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On geometric structures associated with triple systems

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Abstract

In this paper, we show that geometric phenomena can be characterized using the concept of triple systems. In particular, we study a complex structure associated with triple systems.

1 Introduction

In future, it is expected that triple systems will be useful for the characterization of geometric structures in mathematics and physics as well as that of (classical) Yang-Baxter equations ([10],[19],[22]).

Our aim is to use triple systems to investigate the characterization of differential geometry and mathematical physics from the viewpoint of nonassociative algebras that contain a class of Lie algebras or Jordan algebras ([6],[7],[8],[13],[16],[17],[18]). In this paper, we will study about the Lie superalgebras or Lie algebras associated with triple systems, and the relationship with several vector subspaces associated with these algebras. We show that these subspaces have a complex structure as well as differential geometry, and we provide an algebraic characterization of their subspaces. Furthermore, for B_3-type Lie algebras, we will give some examples of triple systems and their correspondence with extended Dynkin diagrams.

A $(2\nu+1)$-graded Lie algebra is a Lie algebra of the form $g = \oplus_{k=-\nu}^{\nu} g_k$ such that $[g_k, g_l] \subset g_{k+l}$. It is well known that 3-graded Lie algebras are essentially bijective with certain theoretic objects called Jordan triple systems or Jordan

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pairs. Kantor remarked that more general graded Lie algebras correspond to generalized Jordan triple systems. In particular, the graded Lie algebra g (or L) given by

$$g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \quad (\text{or } L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \ )$$

has the structure of a triple product on the subspace g_{-1}, and is known as a generalized Jordan triple system (GJTS) of the second order or a (-1,1)-Freudenthal-Kantor triple system (F-K.t.s.)([8],[14]). Also $g_{-1} \oplus g_1$ has the structure of a Lie triple system (in particular, the system over a real number is known to correspond with a symmetric Riemannian space by means of a totally geodesic manifold notation). We will discuss the corresponding geometrical object by using these triple systems. The notation and terminology used for the geometry can be found in ([4], [5], [21]). We will often use the symbols g and L to denote a Lie algebra or Lie superalgebra as is conventionally used ([2],[3],[7],[25]).

From the viewpoint of an algebraic study, our purpose is to propose a unified structural theory for triple systems in nonassociative algebras. In previous works ([12],[14],[15]), we have studied the Peirce decomposition of a GJTS U of the second order by employing a tripotent element e of U (for a tripotent element, $\{eee\} = e$).
The Peirce decomposition of U is described as follows:

$$U = U_{00} \oplus U_{\frac{1}{2}\frac{1}{2}} \oplus U_{11} \oplus U_{\frac{3}{2}\frac{3}{2}} \oplus U_{-\frac{1}{2}0} \oplus U_{01} \oplus U_{\frac{1}{2}2} \oplus U_{13},$$

where $L(a) = \{eea\} = \lambda a$ and $R(a) = \{aee\} = \mu a$ if $a \in U_{\lambda\mu}$.

These viewpoints as above have formed the basis of our study on triple systems. However in this note, our consideration will mainly be from a geometrical viewpoint.

We are concerned with triple systems and algebras which have finite dimensionality over a field Φ of characteristic $\neq 2$ *or* 3, unless otherwise specified.

2 Definitions and Preamble

To make this paper as self-contained as possible, we first recall the definition of a generalized Jordan triple system of the second order (hereafter, referred to as a GJTS of 2nd order), and the construction of Lie algebras associated with a GJTS of 2nd order.

A vector space V over a field Φ, endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto \{xyz\}$, is said to be a *GJTS of 2nd order* if the following two conditions are satisfied:

$(J1) \quad \{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\} \quad (GJTS)$

$(K1) \quad K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0 \quad$ (2nd order), where $L(a,b)c = \{abc\}$ and $K(a,b)c = \{acb\} - \{bca\}$.

Remark. If $K(a,b) \equiv 0$ (identically zero), then this triple system is a Jordan triple system (JTS), i.e., it satisfies the relations $\{acb\} = \{cba\}$ and $GJTS$.

