\|_{p(.)} < \infty.$$

The space $\left(L_w^{p(.)}(\mathbb{R}^n), \|.\|_{p(.),w}\right)$ is a Banach space. Throughout this paper we assume that w is a Beurling weight.

Proposition 2.3. **(i)** The embeddings $L_w^{p(.)}(\mathbb{R}^n) \hookrightarrow L^{p(.)}(\mathbb{R}^n)$ is continous and the inequality

$$\|f\|_{p(.)} \leq \|f\|_{p(.),w}$$

is satisfied for all $f \in L_w^{p(.)}(\mathbb{R}^n)$.

(ii) $C_c(\mathbb{R}^n) \subset L_w^{p(.)}(\mathbb{R}^n)$.

(iii) $C_c(\mathbb{R}^n)$ is dense in $L_w^{p(.)}(\mathbb{R}^n)$.

(iv) $L_w^{p(.)}(\mathbb{R}^n)$ is a BF-space.

(v) $L_w^{p(.)}$ is a Banach module over L^∞ with respect to pointwise multiplication.

Proof. **(i)** Assume $f \in L_w^{p(.)}(\mathbb{R}^n)$. Since $w(x)^{p(x)} \geq 1$, then

$$\begin{aligned} |f(x)|^{p(x)} &\leq |f(x)w(x)|^{p(x)}, \\ \varrho_p(f) &\leq \varrho_{p,w}(f) < \infty. \end{aligned}$$

This implies that $L_w^{p(.)}(\mathbb{R}^n) \subset L^{p(.)}(\mathbb{R}^n)$. Also by using the inequality $|f(x)| \leq |f(x)w(x)|$ and definition of $\|.\|_{p(.)}$, then

$$\|f\|_{p(.)} \leq \|fw\|_{p(.)} = \|f\|_{p(.),w}.$$

(ii) Let $f \in C_c(\mathbb{R}^n)$ be any function such that $\mathrm{supp} f = K$ compact. For $p^* < \infty$ it is known that $C_c(\mathbb{R}^n) \subset L^{p(.)}(\mathbb{R}^n)$ by Lemma 4 in [1] and $\varrho_p(f) < \infty$. Hence we have

$$\begin{aligned} \varrho_{p,w}(f) &= \varrho_p(fw) = \int\limits_K |f(x)|^{p(x)} w(x)^{p(x)} dx \\ &\leq \left(\sup_{x \in K} w(x)^{p^*}\right) \varrho_p(f) < \infty \end{aligned}$$

and $C_c(\mathbb{R}^n) \subset L_w^{p(.)}(\mathbb{R}^n)$.

(iii) It is known that $C_c^\infty(\mathbb{R}^n)$ is dense in $L_w^{p(.)}(\mathbb{R}^n)$ by Corollary 2.5 in [2]. Hence $C_c(\mathbb{R}^n)$ is dense in $L_w^{p(.)}(\mathbb{R}^n)$.

(iv) Let $K \subset \mathbb{R}^n$ be a compact subset and $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$. By Hölder inequality for generalized Lebesgue spaces [20], we write

$$\begin{aligned} \int_K |f(x)|\, dx &\leq C \|\chi_K\|_{q(.)} \|f\|_{p(.)} \\ &\leq C \|\chi_K\|_{q(.),w} \|f\|_{p(.),w} \end{aligned}$$

for all $f \in L_w^{p(.)}(\mathbb{R}^n)$, where χ_K is the charecteristic function of K. It is known that $\|\chi_K\|_{q(.),w} < \infty$ if and only if $\varrho_{q,w}(\chi_K) < \infty$ for $q^* < \infty$. Then we have

$$\varrho_{q,w}(\chi_K) = \int\limits_K w(x)^{q(x)}dx = \left(\sup_{x\in K} w(x)^{q^*}\right)\mu(K) < \infty.$$

That means $L_w^{p(.)}(\mathbb{R}^n) \hookrightarrow L^1_{loc}(\mathbb{R}^n)$.

