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Strong convergence of a hybrid method for pseudomonotone variational inequalities and fixed point problems

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Abstract

In this paper, we suggest a hybrid method for finding a common element of the set of solution of a pseudomonotone, Lipschitz-continuous variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings. The proposed iterative method combines two well-known methods: extragradient method and CQ method. We derive a necessary and sufficient condition for the strong convergence of the sequences generated by the proposed method.

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear operator. By definition, the variational inequality problem VI(C, A) is to find $u \in C$ such that

$$(\text{VI(C,A)}): \quad \langle Au, v-u\rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality is denoted by Ω.

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral and equilibrium problems, which arise in

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several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [1], [8], [9],[11]-[14],[21]-[24], [28]-[31] and the references therein. Let us start with Korpelevich's extragradient method which was introduced by Korpelevich [13] in 1976 and which generates a sequence $\{x_n\}$ via the recursion:

$$\begin{cases} y_n = P_C[x_n - \lambda A x_n], \\ x_{n+1} = P_C[x_n - \lambda A y_n], n \geq 0, \end{cases} \tag{1}$$

where P_C is the metric projection from R^n onto C, $A : C \to H$ is a monotone operator and λ is a constant. Korpelevich [13] proved that the sequence $\{x_n\}$ converges strongly to a solution of $VI(C, A)$. Note that the setting of the problem is the Euclidean space R^n.

Korpelevich's extragradient method has extensively been studied in the literature for solving a more general problem that consists of finding a common point that lies in the solution set of a variational inequality and the set of fixed points of a nonexpansive mapping. This type of problem aries in various theoretical and modeling contexts, see e.g., [2],[4]-[7],[15],[25],[26] and references therein. Especially, Nadezhkina and Takahashi [17] introduced the following iterative method which combines Korpelevich's extragradient method and a CQ method:

$$\begin{aligned} &x_0 = x \in C, \\ &y_n = P_C[x_n - \lambda_n A x_n], \\ &z_n = \alpha_n x_n + (1 - \alpha_n) S P_C[x_n - \lambda_n A y_n], \\ &C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ &Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ &x_{n+1} = P_{C_n \cap Q_n} x, n \geq 0, n \geq 0, \end{aligned}$$

where P_C is the metric projection from H onto C, $A : C \to H$ is a monotone k-Lipschitz-continuous mapping, $S : C \to C$ is a nonexpansive mapping, $\{\lambda_n\}$ and $\{\alpha_n\}$ are two real number sequences. They proved the strong convergence of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ to the same element in $Fix(S) \cap \Omega$. We note that Nadezhkina and Takahashi [17] employed the monotonicity and Lipschitz-continuity of A to define a maximal monotone operator T as follows:

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

where $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$ (see, [19]). However, if the mapping A is a pseudomonotone Lipschitz-continuous, then T is not necessarily a maximal monotone operator. This fact implies that the approach used in [17] cannot be applied. To overcome this difficulty, Ceng, Teboulle and Yao [3] suggested a new iterative method as follows:

$$\begin{aligned}&y_n = P_C[x_n - \lambda_n A x_n],\\ &z_n = \alpha_n x_n + (1-\alpha_n) S_n P_C[x_n - \lambda_n A y_n],\\ &C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\},\\ &\text{find } x_{n+1} \in C_n \text{ such that}\\ &\langle x_n - x_{n+1} + e_n - \sigma_n A x_{n+1}, x_{n+1} - x\rangle \ge -\epsilon_n, \quad \forall x \in C_n,\end{aligned}$$

where $A : C \to H$ is a pseudomonotone, k-lipschitz-continuous and (w,s)-sequentially-continuous mapping, $\{S_i\}_{i=1}^N : C \to C$ are N nonexpansive mappings. Under some mild conditions, they proved that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge weakly to the same element of $\bigcap_{i=1}^N Fix(S_i) \cap \Omega$ if and only if $\liminf_n \langle Ax_n, x - x_n\rangle \ge 0, \forall x \in C$. Note that Ceng, Teboulle and Yao's method has only weak convergence. So, we may ask of whether $i)$ a strong convergence property is available, $ii)$ a denumerable family of maps $(S_i; i \ge 1)$ is allowed.

