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Stability of generalized Newton difference equations*

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Abstract

In the paper we discuss a stability in the sense of the generalized Hyers-Ulam-Rassias for functional equations $\Delta^n_{(p,\ c)}\varphi(x) = h(x)$, which is called generalized Newton difference equations, and give a sufficient condition of the generalized Hyers-Ulam-Rassias stability. As corollaries, we obtain the generalized Hyers-Ulam-Rassias stability for generalized forms of square root spirals functional equations and general Newton functional equations for logarithmic spirals.

1 Introduction

In 1940, S.M. Ulam [24] posed the stability problem of functional equations: When is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation? For Banach spaces, the problem was solved by D. H. Hyers [7] in the case of approximately additive mappings. Thereafter, such idea of stability is called the Hyers-Ulam stability of functional equations. This concept is also generalized in [22]. As in [8, 13, 14] we say a functional equation

$$E_1(\varphi) = E_2(\varphi) \tag{1.1}$$

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has the *generalized Hyers-Ulam-Rassias stability* if for an approximate solution φ_s such that

$$|E_1(\varphi_s)(x) - E_2(\varphi_s)(x)| \leq \phi(x),$$

for some fixed function ϕ, there exists a solution φ of equation (1.1) such that $|\varphi_s(x) - \varphi(x)| \leq \Phi(x)$ for some fixed function Φ depending only on ϕ. For some results on the stability of functional equations have been discussed extensively in many references, e.g., [1, 2, 3, 4, 5, 9, 10, 16, 17, 18, 19, 20, 21].

For the linear functional equation

$$\varphi(f(x)) = g(x)\varphi(x) + h(x), \tag{1.2}$$

in some classes of special function, where f, g, h are given functions and φ is an unknown function, M. Kuczma, B. Choczewski and R. Ger [15] gave some results in details on nonnegative solutions, monotonic solutions, convex and regularly varying solutions, and regular solutions of equation (1.2). The generalized Hyers-Ulam-Rassias stability of equation (1.2) was discussed by T. Trif [23]. The functional equation of square root spiral

$$\varphi(\sqrt{x^2+1}) = \varphi(x) + \arctan\frac{1}{x}, \tag{1.3}$$

is a special case of equation (1.2). K. J. Heuvers, D. S. Moak and B. Boursaw [6] presented the general solution without additional regularity of equation (1.3). After that, the generalized Hyers-Ulam-Rassias stability of equation (1.3) was proved by S.-M. Jung and P. K. Sahoo [11]. One generalization of equation (1.3) is the linear functional equation

$$\varphi(p^{-1}(p(x) + c)) = \varphi(x) + h(x), \tag{1.4}$$

where p, h are given functions, p^{-1} is the inverse of p, φ is an unknown function and $c \neq 0$ is a constant. The paper [25] gave the general solution of equation (1.4), also proved the generalized Hyers-Ulam-Rassias stability and the stability in the sense of Ger for homogeneous equations of equation (1.4).

For convenience, let n be a fixed positive integer, $\mathbb{K}$ be either the field $\mathbb{R}$ of reals numbers or the field $\mathbb{C}$ of complex numbers, $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_+^* := [0, \infty)$, and X stand for a Banach space over $\mathbb{K}$. Suppose that $p : \mathbb{K} \to \mathbb{K}$ is bijective, $c \in \mathbb{K}$ and $c \neq 0$. By $\mathscr{F}$ we denote the set of all functions $\varphi : \mathbb{K} \to X$. Let $\Delta_{(p,\, c)}$ be the difference operator defined by

$$(\Delta_{(p,\, c)}\varphi)(x) = \varphi(p^{-1}(p(x) + c)) - \varphi(x), \quad \forall x \in \mathbb{K}, \tag{1.5}$$

for all $\varphi \in \mathscr{F}$. And we define an operator $\Delta^n_{(p,\, c)} : \mathscr{F} \to \mathscr{F}$ by

$$(\Delta^n_{(p,\, c)}\varphi)(x) = (\Delta_{(p,\, c)}(\Delta^{n-1}_{(p,\, c)}\varphi))(x), \quad \forall x \in \mathbb{K}, \tag{1.6}$$

for all $\varphi \in \mathscr{F}$, where $\Delta^0_{(p,\,c)}\varphi = \varphi$. For instance, we see that

$$\begin{aligned}
(\Delta^2_{(p,\,c)}\varphi)(x) &= \varphi(p^{-1}(p(x)+2c)) - 2\varphi(p^{-1}(p(x)+c)) + \varphi(x),\\
(\Delta^3_{(p,\,c)}\varphi)(x) &= \varphi(p^{-1}(p(x)+3c)) - 3\varphi(p^{-1}(p(x)+2c))\\
&\quad +3\varphi(p^{-1}(p(x)+c)) - \varphi(x).
\end{aligned} \tag{1.7}$$