(v) We know that $L_w^{p(.)}(\mathbb{R}^n)$ is a Banach space. Also it is known that $L^\infty(\mathbb{R}^n)$ is a Banach algebra with respect to pointwise multiplication. Let $(f,g) \in L^\infty(\mathbb{R}^n) \times L_w^{p(.)}(\mathbb{R}^n)$.Then

$$\begin{aligned} \varrho_{p,w}(fg) &= \int\limits_{\mathbb{R}} |f(x)g(x)|^{p(x)} w(x)^{p(x)}dx \\ &\leq \max\left\{1, \|f\|_\infty^{p^*}\right\} \int\limits_{\mathbb{R}} |g(x)w(x)|^{p(x)} dx < \infty. \end{aligned}$$

We also have

$$\begin{aligned} \varrho_{p,w}\Big(\frac{fg}{\|f\|_\infty \|g\|_{p(.),w}}\Big) &\leq \int\limits_{\mathbb{R}} \frac{|f(x)g(x)|^{p(x)}}{\|f\|_\infty^{p(x)} \|g\|_{p(.),w}^{p(x)}}dx \leq \int\limits_{\mathbb{R}} \frac{\|f\|_{L^\infty}^{p(x)} |g(x)|^{p(x)}}{\|f\|_\infty^{p(x)} \|g\|_{p(.),w}^{p(x)}}dx \\ &= \varrho_{p,w}\Big(\frac{g}{\|g\|_{p(.),w}}\Big) \leq 1. \end{aligned}$$

Hence by the definition of the norm $\|.\|_{p(.),w}$ of the weighted variable exponent Lebesgue space, we obtain $\|fg\|_{p(.),w} \leq \|f\|_{L^\infty} \|g\|_{p(.),w}$. The remaining part of the proof is easy. □

Proposition 2.4. **(i)** The space $L_w^{p(.)}(\mathbb{R}^n)$ is strongly character invariant.
(ii) The function $t \to M_t f$ is continuous from $\mathbb{R}^n$ into $L_w^{p(.)}(\mathbb{R}^n)$.

Proof. (i) Let take any $f \in L_w^{p(.)}(\mathbb{R}^n)$. We define a function g such that $g(x) = M_t f(x)$ for all $t \in \mathbb{R}^n$. Hence we have

$$|g(x)| = |M_t f(x)| = |< x, t > f(x)| = |f(x)|$$

and

$$\|M_t f\|_{p(.),w} = \|g\|_{p(.),w} = \|f\|_{p(.),w}.$$

(ii) Since $C_c(\mathbb{R}^n)$ is dense in $L_w^{p(.)}(\mathbb{R}^n)$ by Proposition 2.3, then given any $f \in L_w^{p(.)}(\mathbb{R}^n)$ and $\varepsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$ such that

$$\|f - g\|_{p(.),w} < \frac{\varepsilon}{3}.$$

Let assume that supp$g = K$. Thus for every $t \in \mathbb{R}^n$, we have supp$(M_t g - g) \subset K$. If one uses the inequality

$$\begin{aligned} |M_t g(x) - g(x)| &= |< x,t > g(x) - g(x)| = |g(x)| \, |< x,t > -1| \\ &\leq |g(x)| \sup_{x\in K} |< x,t > -1| = |g(x)| \, \| < .,t > -1 \|_{\infty,K} , \end{aligned}$$

we have

$$\|M_t g - g\|_{p(.),w} \leq \| < .,t > -1\|_{\infty,K} \, \|g\|_{p(.),w} .$$

It is known that $\| < .,t > -1\|_{\infty,K} \to 0$ for $t \to 0$. Also, we have

$$\begin{aligned} \|M_t f - f\|_{p(.),w} &\leq \|M_t f - M_t g\|_{p(.),w} + \|M_t g - g\|_{p(.),w} + \|f - g\|_{p(.),w} \\ &= 2\,\|f - g\|_{p(.),w} + \| < .,t > -1\|_{\infty,K} \, \|g\|_{p(.),w} . \end{aligned}$$

Let us take the neighbourhood U of $0 \in \mathbb{R}^n$ such that

$$\| < .,t > -1\|_{\infty,K} < \frac{\varepsilon}{3\,\|g\|_{p(.),w}}$$

for all $t \in U$. Then we have

$$\|M_t f - f\|_{p(.),w} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3\,\|g\|_{p(.),w}} \|g\|_{p(.),w} = \varepsilon$$

for all $t \in U$. □

Definition 2.5. Let $p_1(.)$ and $p_2(.)$ be exponents on $\mathbb{R}^n$. We say that $p_2(.)$ is non-weaker than $p_1(.)$ if and only if $\Phi_{p_2}(x,t) = t^{p_2(x)}$ is non-weaker than $\Phi_{p_1}(x,t) = t^{p_1(x)}$ in the sense of Musielak [21], i.e. there exist constants K_1, $K_2 > 0$ and $h \in L^1(\mathbb{R}^n)$, $h \geq 0$, such that for a.e. $x \in \mathbb{R}^n$ and all $t \geq 0$

$$\Phi_{p_1}(x,t) \leq K_1 \Phi_{p_2}(x, K_2 t) + h(x).$$

We write $p_1(.) \preceq p_2(.)$.