Motivated and inspired by the works of Nadezhkina and Takahashi [17] and Ceng, Teboulle and Yao [3], in this paper we suggest a hybrid method for finding a common element of the set of solution of a pseudomonotone, Lipschitz-continuous variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings. The proposed iterative method combines two well-known methods: extragradient method and CQ method. We derive a necessary and sufficient condition for the strong convergence of the sequences generated by the proposed method.

2 Preliminaries

In this section, we will recall some basic notations and collect some conclusions that will be used in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called monotone if

$$\langle Au - Av, u - v\rangle \ge 0, \forall u, v \in C.$$

A mapping $A : C \to H$ is called pseudomonotone if, for all $u, v \in C$,

$$\langle Au, v - u \ge 0 \Rightarrow \langle Av, v - u\rangle \ge 0.$$

It is clear that if a mapping A is monotone, then it is pseudomonotone.

Recall that a mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C.$$

Denote by $Fix(S)$ the set of fixed points of S; that is, $Fix(S) = \{x \in C : Sx = x\}$.

It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}.$$

We denote u_0 by $P_C[u]$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\|P_C[x] - P_C[y]\| \leq \|x - y\|$ for all $x, y \in H$.

(ii) $\langle x - P_C[x], y - P_C[x] \rangle \leq 0$ for all $x \in H$, $y \in C$.

(iii) The property (ii) is equivalent to

$$\|x - P_C[x]\|^2 + \|y - P_C[x]\|^2 \leq \|x - y\|, \forall x \in H, y \in C.$$

(iv) In the context of the variational inequality problem, the characterization of the projection implies that

$$u \in \Omega \Leftrightarrow u = P_C[u - \lambda A u], \forall \lambda > 0.$$

Recall that H satisfies the Opial condition [27]; i.e., for any sequence $\{x_n\}$ with x_n converges weakly to x, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_i\}_{i=1}^{\infty}$ be infinite family of nonexpansive mappings of C into itself and let $\{\xi_i\}_{i=1}^{\infty}$ be real number sequences such that $0 \leq \xi_i \leq 1$ for every $i \in \mathbf{N}$. For any $n \in \mathbf{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \xi_n S_n U_{n,n+1} + (1-\xi_n)I,\\
&U_{n,n-1} = \xi_{n-1} S_{n-1} U_{n,n} + (1-\xi_{n-1})I,\\
&\quad\vdots\\
&U_{n,k} = \xi_k S_k U_{n,k+1} + (1-\xi_k)I,\\
&U_{n,k-1} = \xi_{k-1} S_{k-1} U_{n,k} + (1-\xi_{k-1})I,\\
&\quad\vdots\\
&U_{n,2} = \xi_2 S_2 U_{n,3} + (1-\xi_2)I,\\
&W_n = U_{n,1} = \xi_1 S_1 U_{n,2} + (1-\xi_1)I. \qquad (2)
\end{aligned}$$

Such W_n is called the W-mapping generated by $\{S_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$.

We have the following crucial Lemmas 3.1 and 3.2 concerning W_n which can be found in [20]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S_1, S_2, \cdots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} Fix(S_n)$ is nonempty, and let $\xi_1, \xi_2, \cdots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in \mathbf{N}$. Then, for every $x \in C$ and $k \in \mathbf{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S_1, S_2, \cdots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} Fix(S_n)$ is nonempty, and let $\xi_1, \xi_2, \cdots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in N$. Then, $Fix(W) = \bigcap_{n=1}^{\infty} Fix(S_n)$.*

Lemma 2.3. *(see [27]) Using Lemmas 2.1 and 2.2, one can define a mapping W of C into itself as: $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$. If $\{x_n\}$ is a bounded sequence in C, then we have*

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0.$$

We also need the following well-known lemmas for proving our main results.

Lemma 2.4. *([10]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. Then S is demiclosed on C, i.e., if $y_n \to z \in C$ weakly and $y_n - Sy_n \to y$ strongly, then $(I - S)z = y$.*

Lemma 2.5. *([16]) Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C[u]$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \text{ for all } n.$$

Then $x_n \to q$.

We adopt the following notation:

- For a given sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$; that is,

  $\omega_w(x_n) := \{x \in H : \{x_{n_j}\}$ converges weakly to x for some subsequence $\{n_j\}$ of $\{n\}\}$.

- $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x;
- $x_n \to x$ stands for the strong convergence of (x_n) to x.