We can also generalize the concept of the GJTS of 2nd order as follows (for examples, see [8],[9],[11],[16] and references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, if the triple product satisfies

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)),$$
$$K(K(a,b)c,d) - L(d,c)K(a,b) + \varepsilon K(a,b)L(c,d) = 0,$$

where $L(x,y)z = (xyz)$ and $K(a,b)c = (acb) - \delta(bca)$, then it is said to be an (ε,δ)-*Freudenthal-Kantor triple system* (hereafter abbreviated as an (ε,δ)-F-K.t.s).

If $Id \in \mathbf{k} = \{K(a,b)\}_{span}$, it is said to be *unitary*.

Furthermore, if the (ε,δ)-F-K.t.s satisfies

$$dim_\Phi\{K(a,b)\}_{span} = dim_\Phi\{< a,b >\}_{span} = 1,$$

where $< a,b > \in \Phi^*$, then it is said to be *balanced.*

Remark. We set $S(x,y) := L(x,y) + \varepsilon L(y,x)$, and $A(x,y) := L(x,y) - \varepsilon L(y,x)$, then this $S(x,y)$ (resp. $A(x,y)$) is a derivation (resp. anti-derivation) of $U(\varepsilon,\delta)$.

Remark. Following the notation of the metasymplectic geometry due to H.Freudenthal, our concept means that

$$P\text{x}Q \cdots\cdots derivation\ S(x,y)\ of\ U(\varepsilon,\delta),$$

$$\{P,Q\} \cdots\cdots anti\ derivation\ A(x,y)(=K(x,y)\ )\ of\ U(\varepsilon.\delta)\ \cdots balanced\ type.$$

We generally denote the triple products by $\{xyz\}$, (xyz), $[xyz]$, and $< xyz >$. Bilinear forms are denoted by $< x|y >, < x,y >$, and $B(x,y)$.

Remark. Note that the concept of a GJTS of 2nd order coincides with that of a $(-1,1)$-F-K.t.s. Thus we can construct simple Lie algebras ($\delta = 1$)or superalgebras ($\delta = -1$) from these triple systems with $\delta = \pm 1$, by means of the standard embedding method (for example, [2], [3], [8]–[12], [16], [17], [18], [23]).

Proposition 1([9],[17]). *Let $U(\varepsilon,\delta)$ be an (ε,δ)-F-K.t.s. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz > = < JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is a Lie triple system (the case of $\delta = 1$) or an anti-Lie triple system (the case of $\delta = -1$) with respect to the product*

$$[xyz] := < xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > .$$

Corollary. *Let $U(\varepsilon,\delta)$ be an (ε,δ)-F-K.t.s. Then the vector space $T(\varepsilon,\delta)=U(\varepsilon,\delta)\oplus U(\varepsilon,\delta)$ becomes a Lie triple system (the case of $\delta=1$) or an anti-Lie triple system (the case of δ = -1) with respect to the triple product defined by*

$$[\begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}\begin{pmatrix}e\\f\end{pmatrix}]=\begin{pmatrix}L(a,d)-\delta L(c,b) & \delta K(a,c)\\ -\varepsilon K(b,d) & \varepsilon(L(d,a)-\delta L(b,c))\end{pmatrix}\begin{pmatrix}e\\f\end{pmatrix}.$$

Thus we can obtain the standard embedding Lie algebra (the case of $\delta=1$) or Lie superalgebra (the case of $\delta=-1$), $L(\varepsilon,\delta)=D(T(\varepsilon,\delta),T(\varepsilon,\delta))\oplus T(\varepsilon,\delta)$, associated with $T(\varepsilon,\delta)$, where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$. That is, these vector spaces $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ and $T(\varepsilon,\delta)$ imply

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)):=\begin{pmatrix}L(a,b) & \delta K(c,d)\\ -\varepsilon K(e,f) & \varepsilon L(b,a)\end{pmatrix}_{span} \text{ and}$$

$$T(\varepsilon,\delta):=\{\begin{pmatrix}x\\y\end{pmatrix}|x,y\in U(\varepsilon,\delta)\}_{span}, (\textit{ denoted by } T(U)\ ).$$

In fact, we have

$$L_0=\{\begin{pmatrix}L(a,b) & 0\\ 0 & \varepsilon L(b,a)\end{pmatrix}\}_{span}=\{L(a,b)\}_{span},$$

$$L_{-2}=\{\begin{pmatrix}0 & \delta K(c,d)\\ 0 & 0\end{pmatrix}\}_{span}=\{K(c,d)\}_{span}\text{and } L_0=DerU\oplus Anti-DerU.$$

