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On Asymptotically Double Lacunary Statistical Equivalent Sequences in Probabilistic Normed Space

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Abstract

In this paper we study the concept of asymptotically double lacunary statistical convergent sequences in probabilistic normed spaces and prove some basic properties.

1 Introduction and Background

An interesting and important generalization of the notion of metric space was introduced by Menger [11] under the name of statistical metric, which is now called probabilistic metric space. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. The theory of probabilistic metric space was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [18, 19]. Probabilistic normed spaces (briefly, PN-spaces) are linear spaces in which the norm of each vector is an appropriate probability distribution function rather than a number. Such spaces were introduced by Serstnev in 1963 [20]. In [1], Alsina et al. gave a new definition of PN-spaces which includes Serstnev's a special case and leads naturally to the identification of the principle class of PN-spaces, the Menger spaces. An important family

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of probabilistic metric spaces are probabilistic normed spaces. The theory of probabilistic normed spaces is important as a generalization of deterministic results of linear normed space.

It seems therefore reasonable to think if the concept of statistical convergence can be extended to probabilistic normed spaces and in that case enquire how the basic properties are affected. But basic properties do not hold on probabilistic normed spaces. The problem is that the triangle function in such spaces.

Marouf [10] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In [13], Patterson extended those concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices. In [14],Patterson and Savaş extended the definitions presented in [13] to lacunary sequences. This paper extend the definitions presented in [4] to double lacunary sequences in probabilistic normed space.

The notion of double sequences has been investigated from different aspects by Tripathy [23], Tripathy and Dutta ([25], [26]), Tripathy and Sarma ([31], [32]) and many others.

In this paper we study the concept of asymptotically double lacunary statistical convergent sequences on probabilistic normed spaces. Since the study of convergence in PN-spaces is fundamental to probabilistic functional analysis, we feel that the concept of asymptotically double lacunary statistical convergent sequences in a PN-space would provide a more general framework for the subject.

2 Preliminaries

Now we recall some notations and definitions used in this paper.

Definition 1. *Let $K \subseteq \mathbb{N}\times\mathbb{N}$ be a two dimensional set of positive integers and let $K(n,m)$ be the numbers of (k,l) in K such that $k \leq n, l \leq m$. Then the two dimensional of natural density can be defined as follows: The lower asymptotic density of a set $K \subseteq \mathbb{N}\times\mathbb{N}$ is defined as $\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{nm}$.In case the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense [15], then we say that K has a double natural density and is defined as $\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm}$.*

Definition 2 ([12])**.** *A real double sequence $x = (x_{k,l})$ is to be statistically convergent to L, provided that for each $\varepsilon > 0$*

$$P - \lim_{m,n} \frac{1}{mn} \left|\left\{(k,l) : \ k \leq m \text{ and } l \leq n, |x_{k,l} - L| \geq \varepsilon\right\}\right| = 0.$$

In this case we write $S^L - \lim x = L$ or $x_{k,l} \to L\left(S^L\right)$.

Definition 3 ([16])**.** *The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that*

$$k_o = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty \text{ and } l_o = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notation 4. $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, *$\theta_{r,s}$ is determined by*

$$I_{r,s} = \{(k,l) : \ k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\},$$
$$q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \bar{q}_s.$$

Definition 5 ([4])**.** *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence; the two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ \ (k,l) \in I_{r,s} : \ \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \ge \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S^L_{\theta_{r,s}}}{\backsim} y$) and simply asymptotically double lacunary statistical equivalent if $L = 1$. Furthermore, let $S^L_{\theta_{r,s}}$ denote the set of all sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ such that $x \overset{S^L_{\theta_{r,s}}}{\backsim} y$.

For the following concepts, we refer to Menger [20], Schweizer-Sklar [19] and Alsina-Schweizer-Sklar[1].

Definition 6 ([11])**.** *A function $f : \mathbb{R} \to \mathbb{R}^+_o$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. We will denote the set of all distribution functions by D.*

Definition 7 ([11])**.** *A triangular norm, briefly t−norm, is a binary operation on $[0,1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, that is, it is the continuous mapping $\circledast : [0,1] \times [0,1] \to [0,1]$ such that for all $a, b, c \in [0,1]$:*

(1) $a \circledast 1 = a$,

(2) $a \circledast b = b \circledast a$,

(3) $c \circledast d \ge a \circledast b$ *if* $c \ge a$ *and* $d \ge b$,

(4) $(a \circledast b) \circledast c = a \circledast (b \circledast c)$.