For the case $p = \mathrm{id}, c = 1$, S.-M. Jung and J. M. Rassias [12] proved the generalized Hyers-Ulam-Rassias stability of the so-called Newton difference equations

$$\Delta^n_{(id,1)}\varphi(x) = A \ln R_n(x), \tag{1.8}$$

where $A > 0$, $R_1(x) = \frac{x+1}{x}$, $R_k(x) = \frac{R_{k-1}(x+1)}{R_{k-1}(x)}$, $k \in \{2, 3, \ldots, n\}$, and applied their results to the functional equation for logarithmic spirals.

In this paper, we consider the following functional equation

$$\Delta^n_{(p,\,c)}\varphi(x) = h(x), \tag{1.9}$$

for all $x \in X$ and some fixed integer $n > 0$, h is a given function, φ is an unknown function. We refer to equation (1.9) as the generalized Newton difference equation. In fact, if we set $n = 1$, then (1.9) is transformed into equation (1.4). If we set $p(x) = x, h(x) = A \ln R_n(x), c = 1$, then (1.9) becomes to (1.8). We prove the generalized Hyers-Ulam-Rassias stability of equation (1.9), and give a sufficient condition on the generalized Hyers-Ulam-Rassias stability. Applying the result of (1.9), we give the generalized Hyers-Ulam-Rassias stability of equations (1.4) and (1.8) as corollaries.

2 Main results

In the following theorem, we prove the generalized Hyers-Ulam-Rassias stability of (1.9).

Theorem 2.1. *Suppose that $c \in \mathbb{K}$, $c \neq 0$, $p : \mathbb{K} \to \mathbb{K}$ is bijective, $h : \mathbb{K} \to X$ is a given function. If $\varphi : \mathbb{K} \to X$ satisfies*

$$\|\Delta^n_{(p,\,c)}\varphi(x) - h(x)\| \leq \phi_n(x), \quad \forall x \in \mathbb{K}, \tag{2.1}$$

where function $\phi_n : \mathbb{K} \to \mathbb{R}_+$ satisfies the condition

$$\Phi_n(x) := \sum_{k=0}^{\infty} \phi_n(p^{-1}(p(x)+kc)) < \infty, \quad \forall x \in \mathbb{K}, \tag{2.2}$$

for some integer $n \in \mathbb{N}$, then there exists a unique function $\Psi_n : \mathbb{K} \to X$ such that $\Delta_{(p,\, c)}\Psi_n(x) = h(x)$ and

$$\|\Psi_n(x) - \Delta^{n-1}_{(p,\, c)}\varphi(x)\| \leq \Phi_n(x), \quad \forall x \in \mathbb{K}. \tag{2.3}$$

Proof. It follows from (2.1) that

$$\begin{array}{ll} \|\Delta^{n}_{(p,\, c)}\varphi(x) - h(x)\| \leq \phi_n(x) \\ \|\Delta^{n}_{(p,\, c)}\varphi(p^{-1}(p(x)+c)) - h(p^{-1}(p(x)+c))\| \leq \phi_n(p^{-1}(p(x)+c)) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \|\Delta^{n}_{(p,\, c)}\varphi(p^{-1}(p(x)+(m-1)c)) - h(p^{-1}(p(x)+(m-1)c))\| \\ \leq \phi_n(p^{-1}(p(x)+(m-1)c)) \end{array} \tag{2.4}$$

for $x \in \mathbb{K}$ and $m \in \mathbb{N}$. In view of triangular inequalities, the above inequalities yield

$$\|\sum_{k=0}^{m-1}\Delta^{n}_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) - \sum_{k=0}^{m-1} h(p^{-1}(p(x)+kc))\| \leq \sum_{k=0}^{m-1}\phi_n(p^{-1}(p(x)+kc)). \tag{2.5}$$