Remark. For the standard embedding algebras obtained from these triple systems, note that

$$L(\varepsilon,\delta):=L_{-2}\oplus L_{-1}\oplus L_0\oplus L_{-1}\oplus L_{-2}$$

(or $g=g_{-2}\oplus g_{-1}\oplus g_0\oplus g_1\oplus g_2$) is a 5-graded Lie algebra or Lie superalgebra such that

$$L_{-1}=g_{-1}=U(\varepsilon,\delta)$$

and

$$DerT(U):=D(T(\varepsilon,\delta),T(\varepsilon,\delta))=L_{-2}\oplus L_0\oplus L_{-2}$$

with

$$[L_i,L_j]\subseteq L_{i+j}.$$

Also, denote $L(\varepsilon,\delta)$ by $L(U)$.

For the correspondence of the (1,1) F.K.t.s with the (-1,1) F.K.t.s, we obtain the following.

Proposition 2. *Let $(U, < xyz >)$ be a $(1,1)$ F-K.t.s. If there is an endomorphism P of U such that $P < xyz >=< PxPyPz >$ and $P^2 = -Id$, then $(U, \{xyz\})$ is a GJTS of 2nd order (that is, (-1,1)-F-K.t.s.) with respect to the new product defined by $\{xyz\} :=< xPyz >$.*

Proof. Suppose that the product $< xyz >$ is satisfying the two relations;

$$< ab < xyz >>=<< abx > yz > + < x < bay > z > + < xy < abz >>,$$

$$K(K(a,b)c,d) - L(d,c)K(a,b) + K(a,b)L(c,d) = 0,$$

where $L(a,b)c =< abc >$ and $K(a,b)c =< acb > - < bca >$.

By means of $P^2 = -Id$ and the definition $\{xyz\} =< xPyz >$, it is enough to show that the triple product $\{xyz\}$ is satisfied the two identities;

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\},$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0$$

where $L(x,y)z = \{xyz\}$ and $K(a,b)c = \{acb\} - \{bca\}$.
Thus, these relations is obtained by straightforward calculatins, This completes the proof.

We now give an explicit example of a JTS and a Lie triple system.

Example. Let U be a vector space with a symmetric bilinear form $< , >$. Then the triple system $(U, [xyz])$ is a Lie triple system with respect to the product

$$[xyz] =< y,z > x- < z,x > y.$$

That is, this triple system is induced from the JTS

$$\{xyz\} = \frac{1}{2}(< x,y > z+ < y,z > x- < z,x > y)$$

by means of

$$[xyz] = \{xyz\} - \{yxz\}.$$

3 A complex structure associated with triple systems

In this section, we will discuss with a complex structure on the following vector space;

$$T(\varepsilon,\delta) = g_{-1} \oplus g_1.$$

We set

$$E = \begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix}, H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}, J = \delta E - \varepsilon F.$$

Then we obtain by straightfoward calculations,

$$H = [E, F],\ [H, E] = 2E,\ [H, F] = -2F,\ J^2 = -\delta\varepsilon \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix},$$

where J is an operator on $T(\varepsilon, \delta)$.

Next we define the Nijenhuis operator on $T(\varepsilon, \delta)$ as below.

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y], \quad X, Y \in T(\varepsilon, \delta).$$

We study the cases of $\varepsilon\delta = 1$, which can be considered to have an almost complex structure, since $J^2 = -Id$. However, we do not deal with the case of $\varepsilon\delta = -1$ called a paracomplex structure, which will be considered in future work.

By a long but straightfoward calculations, proof omitted, we obtain the following Proposition.

Proposition 3. *Let U be an (ε, δ)- F-K.t.s. and its operations be given by the above definitions. The following are equivalent:*

(i) $N(X, Y) = 0$,

(ii) $L(y, x) - \delta L(x, y) = K(x, y)$.

From these results as well as the differential geometry, we conclude that there exists a complex structure on $T(\varepsilon, \delta)$ if $L(y, x) - \delta L(x, y) = K(x, y)$.

Following [16], we exhibit examples of a (-1,-1)-F-K.t.s with a complex structure, is known as an antistructurable algebra.