3 Main results

In this section we will state and prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a pseudomonotone, k-Lipschitz-continuous and (w, s)-sequentially-continuous mapping and let $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega \neq \emptyset$. Let $x_1 = x_0 \in C$. For $x_1 \in C$, $C_1 = C$, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{C_n\}$ be sequences generated as:*

$$\begin{aligned} &y_n = P_{C_n}[x_n - \lambda_n A x_n], \\ &z_n = \alpha_n x_n + (1 - \alpha_n) W_n P_{C_n}[x_n - \lambda_n A y_n], \\ &C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ &x_{n+1} = P_{C_{n+1}}[x_0], n \geq 1, \end{aligned} \tag{3}$$

where $\{W_n; n \geq 1\}$ are W-mappings of (2). Assume that

(i) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$;

(ii) $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (3) converge strongly to the same point $P_{\bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega}[x_0]$ if and only if $\liminf_n \langle Ax_n, x - x_n \rangle \geq 0$, $\forall x \in C$.

The proof will be divided into several conclusions. Assume in the sequel that all assumptions of Theorem 3.1 are satisfied.

Conclusion 3.2. (1) *Every C_n is closed and convex, $n \geq 1$;*

(2) $\bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega \subset C_{n+1}, \forall n \geq 1$;

(3) $\{x_{n+1}\}$ *is well-defined.*

Proof. First we note that $C_1 = C$ is closed and convex. Assume that C_k is closed and convex. From (3), we can rewrite C_{k+1} as

$$C_{k+1} = \{z \in C_k : \langle z - \frac{x_k + z_k}{2}, z_k - x_k\rangle \geq 0\}.$$

It is clear that C_{k+1} is a half space. Hence, C_{k+1} is closed and convex. By induction, we deduce that C_n is closed and convex for all $n \geq 1$. Next we show that $\bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega \subset C_{n+1}, \forall n \geq 1$.

Set $t_n = P_{C_n}[x_n - \lambda Ay_n]$ for all $n \geq 1$. Pick up $u \in \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega$. From property (iii) of P_C, we have

$$\begin{aligned}\|t_n - u\|^2 &\leq \|x_n - \lambda_n Ay_n - u\|^2 - \|x_n - \lambda_n Ay_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - y_n\rangle + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \end{aligned} \quad (4)$$

Since $u \in \Omega$ and $y_n \in C_n \subset C$, we get

$$\langle Au, y_n - u\rangle \geq 0.$$

This together with the pseudomonotonicity of A imply that

$$\langle Ay_n, y_n - u\rangle \geq 0. \quad (5)$$

Combine (4) with (5) to deduce

$$\begin{aligned}\|t_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\| - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\quad +2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad +2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle. \end{aligned} \quad (6)$$

Note that $y_n = P_{C_n}[x_n - \lambda_n Ax_n]$ and $t_n \in C_n$. Then, by using the property (ii) of P_C, we have

$$\langle x_n - \lambda_n Ax_n - y_n, t_n - y_n\rangle \leq 0.$$

Hence,

$$\begin{aligned}\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\ &\leq \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\ &\leq \lambda_n k\|x_n - y_n\|\|t_n - y_n\|. \qquad (7)\end{aligned}$$

From (6) and (7), we get

$$\begin{aligned}\|t_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|x_n - y_n\|\|t_n - y_n\| \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\ &= \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \qquad (8) \\ &\leq \|x_n - u\|^2.\end{aligned}$$

Therefore, from (8), together with $z_n = \alpha_n x_n + (1-\alpha_n)W_n t_n$ and $u = W_n u$, we get

$$\begin{aligned}\|z_n - u\|^2 &= \|\alpha_n(x_n - u) + (1-\alpha_n)(W_n t_n - u)\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|W_n t_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\ &\leq \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \qquad (9) \\ &\leq \|x_n - u\|^2,\end{aligned}$$

which implies that

$$u \in C_{n+1}.$$

Therefore,

$$\bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega \subset C_{n+1}, \forall n \geq 1.$$

This implies that $\{x_{n+1}\}$ is well-defined. □

Conclusion 3.3. *The sequences* $\{x_n\}, \{z_n\}$ *and* $\{t_n\}$ *are all bounded and* $\lim_{n\to\infty}\|x_n - x_0\|$ *exists.*

Proof. From $x_{n+1} = P_{C_{n+1}}[x_0]$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - y\rangle \geq 0, \forall y \in C_{n+1}.$$

Since $\bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega \subset C_{n+1}$, we also have

$$\langle x_0 - x_{n+1}, x_{n+1} - u\rangle \geq 0, \forall u \in \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega.$$