Let $p_1(.) \preceq p_2(.)$. Then the embedding $L^{p_2(.)}(\mathbb{R}^n) \hookrightarrow L^{p_1(.)}(\mathbb{R}^n)$ was proved by Lemma 2.2 in [6].

Proposition 2.6. **(i)** If $w_1 \prec w_2$, then $L^{p(.)}_{w_2}(\mathbb{R}^n) \hookrightarrow L^{p(.)}_{w_1}(\mathbb{R}^n)$.

(ii) If $w_1 \approx w_2$, then $L^{p(.)}_{w_1}(\mathbb{R}^n) = L^{p(.)}_{w_2}(\mathbb{R}^n)$.

(iii) Let $0 < \mu(\Omega) < \infty$, $\Omega \subset \mathbb{R}^n$. If $w_1 \prec w_2$ and $p_1(.) \leq p_2(.)$, then $L^{p_2(.)}_{w_2}(\Omega) \hookrightarrow L^{p_1(.)}_{w_1}(\Omega)$.

Proof. **(i)** Let $f \in L^{p(.)}_{w_2}(\mathbb{R}^n)$. Since $w_1 \prec w_2$, there exists a $C > 0$ such that $w_1(x) \leq C w_2(x)$ for all $x \in \mathbb{R}^n$. Hence we write

$$|f(x)w_1(x)| \leq C\,|f(x)w_2(x)| .$$

This implies that

$$\|f\|_{p(.),w_1} \leq C\,\|f\|_{p(.),w_2}\,.$$

for all $f \in L^{p(.)}_{w_2}(\mathbb{R}^n)$.

(ii) Obvious.

(iii) Let $f \in L^{p_2(.)}_{w_2}(\Omega)$ be given. By using (i), we have $f \in L^{p_2(.)}_{w_1}(\Omega)$ and $fw_1 \in L^{p_2(.)}(\Omega)$. Since $p_1(.) \leq p_2(.)$, then $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ by Theorem 2.8 in [20] and

$$\begin{aligned} \|fw_1\|_{p_1(.)} &\leq C_1\,\|fw_1\|_{p_2(.)} \\ &\leq C_1C_2\,\|f\|_{p_2(.),w_2}\,. \end{aligned}$$

Hence $L^{p_2(.)}_{w_2}(\Omega) \hookrightarrow L^{p_1(.)}_{w_1}(\Omega)$. □

Proposition 2.7. If $p_1(.) \preceq p_2(.)$ and $w_1 \prec w_2$, then $L^{p_2(.)}_{w_2}(\mathbb{R}^n) \hookrightarrow L^{p_1(.)}_{w_1}(\mathbb{R}^n)$.

Proof. Since $p_1(.) \preceq p_2(.)$, then $L^{p_2(.)}_{w_2}(\mathbb{R}^n) \hookrightarrow L^{p_1(.)}_{w_2}(\mathbb{R}^n)$ by Theorem 8.5 of [21]. Also by using Proposition 2.6, we have $L^{p_1(.)}_{w_2}(\mathbb{R}^n) \hookrightarrow L^{p_1(.)}_{w_1}(\mathbb{R}^n)$. □

Remark 2.8. By the closed graph theorem in Banach space, to prove that there is a continuous embedding $L^{p_2(.)}_{w_2}(\mathbb{R}^n) \hookrightarrow L^{p_1(.)}_{w_1}(\mathbb{R}^n)$, one need only prove $L^{p_2(.)}_{w_2}(\mathbb{R}^n) \subset L^{p_1(.)}_{w_1}(\mathbb{R}^n)$.

Let w_1, w_2 be weights on $\mathbb{R}^n$. The space $L^{p_1(.)}_{w_1}(\mathbb{R}^n) \cap L^{p_2(.)}_{w_2}(\mathbb{R}^n)$ is defined as the set of all measurable functions f, for which

$$\|f\|^{p_1(.),p_2(.)}_{w_1,w_2} = \|f\|_{p_1(.),w_1} + \|f\|_{p_2(.),w_2} < \infty.$$

Proposition 2.9. Let w_1, w_2, w_3 and w_4 be weights on $\mathbb{R}^n$. If $w_1 \prec w_3$ and $w_2 \prec w_4$, then $L^{p_1(.)}_{w_3}(\mathbb{R}^n) \cap L^{p_2(.)}_{w_4}(\mathbb{R}^n) \hookrightarrow L^{p_1(.)}_{w_1}(\mathbb{R}^n) \cap L^{p_2(.)}_{w_2}(\mathbb{R}^n)$.