Remark. By the well known fact that the Lie triple systems have a correspondence with symmetric spaces, $T(\varepsilon, \delta)$ is closely related to symmetric space.

By using above J, we may define an operator $\tilde{J}$ on $L(\varepsilon, \delta)$ as follows:

$$\begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{pmatrix} \oplus \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow J \begin{pmatrix} L(a, b) & \delta K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{pmatrix} J^{-1} \oplus J \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then this $\tilde{J}$ satisfies

$$\tilde{J}[X, Y] = [\tilde{J}X, \tilde{J}Y],$$

for all $X, Y \in L(\varepsilon, \delta)$.

Thus $\tilde{J}$ is said to be an automorphism of $L(\varepsilon, \delta)$ induced from the almost complex structure J.

Remark. Note that if U is unitary, then $L(\varepsilon, \delta)$ contains the subalgebra $sl_2 = \{H, E, F\}_{span}$, because $Id \in \mathbf{k} = \{K(a, b)\}_{span} = g_{-2}$.

In this section's final comment,we will introduce a deformed Nijenhuis operator on $H(\varepsilon, \delta) = g_0 \oplus g_1 \oplus g_2$.

$$N_H(X, Y) = [J_H X, J_H Y] - J_H[J_H X, Y] - J_H[X, J_H Y] + J_H^2[X, Y],$$

$X, Y \in H(\varepsilon, \delta)$, where $J_H : (x_0, x_1, x_2) \to (\alpha x_0, \alpha x_1, \alpha x_2)$.

Proposition 4. *For $H(\varepsilon, \delta)$, if there is a element $\alpha \in \Phi$ such that $\alpha^2 = -1$, then the space $H(\varepsilon, \delta)$ has a complex structure.*

Proof. From $\alpha^2 = -1$, it follows that the identity $N_H(X, Y) = 0$ holds. Hence, we have a complex structure.

4 Several algebraic structures

In this section, we will study some algebraic structures on (i) $T(\varepsilon, \delta)$, (ii) $B(\varepsilon, \delta) = g_{-2} \oplus g_{-1} \oplus g_1 \oplus g_2$ and (iii) $H(\varepsilon, \delta)$.

We denote any element of $g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ by $(x_{-2}, x_{-1}, x_0, x_1, x_2)$ or $x_{-2} + x_{-1} + x_0 + x_1 + x_2$.

Before going into further details, we recall the definition of a generalized structurable algebra A (e.g., [13]) as having a bilinear derivation $D(x, y)$ satisfying

$$D(x, y)(uv) = (D(x, y)u)v + u(D(x, y)v)$$

$$D(xy, z) + D(yz, x) + D(zx, y) = 0.$$

Example.([13]) Any Lie algebra, Jordan algebra, alternative algebra or structurable algebra is a generalized structurable algebra.

In fact, the derivation in the case of a Lie algebra is $ad[x, y]$ and in the case of a Jordan algebra is $[L(x), R(y)]$ respectively.

By means of the concept of a generalized structurable algebra, we introduce the following algebraic structure.

For (i) $T(\varepsilon, \delta)$, by defining

$$X \circ Y = 0 \text{ and } D(X, Y)Z = [XYZ],$$

i.e., $D(X, Y) = ad([x_{-1}, y_1] + [x_1, y_{-1}])$, where $X = x_{-1} + x_1$ and $Y = y_{-1} + y_1$,

we obtain a structure with a trivial algebra which bilinear product $\circ$ is identically zero. However, this space is a Lie triple system ($\delta = 1$) or an anti-Lie triple system ($\delta = -1$) (c.f., section 2), that is,

$$D(X, Y)Z = [XYZ] = -\delta[YXZ] = -\delta D(Y, X)Z,$$

$$D(X, Y)Z + D(Y, Z)X + D(Z, X)Y = 0,$$

$$[D(X, Y), D(U, V)] = D(D(X, Y)U, V) + D(U, (D(X, Y)V).$$

Remark. Note that

$$D(X, Y) = -\delta D(Y, X) \quad for \;\; all \;\; X, Y \in T(\varepsilon, \delta).$$

Next, for the case (ii) $B(\varepsilon, \delta)$, we define as below

$$X \circ Y = [x_{-1}, y_{-1}] + [x_{-1}, y_2] + [x_{-2}, y_1] + [x_1, y_{-2}] + [x_1, y_1] + [x_2, y_{-1}]$$

and

$$D_B(X,Y) = ad([x_{-1}, y_1] + [x_{-2}, y_2] + [x_1, y_{-1}] + [x_2, y_{-2}]),$$

where $X = x_{-2} + x_{-1} + x_1 + x_2$, and $Y = y_{-2} + y_{-1} + y_1 + y_2$.