So, for $u \in \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_{n+1}, x_{n+1} - u \rangle \\ &= \langle x_0 - x_{n+1}, x_{n+1} - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_{n+1}\|^2 + \langle x_0 - x_{n+1}, x_0 - u \rangle \\ &\leq -\|x_0 - x_{n+1}\|^2 + \|x_0 - x_{n+1}\| \|x_0 - u\|. \end{aligned}$$

Hence,

$$\|x_0 - x_{n+1}\| \leq \|x_0 - u\|, \forall u \in \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega, \tag{10}$$

which implies that $\{x_n\}$ is bounded. From (8) and (9), we can deduce that $\{z_n\}$ and $\{t_n\}$ are also bounded.

From $x_n = P_{C_n}[x_0]$ and $x_{n+1} = P_{C_{n+1}}[x_0] \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{11}$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \end{aligned}$$

and therefore

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

This together with the boundedness of the sequence $\{x_n\}$ imply that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. □

Conclusion 3.4. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|x_n - t_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - W_n x_n\| = \lim_{n\to\infty} \|x_n - W x_n\| = 0$.

Proof. It is well-known that in Hilbert spaces H, the following identity holds:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \quad \forall x, y \in H.$$

Therefore,

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle. \end{aligned}$$

It follows from (11) that

$$\|x_{n+1}-x_n\|^2 \leq \|x_{n+1}-x_0\|^2 - \|x_n-x_0\|^2.$$

Since $\lim_{n\to\infty}\|x_n-x_0\|$ exists, we get $\|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2 \to 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Since $x_{n+1}\in C_n$, we have

$$\|z_n-x_{n+1}\| \leq \|x_n-x_{n+1}\|,$$

and hence

$$\begin{aligned}\|x_n-z_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-z_n\|\\ &\leq 2\|x_{n+1}-x_n\|\\ &\to 0.\end{aligned}$$

For each $u\in\bigcap_{n=1}^{\infty}Fix(S_n)\cap\Omega$, from (9), we have

$$\begin{aligned}\|x_n-y_n\|^2 &\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|^2-\|z_n-u\|^2)\\ &\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|+\|z_n-u\|)\|x_n-z_n\|.\end{aligned}$$

Since $\|x_n-z_n\|\to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n-y_n\|\to 0$.

We note that

$$\begin{aligned}\|t_n-u\|^2 &\leq \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+2\lambda_nk\|x_n-y_n\|\|t_n-y_n\|\\ &\leq \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+\|x_n-y_n\|^2+\lambda_n^2k^2\|y_n-t_n\|^2\\ &= \|x_n-u\|^2+(\lambda_n^2k^2-1)\|y_n-t_n\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|z_n-u\|^2 &\leq \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|t_n-u\|^2\\ &\leq \alpha_n\|x_n-u\|^2+(1-\alpha_n)(\|x_n-u\|^2+(\lambda_n^2k^2-1)\|y_n-t_n\|^2)\\ &= \|x_n-u\|^2+(1-\alpha_n)(\lambda_n^2k^2-1)\|y_n-t_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|t_n-y_n\|^2 &\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|^2-\|z_n-u\|^2)\\ &\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|+\|z_n-u\|)\|x_n-z_n\|\\ &\to 0.\end{aligned}$$

Since A is k-Lipschitz-continuous, we have $\|Ay_n - At_n\| \to 0$. From

$$\|x_n - t_n\| \leq \|x_n - y_n\| + \|y_n - t_n\|,$$

we also have

$$\|x_n - t_n\| \to 0.$$

Since $z_n = \alpha_n x_n + (1-\alpha_n)W_n t_n$, we have

$$(1-\alpha_n)(W_n t_n - t_n) = \alpha_n(t_n - x_n) + (z_n - t_n).$$

Then,

$$\begin{aligned}(1-c)\|W_n t_n - t_n\| &\leq (1-\alpha_n)\|W_n t_n - t_n\| \\ &\leq \alpha_n\|t_n - x_n\| + \|z_n - t_n\| \\ &\leq (1+\alpha_n)\|t_n - x_n\| + \|z_n - x_n\|\end{aligned}$$

and hence $\|t_n - W_n t_n\| \to 0$. Observe also that

$$\begin{aligned}\|x_n - W_n x_n\| &\leq \|x_n - t_n\| + \|t_n - W_n t_n\| + \|W_n t_n - W_n x_n\| \\ &\leq \|x_n - t_n\| + \|t_n - W_n t_n\| + \|t_n - x_n\| \\ &\leq 2\|x_n - t_n\| + \|t_n - W_n t_n\|.\end{aligned}$$