Proof. Obvious. □

Corollary 2.10. If $w_1 \approx w_3$ and $w_2 \approx w_4$, then $L^{p_1(.)}_{w_3}(\mathbb{R}^n) \cap L^{p_2(.)}_{w_4}(\mathbb{R}^n) = L^{p_1(.)}_{w_1}(\mathbb{R}^n) \cap L^{p_2(.)}_{w_2}(\mathbb{R}^n)$.

Proposition 2.11. If $p_1(x) \leq p_2(x) \leq p_3(x)$ and $w_2 \prec w_1$, then

$$L^{p_1(.)}_{w_1}(\mathbb{R}^n) \cap L^{p_3(.)}_{w_1}(\mathbb{R}^n) \hookrightarrow L^{p_2(.)}_{w_2}(\mathbb{R}^n)\,.$$

Proof. Since $p_1(x) \leq p_2(x) \leq p_3(x)$, then we write

$$\begin{aligned} |f(x)w_1(x)|^{p_2(x)} &\leq |f(x)w_1(x)|^{p_1(x)} \chi_{\{x:|f(x)w_1(x)|\leq 1\}} + \\ &+ |f(x)w_1(x)|^{p_3(x)} \chi_{\{x:|f(x)w_1(x)|\geq 1\}}. \end{aligned}$$

Hence $L^{p_1(.)}_{w_1}(\mathbb{R}^n) \cap L^{p_3(.)}_{w_1}(\mathbb{R}^n) \hookrightarrow L^{p_2(.)}_{w_1}(\mathbb{R}^n)$. Also by using Proposition 2.6, we have $L^{p_2(.)}_{w_1}(\mathbb{R}^n) \hookrightarrow L^{p_2(.)}_{w_2}(\mathbb{R}^n)$. □

Corollary 2.12. Let $1 \leq p_* \leq p(x) \leq p^* < \infty$ for all $x \in \mathbb{R}^n$ and $w_2 \prec w_1$, then

$$L^{p_*}_{w_1}(\mathbb{R}^n) \cap L^{p^*}_{w_1}(\mathbb{R}^n) \hookrightarrow L^{p(.)}_{w_2}(\mathbb{R}^n).$$

Proof. The proof is completed by Proposition 2.11. □

For any $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\widehat{f}$ and defined by

$$\widehat{f}(x) = \int_{\mathbb{R}^n} e^{-it.x} f(t)dt.$$

It is known that $\widehat{f}$ is a continuos function on $\mathbb{R}^n$, which vanishes at infinity and the inequality $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$ is satisfied. Let the Fourier algebra $\left\{\widehat{f} : f \in L^1(\mathbb{R}^n)\right\}$ with by $A(\mathbb{R}^n)$ and is given the norm $\left\|\widehat{f}\right\|_A = \|f\|_1$.

Let ω be an arbitrary Beurling's weight function on $\mathbb{R}^n$. We next introduce the homogeneous Banach space

$$A^\omega(\mathbb{R}^n) = \left\{\widehat{f} : f \in L^1_\omega(\mathbb{R}^n)\right\}$$

with the norm $\left\|\widehat{f}\right\|_\omega = \|f\|_{1,\omega}$. It is known that $A^\omega(\mathbb{R}^n)$ is a Banach algebra under pointwise multiplication [23]. We set $A^\omega_0(\mathbb{R}^n) = A^\omega(\mathbb{R}^n) \cap C_c(\mathbb{R}^n)$ and equip it with the inductive limit topology of the subspaces $A^\omega_K(\mathbb{R}^n) = A^\omega(\mathbb{R}^n) \cap C_K(\mathbb{R}^n)$, $K \subset \mathbb{R}^n$ compact, equipped with their $\|.\|_\omega$ norms. For every $h \in A^\omega_0(\mathbb{R}^n)$ we define the semi-norm q_h on $A^\omega_0(\mathbb{R}^n)'$ by $q_h(h') = |< h, h' >|$, where $A^\omega_0(\mathbb{R}^n)'$ is the topological dual of $A^\omega_0(\mathbb{R}^n)$. The locally convex topology on $A^\omega_0(\mathbb{R}^n)'$ defined by the family $(q_h)_{h\in A^\omega_0(\mathbb{R}^n)}$ of seminorms is called the topology $\sigma\left(A^\omega_0(\mathbb{R}^n)', A^\omega_0(\mathbb{R}^n)\right)$ or the weak star topology.

Lemma 2.13. Let $r^* < \infty$. Then $A^\omega_K(\mathbb{R}^n)$ is continuously embedded into $L^{r(.)}_w(\mathbb{R}^n)$ for every compact subsets $K \subset \mathbb{R}^n$, i.e $A^\omega_K(\mathbb{R}^n) \hookrightarrow L^{r(.)}_w(\mathbb{R}^n)$.