Example 8. *The $\circledast$ operations $a \circledast b = \max\{a + b - 1, 0\}$, $a \circledast b = a.b$ and $a \circledast b = \min\{a, b\}$ on $[0,1]$ are t−norms.*

Definition 9 ([18, 19])**.** *A triple $(X, N, \ast)$ is called a probabilistic normed space or shortly PN-space if X is a real vector space, N is a mapping from X into D (for $x \in X$, the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value of N_x at $t \in \mathbb{R}$) and $\ast$ is a $t-$norm satisfying the following conditions:*

(PN-1) $N_x(0) = 0$,
(PN-2) $N_x(t) = 1$ *for all* $t > 0$ *if and only if* $x = 0$,
(PN-3) $N_{\alpha x}(t) = N_x\left(\frac{t}{|\alpha|}\right)$ *for all* $\alpha \in R \backslash \{0\}$,
(PN-4) $N_{x+y}(s+t) \geq N_x(s) \ast N_x(t)$ *for all* $x, y \in X$ *and* $s, t \in \mathbb{R}_o^+$.

Example 10. *Suppose that $(X, \|.\|)$ is a normed space $\mu \in D$ with $\mu(0) = 0$ and $\mu \neq h$, where*

$$h(t) = \begin{cases} 0 & , \quad t \leq 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Define

$$N_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ \mu\left(\frac{t}{\|x\|}\right) & , \quad x \neq 0 \end{cases},$$

where $x \in X, t \in \mathbb{R}$. Then $(X, N, \ast)$ is a PN-space. For example if we define the functions μ and ν on $\mathbb{R}$ by

$$\mu(x) = \begin{cases} 0 & , \quad x \leq 0 \\ \frac{x}{1+x} & , \quad x > 0 \end{cases}, \quad \nu(x) = \begin{cases} 0 & , \quad x \leq 0 \\ e^{-\frac{1}{x}} & , \quad x > 0 \end{cases}$$

then we obtain the following well-known $\ast$ norms:

$$N_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ \frac{t}{t+\|x\|} & , \quad x \neq 0 \end{cases}, \quad M_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ e^{\left(-\frac{\|x\|}{t}\right)} & , \quad x \neq 0 \end{cases}.$$

We recall the concept of double convergence sequences in a probabilistic normed space.

Definition 11 ([8])**.** *Let $(X, N, \ast)$ be a PN-space. Then a double sequence $x = (x_{k,l})$ is said to be convergent to $L \in X$ with respect to the probabilistic norm N if, for every $\varepsilon > 0$ and $\theta \in (0, 1)$, there exists a positive integer k_o such that $N_{x_{k,l}-L}(\varepsilon) > 1-\theta$ whenever $k, l \geq k_o$. It is denoted by $N_2-\lim x = L$ or $x_{k,l} \overset{N_2}{\rightarrow} L$ as $k, l \rightarrow \infty$.*

3 Asymptotically Double Lacunary Convergence in PN-spaces

The idea of statistical convergence was first introduced by Steinhaus in 1951 [21] and then studied by various authors, e.g. Fast [5], Salat [17], Fridy [6], Esi [3], Tripathy ([22], [23]), Tripathy and Mahanta ([29], [30]), Tripathy and Dutta ([25], [26]), Tripathy and Sarma [31], Tripathy and Hazarika ([27], [28], Tripathy and Baruah [24], and many others. In normed space by Kolk [9]. Karakus [7] has studied the concept of statistical convergence in probabilistic normed spaces for single sequences. Recently Karakuş and Demirci [8] have studied this concept for double sequences.

Definition 12 ([8]). *Let $(X, N, \circledast)$ be a PN-space. Then a double sequence $x = (x_{k,l})$ is said to be statistically convergent to $L \in X$ with respect to the probabilistic norm N provided that for every $\varepsilon > 0$ and $\gamma \in (0,1)$*

$$\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : N_{x_{k,l}-L}(\varepsilon) \leq 1-\gamma \right\}\right) = 0,$$

or equivalently

$$\lim_{n,m} \frac{1}{nm} \left|\left\{k \leq n, l \leq m :\ N_{x_{k,l}-L}(\varepsilon) \leq 1-\gamma \right\}\right| = 0.$$

In this case we write $st_{N_2} - \lim x = L$.