This space $B(\varepsilon, \delta)$ is a generalized structurable algebra with respect to the above product $X \circ Y$ and derivation $D_B(X,Y)$.

Remark. Note that

$$D_B(X,Y) = \begin{cases} -\delta D_B(Y,X) & if\ X, Y \in g_{-1} \oplus g_1 \\ -D_B(Y,X) & if\ X\ or\ Y \in g_{-2} \oplus g_2. \end{cases}$$

For (iii) $H(\varepsilon, \delta)$, we define as below

$$X \circ Y = [x_0, y_0] + [x_1, y_0] + [x_2, y_0] + [x_1, y_1] + [x_0, y_2] + [x_0, y_1]$$

$$D_H(X,Y) = ad([x_0, y_0]),$$

where $X = x_0 + x_1 + x_2$ and $Y = y_0 + y_1 + y_2$.

This space $H(\varepsilon, \delta)$ is a generalized structurable algebra with respect to the above product $X \circ Y$ and the derivation $D_H(X,Y) = ad([x_0, y_0])$.

Remark. Note that

$$D_H(X,Y) = \begin{cases} -\delta D_H(Y,X) & if\ X, Y \in g_1 \\ -D_H(Y,X) & if\ X\ or\ Y \in g_0 \oplus g_2. \end{cases}$$

5 Construction of B_3-type Lie algebras from several triple systems

In this section, we will discuss the construction of simple B_3-type Lie algebras associated with several triple systems (the details will be described in a forthcoming paper).

a) The case of a JTS,
b) The case of a balanced GJTS,
c) The case of a GJTS of 2nd order,
d) The case of a derivation induced from a JTS.

To consider these cases, we start with an extended Dynkin diagram for a B_3-type Lie algebra.

```
1         2         2
○  ···    ○   =>    ○
          |
          ○
         −ρ
```

where we denote $-\rho = \alpha_1 + 2\alpha_2 + 2\alpha_3$.

For the root system Δ of $B_3 = so(7)$ type and dim $B_3 = 21$ with simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$, it is well known that

$\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_2+2\alpha_3, \alpha_1+\alpha_2+2\alpha_3, \alpha_1+2\alpha_2+2\alpha_3\}$, and

$$B_3 = g = \mathbf{h} \oplus \Sigma_{\alpha\in\Delta}(g_\alpha \oplus g_{-\alpha}),$$

where $\mathbf{h}$ is the Cartan subalgebra of B_3.

5.1 The case of a JTS

First, we study the case of $g_{-1} = U = Mat(1,5;\Phi)$. (Hereafter, we assume Φ=C complex number field.)

In this case, g_{-1} is a JTS with respect to the product

$$\{xyz\} = x\ {}^t yz + y\ {}^t zx - z\ {}^t xy,$$

where ${}^t x$ denotes the transpose matrix of x.

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_3-type Lie algebra with $g_{-1} \oplus g_0 \oplus g_1$. Thus, we have

$$\begin{aligned} g_0 &= DerU \oplus Anti - DerU \\ &= B_2 \oplus \Phi H,\ where\ H := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \end{aligned}$$

$$Der(g_{-1} \oplus g_1) \cong \{\circ \cdots \circ \Longrightarrow \circ\} = B_3, \quad (\odot\ omitted).$$

$$\begin{array}{ccccc} 1 & & 2 & & 2 \\ \odot & \cdots & \circ & => & \circ \\ & & | & & \\ & & \circ & & \\ & & -\rho & & \end{array}$$

Furthermore, we obtain

$$\begin{aligned} &DerU = \{L(x,y) - L(y,x)\}_{span} = B_2, \\ &Anti - DerU = \{L(x,y) + L(y,x)\}_{span} = \Phi H, \end{aligned}$$

$$g_0 = \{\begin{pmatrix} L(x,y) & 0 \\ 0 & -L(y,x) \end{pmatrix}\}_{span} = \{S(x,y) + A(x,y)\}_{span},$$

where

$$S(x,y) = L(x,y) - L(x,y), A(x,y) = L(x,y) + L(y,x).$$

Here, g_{-1} corresponds to the root system

$$\{\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3\}.$$

5.2 The case of a balanced GJTS

Second, we study the case of $g_{-1} = U = Mat(2,3;\Phi)$.