Proof. Using the classical result $A^\omega_K(\mathbb{R}^n) \hookrightarrow L^{r_*}_w(\mathbb{R}^n) \cap L^{r^*}_w(\mathbb{R}^n)$ and $L^{r_*}_w(\mathbb{R}^n) \cap L^{r^*}_w(\mathbb{R}^n) \hookrightarrow L^{r(.)}_w(\mathbb{R}^n)$ by Corollary 2.12, then $A^\omega_K(\mathbb{R}^n) \hookrightarrow L^{r(.)}_w(\mathbb{R}^n)$. □

Theorem 2.14. $L_w^{p(.)}(\mathbb{R}^n)$ is continuously embedded into $A_0^{\omega}(\mathbb{R}^n)'$.

Proof. Let $f \in L_w^{p(.)}(\mathbb{R}^n)$ and $h \in A_0^{\omega}(\mathbb{R}^n)$. By definition of $A_0^{\omega}(\mathbb{R}^n)$, there exists a compact subset $K \subset \mathbb{R}^n$ such that $h \in A_K^{\omega}(\mathbb{R}^n)$. Suppose that $\frac{1}{p(.)} + \frac{1}{r(.)} = 1$. Then by Hölder inequality for variable exponent Lebesgue spaces and by Lemma 2.13, there exists a $C > 0$ such that

$$\begin{aligned} |< f, h >| &= \left| \int_{\mathbb{R}^n} f(x)h(x)dx \right| \leq \int_{\mathbb{R}^n} |f(x)h(x)|\, dx \\ &\leq C \|f\|_{p(.)} \|h\|_{r(.)} \leq C \|f\|_{p(.),w} \|h\|_{r(.),\omega} < \infty. \end{aligned} \tag{1}$$

Hence the integral

$$< f, h >= \int_{\mathbb{R}^n} f(x)h(x)dx$$

is well defined. Now define the linear functional $< f, . >: A_0^{\omega}(\mathbb{R}^n) \to \mathbb{C}$ for $f \in L_w^{p(.)}(\mathbb{R}^n)$ such that

$$< f, h >= \int_{\mathbb{R}^n} f(x)h(x)dx.$$

It is known that the functional $< f, . >$ is continuous from $A_0^{\omega}(\mathbb{R}^n)$ into $\mathbb{C}$ if and only if $< f, . > \big|_{A_K^{\omega}}$ is continuous from $A_K^{\omega}(\mathbb{R}^n)$ into $\mathbb{C}$ for all compact subsets $K \subset \mathbb{R}^n$. By Lemma 2.13, there exists a $M_K > 0$ such that

$$\|h\|_{r(.),w} \leq M_K \|h\|_{\omega}. \tag{2}$$

By (1) and (2),

$$\begin{aligned} |< f, h >| &\leq C \|f\|_{p(.),w} \|h\|_{r(.),\omega} \\ &\leq CM_K \|f\|_{p(.),w} \|h\|_{\omega} = D_K \|h\|_{\omega} \end{aligned} \tag{3}$$

where $D_K = CM_K \|f\|_{p(.),w}$. Then we have the inclusion $L_w^{p(.)}(\mathbb{R}^n) \subset A_0^{\omega}(\mathbb{R}^n)'$. Define the unit map $I : L_w^{p(.)}(\mathbb{R}^n) \to A_0^{\omega}(\mathbb{R}^n)'$. Let $h \in A_0^{\omega}(\mathbb{R}^n)$ be given. Then there exists a compact subset $K \subset \mathbb{R}^n$ such that $h \in A_K^{\omega}(\mathbb{R}^n)$. Take any semi-norm $q_h \in (q_h)$, $h \in A_0^{\omega}(\mathbb{R}^n)$ on $A_0^{\omega}(\mathbb{R}^n)'$. By using (3) we obtain

$$q_h(I(f)) = q_h(f) = |< f, h >| \leq B_K \|f\|_{p(.),w},$$

where $B_K = CM_K \|h\|_{\omega}$. Then I is continuous map from $L_w^{p(.)}(\mathbb{R}^n)$ into $A_0^{\omega}(\mathbb{R}^n)'$. The proof is completed. ☐

3 Weighted Variable Exponent Amalgam Spaces $W(L_w^{p(\cdot)}, L_\upsilon^q)$

The space $\left(L_w^{p(\cdot)}(\mathbb{R}^n)\right)_{loc}$ consists of all (classes of) measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$, where χ_K is the characteristic function of K. Since the general hypotheses for the amalgam space $W(L_w^{p(\cdot)}, L_\upsilon^q)$ are satisfied by Lemma 2.13 and Theorem 2.14, then $W(L_w^{p(\cdot)}, L_\upsilon^q)$ is well defined as follows as in [10].