Definition 13. *Let $(X, N, \circledast)$ be a PN-space and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. The two non-negative sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double statistical equivalent of multiple L in PN-space X if for every $\varepsilon > 0$ and $\gamma \in (0,1)$*

$$\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon) \leq 1-\gamma \right\}\right) = 0, \tag{1}$$

or equivalently

$$\lim_{n,m} \frac{1}{nm} \left|\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : k \leq m \text{ and } l \leq n,\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon) \leq 1-\gamma \right\}\right| = 0.$$

In this case we write $x \overset{S^L(PN)}{\backsim} y$ and simply asymptotically double statistical equivalent if $L = 1$. Furthermore, let $S(PN)$ denote the set of all sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ such that $x \overset{S^L(PN)}{\backsim} y$.

Definition 14. *Let $(X, N, \circledast)$ be a PN-space and $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. The two non-negative sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double lacunary statistical equivalent of multiple L in PN-space X if for every $\varepsilon > 0$ and $\gamma \in (0,1)$*

$$\delta_\theta\left(\left\{(k,l) \in I_{r,s} : N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon) \leq 1-\gamma \right\}\right) = 0$$

or equivalently

$$\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\ \right\}\right|=0.$$

In this case we write $x\overset{S_\theta^L(PN)}{\backsim}y$ *and simply asymptotically double lacunary statistical equivalent if* $L=1$. *Furthermore, let* $S_\theta(PN)$ *denote the set of all sequences* $x=(x_{k,l})$ *and* $y=(y_{k,l})$ *such that* $x\overset{S_\theta^L(PN)}{\backsim}y$.

By using (1) and well-known density properties, we easily get the following lemma.

Lemma 15. *Let* $(X,N,\circledast)$ *be a PN-space. Then, for every* $\varepsilon>0$ *and* $\gamma\in(0,1)$, *the following statements are equivalent:*

(i) $\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\ \right\}\right|=0,$

(ii) $\delta_\theta\left(\left\{(k,l)\in I_{r,s}:N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\ \right\}\right)=0,$

(iii) $\delta_\theta\left(\left\{(k,l)\in I_{r,s}:N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)>1-\gamma\ \right\}\right)=1,$

(iv) $\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)>1-\gamma\right\}\right|=1.$

Theorem 16. *Let* $(X,N,\circledast)$ *be a PN-space. If two sequences* $x=(x_{k,l})$ *and* $y=(y_{k,l})$ *are asymptotically double lacunary statistical equivalent of multiple* L *with respect to the probabilistic norm* N, *then* L *is unique.*

Proof. Assume that $x\overset{S_\theta^{L_1}(PN)}{\backsim}y$ and $x\overset{S_\theta^{L_2}(PN)}{\backsim}y$, $(L_1\neq L_2)$. For a given $\lambda>0$ choose $\gamma\in(0,1)$ such that $(1-\gamma)\circledast(1-\gamma)>1-\lambda$. Then, for any $\varepsilon>0$, define the following sets:

$$K_1=\left\{(k,l)\in I_{r,s}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L_1}(\varepsilon)\leq 1-\gamma\ \right\}$$

and

$$K_2=\left\{(k,l)\in I_{r,s}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L_2}(\varepsilon)\leq 1-\gamma\ \right\}.$$

Then, clearly

$$\lim_{r,s}\frac{|K_1\cap K_2|}{h_{r,s}}=1,$$

so $K_1\cap K_2$ is a non-empty set. Since $x\overset{S_\theta^{L_1}(PN)}{\backsim}y$, $\delta_\theta(K_1)=0$ and $x\overset{S_\theta^{L_2}(PN)}{\backsim}y$, $\delta_\theta(K_2)=0$ for all $\varepsilon>0$. Now let $K=K_1\cap K_2$. Then we observe that

$\delta_\theta(K) = 0$ which implies $\delta_\theta(\mathbb{N}\times\mathbb{N} - K) = 1$. If $(k,l)\in\mathbb{N}\times\mathbb{N} - K$, then we have

$$N_{L_1-L_2}(\varepsilon) = N_{\frac{x_k}{y_k}-L_1-L_2-\frac{x_k}{y_k}}(\varepsilon) \geq N_{\frac{x_k}{y_k}-L_1}\left(\frac{\varepsilon}{2}\right) \circledast N_{\frac{x_k}{y_k}-L_2}\left(\frac{\varepsilon}{2}\right)$$
$$> (1-\gamma)\circledast(1-\gamma) \geq 1-\lambda\ .$$

Since $\lambda > 0$ was arbitrary, we get $N_{L_1-L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which gives $L_1 = L_2$. This completes the proof. □

Theorem 17. *Let* $(X, N, \circledast)$ *be a PN-space and* $\theta_{r,s} = \{(k_r, l_s)\}$ *be a double lacunary sequence with* $\limsup_r q_r < \infty$ *and* $\limsup_s \overline{q}_s < \infty$ *then* $S_\theta(PN) \subset S(PN)$.