Substitute $p^{-1}(p(x)+\ell c)$ for x in (2.5) and then substitute k for $k+\ell$ in the resulting inequality to obtain

$$\|\sum_{k=\ell}^{\ell+m-1}\Delta^{n}_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) - \sum_{k=\ell}^{\ell+m-1} h(p^{-1}(p(x)+kc))\| \leq \sum_{k=\ell}^{\ell+m-1}\phi_n(p^{-1}(p(x)+kc)) \tag{2.6}$$

for all $x \in \mathbb{K}$ and $\ell, m \in \mathbb{N}$.

By some manipulation, we further have

$$\begin{aligned} &\| \sum_{k=0}^{\ell+m-1}\Delta^{n}_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) - \sum_{k=0}^{\ell+m-1} h(p^{-1}(p(x)+kc)) + \Delta^{n-1}_{(p,\, c)}\varphi(x) \\ &- \sum_{k=0}^{\ell-1}\Delta^{n}_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) + \sum_{k=0}^{\ell-1} h(p^{-1}(p(x)+kc)) - \Delta^{n-1}_{(p,\, c)}\varphi(x)\| \\ &\leq \sum_{k=\ell}^{\ell+m-1}\phi_n(p^{-1}(p(x)+kc)) \end{aligned} \tag{2.7}$$

for all $x \in \mathbb{K}$ and $\ell, m \in \mathbb{N}$. Thus, considering (2.2), we see that the sequence

$$\{\sum_{k=0}^{m-1}[\Delta^{n}_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) - h(p^{-1}(p(x)+kc))] + \Delta^{n-1}_{(p,\, c)}\varphi(x)\}_{m=1}^{\infty} \tag{2.8}$$

is a Cauchy sequence for all $x \in \mathbb{K}$. Hence, we can define a function $\Psi_n : \mathbb{K} \to X$ by

$$\Psi_n(x) = \sum_{k=0}^{\infty} [\Delta^n_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) - h(p^{-1}(p(x)+kc))] + \Delta^{n-1}_{(p,\, c)}\varphi(x). \quad (2.9)$$

By (2.9), we obtain

$$\begin{aligned} \Delta_{(p,\, c)}\Psi_n(x) &= \Psi_n(p^{-1}(p(x)+c)) - \Psi_n(x) \\ &= \sum_{k=1}^{\infty} [\Delta^n_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) - h(p^{-1}(p(x)+kc))] \\ &+ \Delta^{n-1}_{(p,\, c)}\varphi(p^{-1}(p(x)+c)) \\ &- \sum_{k=0}^{\infty} [\Delta^n_{(p,\, c)}\varphi(p^{-1}(p(x)+kc)) - h(p^{-1}(p(x)+kc))] - \Delta^{n-1}_{(p,\, c)}\varphi(x) \\ &= h(x) \qquad (2.10) \end{aligned}$$

for all $x \in \mathbb{K}$. In view of (2.2) and (2.9), if we let m go to infinity in (2.5), then we obtain (2.3).

It only remains to prove the uniqueness of the function Ψ_n. If a function $H : \mathbb{K} \to X$ satisfies $\Delta_{(p,\, c)}H(x) = h(x)$ for each $x \in \mathbb{K}$, then we can easily show that

$$H(p^{-1}(p(x)+mc)) - H(x) = \sum_{k=0}^{m-1} h(p^{-1}(p(x)+kc)) \qquad (2.11)$$

for all $x \in \mathbb{K}$ and $m \in \mathbb{N}$. Now, assume that $G_n : \mathbb{K} \to X$ satisfies $\Delta_{(p,\, c)}G_n(x) = h(x)$ and the inequality (2.3) in place of Ψ_n. By (2.2), (2.3) and (2.11), we get

$$\begin{aligned} \|\Psi_n(x) - G_n(x)\| &= \|\Psi_n(p^{-1}(p(x)+mc)) - G_n(p^{-1}(p(x)+mc))\| \\ &\leq 2\Phi_n(p^{-1}(p(x)+mc)) \longrightarrow 0, \quad as \quad m \longrightarrow \infty, \qquad (2.12) \end{aligned}$$

for all $x \in \mathbb{K}$, which proves the uniqueness of Ψ_n. This completes the proof. $\square$

Now we give a sufficient condition of the generalized Hyers-Ulam-Rassias stability of (1.9).