Let us fix an open set $Q \subset \mathbb{R}^n$ with compact closure. The *variable exponent amalgam space* $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$ consists of all elements $f \in \left(L_w^{p(\cdot)}(\mathbb{R}^n)\right)_{loc}$ such that $\mathcal{F}_f(z) = \|f\chi_{z+Q}\|_{p(.),w}$ belongs to $L_\upsilon^q(\mathbb{R}^n)$; the norm of $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$ is

$$\|f\|_{W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)} = \|\mathcal{F}_f\|_{q,\upsilon}.$$

Given a discrete family $X = (x_i)_{i \in I}$ in $\mathbb{R}^n$ and a weighted space $L_w^q(\mathbb{R}^n)$, the *associated weighted sequence space* over X is the appropriate weighted ℓ^q -space ℓ_w^q, the *discrete* w being given by $w(i) = w(x_i)$ for $i \in I$, (see Lemma 3.5 in [12]).

The following theorem, based on Theorem 1 in [10], describes the basic properties of $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$.

Theorem 3.1. (i) $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$ is a Banach space with norm $\|.\|_{W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)}$.

(ii) $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$ is continuously embedded into $\left(L_w^{p(\cdot)}(\mathbb{R}^n)\right)_{loc}$.

(iii) The space

$$\Lambda_0 = \left\{ f \in L_w^{p(\cdot)}(\mathbb{R}^n) : \operatorname{supp}(f) \text{ is compact} \right\}$$

is continuously embedded into $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$.

(iv) $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$ does not depend on the particular choice of Q, i.e. different choices of Q define the same space with equivalent norms.

By (iii) and Proposition 2.3 it is easy to see that $C_c(\mathbb{R}^n)$ is continuously embedded into $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$.

Now by using the techniques in [14], we prove the following proposition.

Proposition 3.2. $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$ is a BF-space on $\mathbb{R}^n$.

Proposition 3.3. $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$ is strongly character invariant and the map $t \to M_t f$ is continuous from $\mathbb{R}^n$ into $W\left(L_w^{p(\cdot)}, L_\upsilon^q\right)$.

Proof. It is known that $L_w^{p(.)}(\mathbb{R}^n)$ is strongly character invariant and the function $t \to M_t f$ is continuous from $\mathbb{R}^n$ into $L_w^{p(.)}(\mathbb{R}^n)$ by Proposition 2.4. Hence the proof is completed by Lemma 1.5. in [24]. □

Proposition 3.4. w_1, w_2, w_3, υ_1, υ_2 and υ_3 be weight functions. Suppose that there exist constants $C_1, C_2 > 0$ such that

$$\forall h \in L_{w_1}^{p_1(.)}(\mathbb{R}^n), \forall k \in L_{w_2}^{p_2(.)}(\mathbb{R}^n), \quad \|hk\|_{p_3(.),w_3} \leq C_1 \|h\|_{p_1(.),w_1} \|k\|_{p_2(.),w_2}$$

and

$$\forall u \in L_{\upsilon_1}^{q_1}(\mathbb{R}^n), \forall \vartheta \in L_{\upsilon_2}^{q_2}(\mathbb{R}^n), \quad \|u\vartheta\|_{q_3,\upsilon_3} \leq C_2 \|u\|_{q_1,\upsilon_1} \|\vartheta\|_{q_2,\upsilon_2}$$

Then there exists $C > 0$ such that

$$\|fg\|_{W\left(L_{w_3}^{p_3(.)},L_{\upsilon_3}^{q_3}\right)} \leq C \|f\|_{W\left(L_{w_1}^{p_1(.)},L_{\upsilon_1}^{q_1}\right)} \|g\|_{W\left(L_{w_2}^{p_2(.)},L_{\upsilon_2}^{q_2}\right)}$$

for all $f \in W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right)$ and $g \in W\left(L_{w_2}^{p_2(.)}, L_{\upsilon_2}^{q_2}\right)$. In other words

$$W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right) W\left(L_{w_2}^{p_2(.)}, L_{\upsilon_2}^{q_2}\right) \subset W\left(L_{w_3}^{p_3(.)}, L_{\upsilon_3}^{q_3}\right).$$