Proof. Since $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$, so there exists $H > 0$ such that $q_r < H$ and $\overline{q}_s < H$ for all r and s. Let $x \overset{S^L_\theta(PN)}{\backsim} y$ and $\varepsilon > 0$, be given. Then there exists $r_o > 0$ and $s_o > 0$ such that for every $i \geq r_o$ and $j \geq s_o$

$$B_{i,j} = \frac{1}{h_{i,j}}\left|\left\{\ (k,l)\in I_{i,j}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\right\}\right| < \varepsilon.$$

Let $M = \max\{B_{i,j}:\ 1\leq i\leq r_o \text{ and } 1\leq j\leq s_o\}$ and m and n be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$.Thus we obtain the following

$$\frac{1}{mn}\left|\left\{(k,l)\in I_{i,j}:\ k\leq m \text{ and } l\leq n, N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\right\}\right|$$
$$\leq\frac{1}{k_{r-1}l_{s-1}}\left|\left\{(k,l)\in I_{i,j}:\ k\leq k_r \text{ and } l\leq l_s, N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\right\}\right|$$
$$\leq\frac{1}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r_o,s_o} h_{t,u}B_{t,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}B_{t,u}$$
$$\leq\frac{M}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r_o,s_o} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}B_{t,u}$$
$$\leq\frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}B_{t,u}$$
$$\leq\frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \left(\sup_{t\geq r_o\cup u\geq s_o} B_{t,u}\right)\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}$$
$$\leq\frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \frac{\varepsilon}{k_{r-1}l_{s-1}}\sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}$$
$$\leq\frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \varepsilon H^2.$$

Since k_r and l_s both approach to infinity as both m and n approach to infinity, it follows that

$$\frac{1}{mn}\left|\left\{(k,l)\in I_{i,j}:\ k\leq m \text{ and } l\leq n, N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\right\}\right|\to 0.$$

This completes the proof. □

Theorem 18. *Let* $(X,N,\circledast)$ *be a PN-space and* $\theta_{r,s}=\{(k_r,l_s)\}$ *be a double lacunary sequence with* $\liminf_r q_r>1$ *and* $\liminf_s \overline{q}_s>1$ *then* $S(PN)\subset S_\theta(PN)$.

Proof. Suppose that $\liminf_r q_r>1$ and $\liminf_s \overline{q}_s>1$ then there exists $\delta>0$ such that $q_r>1+\delta$ and $\overline{q}_s>1+\delta$. This implies that $\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}$ and $\frac{\overline{h}_s}{l_s}\geq\frac{\delta}{1+\delta}$.Since $h_{r,s}=k_rl_s-k_{r-1}l_{s-1}$, we are granted the following

$$\frac{k_rl_s}{h_{r,s}}\leq\frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1}l_{s-1}}{h_{r,s}}\leq\frac{1}{\delta}.$$

Then for $x\overset{S^L(PN)}{\backsim}y$, we can write for every $\varepsilon>0$ and for sufficiently large r and s, we have

$$\begin{aligned}&\frac{1}{k_rl_s}\left|\left\{\ (k,l)\in I_{r,s}:k\leq k_r \text{ and } l\leq l_s,\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\right\}\right|\\ \geq&\frac{1}{k_rl_s}\left|\left\{\ (k,l)\in I_{r,s}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\right\}\right|\\ \geq&\frac{\delta}{1+\delta}.\frac{1}{h_{r,s}}\left|\left\{\ (k,l)\in I_{r,s}:\ N_{\frac{x_{k,l}}{y_{k,l}}-L}(\varepsilon)\leq 1-\gamma\right\}\right|.\end{aligned}$$

This completes the proof. □

Corollary 19. *Let* $(X,N,\circledast)$ *be a PN-space and* $\theta_{r,s}=\{(k_r,l_s)\}$ *be a double lacunary sequence with* $1<\liminf_{r,s}q_{rs}\leq\limsup_{r,s}q_{rs}<\infty$ *, then* $S(PN)$ $=S_\theta(PN)$.

Proof. The result follows from Theorem 17 and 18. □

Conclusion 20. *The idea of probabilistic norm is very useful to deal with the convergence problems of sequences of real numbers. The main purpose of this paper is to generalize the results on statistical convergence proved by Karakuş and Demirci [8]. We have introduced a more wider class of asymptotically double lacunary statistically convergent sequences in a PN-space to deal with the double sequences which are not covered in [8].*

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