In this case, g_{-1} is a balanced GJTS of 2nd order w.r.t. the product

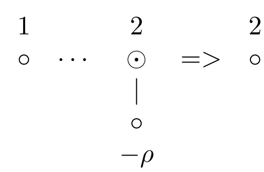
$$\{xyz\} := z\ {}^t yx + x\ {}^t yz - zJ_3\ {}^t xyJ_3,$$

where

$$J_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

By straightforward calculations, it can be shown that $L(U) = g$ is a 5-graded B_3-type Lie algebra with $g_{-2} \oplus \cdots \oplus g_2$ and $dim\ g_{-2} = 1$. Thus, we have

$$g_0 = DerU \oplus Anti - DerU = A_1 \oplus A_1 \oplus \Phi H,\ where\ H := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$$
$$Der(g_1 \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_1 \oplus A_1 \oplus A_1(\odot\ omitted)\ \cong DerT(U).$$



Furthermore, we obtain

$$g_{-2} = \{K(x,y)\}_{span} = \Phi Id \cdots \text{ this means one dimensional,}$$

i.e., balanced. This g_{-1} corresponds to the root system

$$\{\alpha_2,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3,\ \alpha_2 + 2\alpha_3,\ \alpha_1 + \alpha_2 + 2\alpha_3\},$$

g_{-2} corresponds to the highest root

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3\},$$

and $g/(g_{-2} \oplus g_0 \oplus g_2) \cong T(= g_{-1} \oplus g_1)$ is the tangent space of a quaternion symmetric space of dimension 12, since T is a Lie triple system associated with g_{-1} (e.g., [1]).

5.3 The case of a GJTS of 2nd order

Third, we study the case of $g_{-1} = U = Mat(1, 3; \Phi)$.

In this case, g_{-1} is a GJTS of 2nd order with respect to the product

$$\{xyz\} = x\ {}^tyz + z\ {}^tyx - y\ {}^txz.$$

By straightforward calculations, it can be shown that $L(U)$ is a 5-graded B_3-type Lie algebra with $g_{-2} \oplus \cdots \oplus g_2$ and $dim\ g_{-2} = 3$,

$$\begin{array}{l} g_0 = DerU \oplus Anti - DerU = A_2 \oplus \Phi H,\ where\ H := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \\ Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_3(\odot\ omitted)\ \cong DerT(U). \end{array}$$

$$\begin{array}{ccccc} 1 & & 2 & & 2 \\ \circ & \cdots & \circ & => & \odot \\ & & | & & \\ & & \circ & & \\ & & -\rho & & \end{array}$$

Furthermore, we obtain

$$g_{-2} = \{K(x, y)\}_{span} = Alt(3, 3; \Phi).$$

That is, the triple system $g_{-1}(resp.\ g_{-2})$ corresponds to the root system

$\{\alpha_3,\ \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3\}(resp.\ \{\alpha_2+2\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3\})$,

implying that

$$\underbrace{\circ \cdots\cdots \circ}\cdots \Longrightarrow \odot\ (\odot \text{ omitted) and}$$

$$g_0 = A_2 \oplus \Phi H.$$

Remark. Following [9], for the case of a GJT of 2nd order, note that $g_{-2}(\cong \mathbf{k})$ has the structure of the JTS associated with a GJTS of 2nd order.

5.4 The case of a derivation induced from a JTS

Finally, we study the case of $g_{-1} = U = Mat(1, 7; \Phi)$.

In this case, g_{-1} is a JTS with respect to the product

$$\{xyz\} = x\ {}^tyz + y\ {}^tzx - z\ {}^txy.$$

For this case, we obtain

$$\begin{array}{lll} DerU = & \{L(x, y) - L(y, x)\}_{span} = Alt(7, 7; \Phi) & \cong B_3, \\ Anti - DerU & \cong\ \Phi H,\ \text{which is one dimensional.} & \end{array}$$

The standard embedding Lie algebra is a 3-graded B_4-type Lie algebra with $g_{-1} \oplus g_0 \oplus g_1$.