Proof. If $f \in W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right)$ and $g \in W\left(L_{w_2}^{p_2(.)}, L_{\upsilon_2}^{q_2}\right)$, then we have

$$\begin{aligned}
\|fg\|_{W\left(L_{w_3}^{p_3(.)},L_{\upsilon_3}^{q_3}\right)} &= \left\| \|fg\chi_{z+Q}\|_{p_3(.),w_3} \right\|_{q_3,\upsilon_3} \\
&= \left\| \|(f\chi_{z+Q})(g\chi_{z+Q})\|_{p_3(.),w_3} \right\|_{q_3,\upsilon_3} \\
&\leq C_1 \left\| \|f\chi_{z+Q}\|_{p_1(.),w_1} \|g\chi_{z+Q}\|_{p_2(.),w_2} \right\|_{q_3,\upsilon_3} \\
&= C_1 \|\mathcal{F}_f \mathcal{F}_g\|_{q_3,\upsilon_3} \leq C_1 C_2 \|\mathcal{F}_f\|_{q_1,\upsilon_1} \|\mathcal{F}_g\|_{q_2,\upsilon_2} \\
&= C \|f\|_{W\left(L_{w_1}^{p_1(.)},L_{\upsilon_1}^{q_1}\right)} \|g\|_{W\left(L_{w_2}^{p_2(.)},L_{\upsilon_2}^{q_2}\right)}
\end{aligned}$$

and the proof is complete. □

Proposition 3.5. (i) If $p_1(.) \leq p_2(.)$, $q_2 \leq q_1$, $w_1 \prec w_2$ and $\upsilon_1 \prec \upsilon_2$, then

$$W\left(L_{w_2}^{p_2(.)}, L_{\upsilon_2}^{q_2}\right) \subset W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right).$$

(ii) If $p_1(.) \leq p_2(.)$, $q_2 \leq q_1$, $w_1 \prec w_2$ and $\upsilon_1 \prec \upsilon_2$, then

$$W\left(L_{w_1}^{p_1(.)} \cap L_{w_2}^{p_2(.)}, L_{\upsilon_2}^{q_2}\right) \subset W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right).$$

Proof. (i) Let $f \in W\left(L_{w_2}^{p_2(.)}, L_{\upsilon_2}^{q_2}\right)$ be given. Since $p_1(.) \le p_2(.)$ and $w_1 \prec w_2$ then $L_{w_2}^{p_2(.)}(z+Q) \hookrightarrow L_{w_1}^{p_1(.)}(z+Q)$ and

$$\begin{aligned} \|f\chi_{z+Q}\|_{p_1(.),w_1} &\le C\left(\mu\left(z+Q\right)+1\right)\|f\chi_{z+Q}\|_{p_2(.),w_2} \\ &\le C\left(\mu\left(Q\right)+1\right)\|f\chi_{z+Q}\|_{p_2(.),w_2} \end{aligned}$$

for all $z \in \mathbb{R}^n$ by Theorem 2.8 in [20], where μ is the Lebesgue measure. Hence by the solidity of $L_{\upsilon_2}^{q_2}(\mathbb{R}^n)$ we have

$$W\left(L_{w_2}^{p_2(.)}, L_{\upsilon_2}^{q_2}\right) \subset W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_2}^{q_2}\right).$$

It is known by Proposition 3.7 in [12], that

$$W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_2}^{q_2}\right) \subset W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right)$$

if and only if $\ell_{\upsilon_2}^{q_2} \subset \ell_{\upsilon_1}^{q_1}$, where $\ell_{\upsilon_2}^{q_2}$ and $\ell_{\upsilon_1}^{q_1}$ are the associated sequence spaces of $L_{\upsilon_2}^{q_2}(\mathbb{R}^n)$ and $L_{\upsilon_1}^{q_1}(\mathbb{R}^n)$ respectively. Since $q_2 \le q_1$ and $\upsilon_1 \prec \upsilon_2$, then $\ell_{\upsilon_2}^{q_2} \subset \ell_{\upsilon_1}^{q_1}$ [14]. This completes the proof.

(ii) The proof of this part is easy by (i). □

The following Proposition was proved by [3].

Proposition 3.6. Let B be any solid space. If $q_2 \le q_1$ and $\upsilon_1 \prec \upsilon_2$, then we have

$$W\left(B, L_{\upsilon_1}^{q_1} \cap L_{\upsilon_2}^{q_2}\right) = W\left(B, L_{\upsilon_2}^{q_2}\right).$$

Corollary 3.7. **(i)** If $p_1^*, p_2^* < \infty$, $L_{w_1}^{p_1(.)}(\mathbb{R}^n) \subset L_{w_2}^{p_2(.)}(\mathbb{R}^n)$, $q_2 \le q_1$, $q_4 \le q_3$, $q_4 \le q_2$, $\upsilon_1 \prec \upsilon_2$, $\upsilon_3 \prec \upsilon_4$ and $\upsilon_2 \prec \upsilon_4$, then

$$W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_3}^{q_3} \cap L_{\upsilon_4}^{q_4}\right) \subset W\left(L_{w_2}^{p_2(.)}, L_{\upsilon_1}^{q_1} \cap L_{\upsilon_2}^{q_2}\right).$$