Corollary 2.1. *Suppose that $c \in \mathbb{K}$, $c \neq 0$, $p : \mathbb{K} \to \mathbb{K}$ is bijective, and $h : \mathbb{K} \to X$ is a given function. If $\varphi : \mathbb{K} \to X$ satisfies $\|\Delta^n_{(p,\, c)}\varphi(x) - h(x)\| \leq \phi_n(x)$ for all $x \in \mathbb{K}$, where function $\phi_n : \mathbb{K} \to \mathbb{R}_+$ is a fixed function, for some integer $n \in \mathbb{N}$. If*

$$\liminf_{k\to\infty} \frac{\phi_n(p^{-1}(p(x)+(k-1)c))}{\phi_n(p^{-1}(p(x)+kc))} > 1, \quad \forall x \in \mathbb{K}, \qquad (2.13)$$

then equation (1.9) *has the generalized Hyers-Ulam-Rassias stability.*

Proof. Consider the sequence $\{U_k(x)\}$ defined by $U_k(x) := \phi_n(p^{-1}(p(x)+kc))$. By (2.13), we have

$$\begin{aligned}\limsup_{k\to\infty}\frac{U_k}{U_{k-1}} &= \limsup_{k\to\infty}\frac{\phi_n(p^{-1}(p(x)+kc))}{\phi_n(p^{-1}(p(x)+(k-1)c))}\\ &= \frac{1}{\liminf\limits_{k\to\infty}\frac{\phi_n(p^{-1}(p(x)+(k-1)c))}{\phi_n(p^{-1}(p(x)+kc))}}\\ &< 1, \quad \forall x\in\mathbb{K}.\end{aligned}$$

By ratio test we see that the series (2.2) converges for all $x \in \mathbb{K}$. By Theorem 2.1 we get the generalized Hyers-Ulam-Rassias stability. This completes the proof of the corollary. □

By Theorem 2.1, we can obtain directly the generalized Hyers-Ulam-Rassias stability of equations (1.4) and (1.8).

Corollary 2.2. (cf.[25]). *Suppose that $c \in \mathbb{K}$, $c \neq 0$, $p : \mathbb{K} \to \mathbb{K}$ is bijective, $h : \mathbb{K} \to X$ is a given function. If $\varphi_s : \mathbb{K} \to X$ satisfies*

$$\|\varphi_s(p^{-1}(p(x)+c)) - \varphi_s(x) - h(x)\| \le \phi(x), \quad \forall x \in \mathbb{K}, \tag{2.14}$$

where function $\psi : \mathbb{K} \to \mathbb{R}_+$ satisfies

$$\Phi(x) := \sum_{k=0}^{\infty} \phi(p^{-1}(p(x)+kc)) < \infty, \quad \forall x \in \mathbb{K}, \tag{2.15}$$

then there exists a unique solution $\varphi : \mathbb{K} \to X$ of equation (1.4) such that

$$\|\varphi(x) - \varphi_s(x)\| \le \Phi(x), \quad \forall x \in \mathbb{K}. \tag{2.16}$$

Corollary 2.3. (cf.[12]). *If a function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ satisfies*

$$|\Delta^n_{(id,1)}\varphi(x) - A\ln R_n(x)| \le \gamma_n(x), \quad \forall x \in \mathbb{R}_+, \tag{2.17}$$

and some integer $n \in \mathbb{N}$, where $\gamma_n : \mathbb{R}_+ \to \mathbb{R}^*_+$ is a function which satisfies

$$\Upsilon_n(x) := \sum_{k=0}^{\infty} \gamma_n(x+k) < \infty, \quad \forall x \in \mathbb{R}_+, \tag{2.18}$$

then there exists a unique function $\Psi_n : \mathbb{R}_+ \to \mathbb{R}$ such that $\Delta_{(id,1)}\Psi_n(x) = A\ln R_n(x)$ and

$$|\Psi_n(x) - \Delta^{n-1}_{(id,1)}\varphi(x)| \le \Upsilon_n(x), \quad \forall x \in \mathbb{R}_+. \tag{2.19}$$

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