Furthermore, we have

$$\odot \cdots \underbrace{\circ \cdots \circ \Longrightarrow \circ}\ (\odot \text{ omitted})$$

$$g_0 = B_3 \oplus \Phi H.$$

This case is obtained from $DerU$ such that $U = Mat(1, 7; \Phi)$ with the JTS structure.

Remark. In the above constructions, note that there exist four different constructions for B_3-type Lie algebras. These results may be applicable to mathematical physics, for example, quark theory and gravity theory.

6 Concluding Remarks

To briefly summarize our study, we note the following.

6.1 Geometrical viewpoint

The inner structures of triple systems are closely related to the characterization of root systems of Lie algebras. In particular, we have the following:

(i) There exists a correspondence between simple balanced (-1,1) -Freudenthal-Kantor triple systems and quaternionic Riemannian symmetric spaces ([1],[12],[17]). That is, there exists a correspondence between simple balanced GJTS of 2nd order and quaternionic Riemannian symmetric spaces.

(ii) There exists a relationship between Lie triple systems and totally geodesic manifolds ([5],[21]).

(iii) There exists a relationship between symmetric domains and positive definite Hermitian Jordan triple systems ([24]).

Thus, triple systems appear to be a useful tool and concept for characterizing geometrical phenomena.

6.2 Another viewpoint

For the theory of Peirce decompositions, we refer the reader to ([12],[14],[15]), and for the mathematical physics, we refer the reader to ([6],[19],[22]).

For sl_2 subalgebras, let $U(\varepsilon, \delta)$ be an (ε, δ)-F-K.t.s. and $L(\varepsilon, \delta)$ be the standard embedding Lie algebra or superalgebra. If $P \in \{K(a, b)\}_{span}$ and $P^2 = \mu Id$, $\sqrt{\mu} \in \Phi$, then we have the Lie subalgebras:

$$sl_2(\Phi) \leq D(T(\varepsilon, \delta)) \leq L(\varepsilon, \delta) \text{ and } sl_2(\Phi) := \{H, E, F\}_\Phi,$$

where $E = \frac{1}{\sqrt{\mu}}\begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}$, $F = \frac{1}{\sqrt{\mu}}\begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}$, $H := [E, F] = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$.

This means that there is a generalization of $sl_2(\Phi)$ loop algebra in $L(\varepsilon, \delta)$, which we will discuss in a future paper.

6.3 7-graded Lie algebra and superalgebra

Let $g = \sum_{i=-3}^{3} g_i$ be a 7 graded Lie algebra or superalgebra such that $[g_i, g_j] \subseteq g_{i+j}$ if $|i+j| \leq 3$ and $[g_i, g_j] = 0$ if $|i+j| \geq 4$.

If we define

$$A = g_{-2} \oplus g_{-1} \oplus g_1 \oplus g_2,$$

$$\begin{aligned} D(A, A) &= g_{-3} \oplus g_0 \oplus g_3 \\ &= ad([g_{-2}, g_{-1}] + [g_{-1} + g_{-2}] + [g_{-2}, g_2] + [g_{-1}, g_1] + \\ &\quad + [g_2, g_1] + [g_1, g_2] + [g_2, g_{-2}] + [g_1, g_{-1}]), \end{aligned}$$

where we denote $ad\ xy = [x, y]$, then A is a generalized structurable algebra with respect to the product

$$X \circ Y = [x_{-1}, y_{-1}] + [x_{-2}, y_1] + [x_{-1}, y_2] + [x_1, y_1] + [x_2, y_{-1}] + [x_1, y_{-2}]$$

where $X = x_{-2} + x_{-1} + x_1 + x_2$ and $Y = y_{-2} + y_{-1} + y_1 + y_2$, and the derivation $D(X, Y)$. That is, we have

$$D(X \circ Y, Z) + D(Y \circ Z, X) + D(Z \circ X, Y) = 0.$$

Finally, we emphasize that many mathematical and physical subjects involving geometic phenomena may be characterized by applying concept of triple systems.

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