(ii) If $p_1(x) \le p_3(x)$, $p_2(x) \le p_4(x)$, $q_2 \le q_1$, $q_4 \le q_3$, $q_4 \le q_2$, $w_1 \prec w_3$, $w_2 \prec w_4$, $\upsilon_1 \prec \upsilon_2$, $\upsilon_3 \prec \upsilon_4$ and $\upsilon_2 \prec \upsilon_4$, then

$$W\left(L_{w_3}^{p_3(.)} \cap L_{w_4}^{p_4(.)}, L_{\upsilon_3}^{q_3} \cap L_{\upsilon_4}^{q_4}\right) \subset W\left(L_{w_1}^{p_1(.)} \cap L_{w_2}^{p_2(.)}, L_{\upsilon_1}^{q_1} \cap L_{\upsilon_2}^{q_2}\right).$$

Proposition 3.8. If $1 \le q \le \infty$ and $\upsilon \in L^q(\mathbb{R}^n)$, then $L_w^{p(.)}(\mathbb{R}^n) \subset W\left(L_w^{p(.)}, L_\upsilon^q\right)$.

Proof. If $1 \le q < \infty$ and $\upsilon \in L^q(\mathbb{R}^n)$, we have

$$\begin{aligned} \|f\|_{W\left(L_w^{p(.)},L_\upsilon^q\right)} &= \left\| \|f\chi_{z+Q}\|_{p(.),w} \right\|_{q,\upsilon} \\ &= \left\{ \int\limits_{\mathbb{R}^n} \|f\chi_{z+Q}\|_{p(.),w}^q \,\upsilon^q(z)dz \right\}^{\frac{1}{q}} \\ &\le \left\{ \int\limits_{\mathbb{R}^n} \|f\|_{p(.),w}^q \,\upsilon^q(z)dz \right\}^{\frac{1}{q}} \\ &= \|f\|_{p(.),w} \|\upsilon\|_q . \end{aligned}$$

Hence $L_w^{p(.)}(\mathbb{R}^n) \subset W\left(L_w^{p(.)}, L_\upsilon^q\right)$. Similarly, for $q = \infty$, we obtain

$$\|f\|_{W\left(L_w^{p(.)},L_\upsilon^\infty\right)} = \left\| \|f\chi_{z+Q}\|_{p(.),w} \,\upsilon \right\|_\infty \le \|f\|_{p(.),w} \|\upsilon\|_\infty .$$

Then $L_w^{p(.)}(\mathbb{R}^n) \subset W\left(L_w^{p(.)}, L_\upsilon^\infty\right)$. □

Proposition 3.9. Let 1¡$q_0, q_1 < \infty$. If $p_0(.)$ and $p_1(.)$ are variable exponents with $1 < p_{j,*} \le p_j^* < \infty$, $j = 0, 1$. Then, for $\theta \in (0,1)$, we have

$$\left[W\left(L_{w_0}^{p_0(.)}, L_{\upsilon_0}^{q_0}\right), W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right)\right]_{[\theta]} = W\left(L_w^{p_\theta(.)}, L_\upsilon^{q_\theta}\right)$$

where $\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $w = w_0^{1-\theta} w_1^\theta$ and $\upsilon = \upsilon_0^{1-\theta}\upsilon_1^\theta$.

Proof. By Theorem 2.2 in [11] the interpolation space $\left[W\left(L_{w_0}^{p_0(.)}, L_{\upsilon_0}^{q_0}\right), W\left(L_{w_1}^{p_1(.)}, L_{\upsilon_1}^{q_1}\right)\right]_{[\theta]}$ is $W\left(\left[L_{w_0}^{p_0(.)}, L_{w_1}^{p_1(.)}\right]_{[\theta]}, \left[L_{\upsilon_0}^{q_0}, L_{\upsilon_1}^{q_1}\right]_{[\theta]}\right)$. We know that $\left[L_{\upsilon_0}^{q_0}, L_{\upsilon_1}^{q_1}\right]_{[\theta]} = L_\upsilon^{q_\theta}$ and by Corollary A.2. in [7] that $\left[L_{w_0}^{p_0(.)}, L_{w_1}^{p_1(.)}\right]_{[\theta]} = L_w^{p_\theta(.)}$. This completes the proof. □

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