So, we have $\|x_n - W_n x_n\| \to 0$. On the other hand, since $\{x_n\}$ is bounded, from Lemma 2.3, we have $\lim_{n\to\infty}\|W_n x_n - W x_n\| = 0$. Therefore, we have

$$\lim_{n\to\infty}\|x_n - W x_n\| = 0.$$

□

Proof. Proof of Theorem 3.1, continued. First, we prove the necessity. Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same element $\tilde{u} \in \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega$. From the (w,s)-sequential continuity of A, we have $Ax_n \to A\tilde{u}$. Observe that, for every $x \in C$,

$$\begin{aligned}|\langle Ax_n, x - x_n\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| &\leq |\langle Ax_n, x - x_n\rangle - \langle A\tilde{u}, x - x_n\rangle| \\ &\quad + |\langle A\tilde{u}, x - x_n\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| \\ &= |\langle Ax_n - A\tilde{u}, x - x_n\rangle| + |\langle A\tilde{u}, \tilde{u} - x_n\rangle| \\ &\leq \|Ax_n - A\tilde{u}\|\|x - x_n\| + |\langle A\tilde{u}, \tilde{u} - x_n\rangle|.\end{aligned}$$

This implies that

$$\liminf_{n\to\infty}\langle Ax_n, x - x_n\rangle = \lim_{n\to\infty}\langle Ax_n, x - x_n\rangle = \langle A\tilde{u}, x - \tilde{u}\rangle, \forall x \in C.$$

Consequently, the necessity holds.

Next, we prove the sufficiency. By Conclusions 3.3-3.5, we have

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0.$$

Furthermore, since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}$ which converges weakly to some $\tilde{u} \in C$; hence, we have $\lim_{j\to\infty} \|x_{n_j} - Wx_{n_j}\| = 0$. Note that, from Lemma 2.4, it follows that $I - W$ is demiclosed at zero. Thus $\tilde{u} \in Fix(W)$. Observe that, for every $x \in C$,

$$\begin{aligned} & |\langle Ax_{n_j}, x - x_{n_j}\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| \\ \le\ & |\langle Ax_{n_j}, x - x_{n_j}\rangle - \langle A\tilde{u}, x - x_{n_j}\rangle| + |\langle A\tilde{u}, x - x_{n_j}\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| \\ =\ & |\langle Ax_{n_j} - A\tilde{u}, x - x_{n_j}\rangle| + |\langle A\tilde{u}, \tilde{u} - x_{n_j}\rangle| \\ \le\ & \|Ax_{n_j} - A\tilde{u}\|\|x - x_{n_j}\| + |\langle A\tilde{u}, \tilde{u} - x_{n_j}\rangle|. \end{aligned}$$

From the (w, s)-sequential continuity of A, it follows that $\lim_{j\to\infty} \|Ax_{n_j} - A\tilde{u}\| = 0$. Hence, we have

$$\langle A\tilde{u}, x - \tilde{u}\rangle = \lim_{j\to\infty} \langle Ax_{n_j}, x - x_{n_j}\rangle \ge \liminf_{n\to\infty} \langle Ax_n, x - x_n\rangle \ge 0, \forall x \in C.$$

This implies that $\tilde{u} \in \Omega$. Consequently, $\tilde{u} \in \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega$. That is, $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega$.

In (10), if we take $u = P_{\bigcap_{n=1}^{\infty} Fix(S_n)\cap\Omega}[x_0]$, we get

$$\|x_0 - x_{n+1}\| \le \|x_0 - P_{\bigcap_{n=1}^{\infty} Fix(S_n)\cap\Omega}[x_0]\|. \tag{12}$$

Notice that $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega$. Then, (12) and Lemma 2.5 ensure the strong convergence of $\{x_{n+1}\}$ to $P_{\bigcap_{n=1}^{\infty} Fix(S_n)\cap\Omega}[x_0]$. Consequently, $\{y_n\}$ and $\{z_n\}$ also converge strongly to $P_{\bigcap_{n=1}^{\infty} Fix(S_n)\cap\Omega}[x_0]$. This completes the proof. □

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