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Characterizations of Generalized Quasi-Einstein Manifolds

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Abstract

We give characterizations of generalized quasi-Einstein manifolds for both even and odd dimensions.

1 Introduction

A Riemannian manifold (M, g), $(n \geq 2)$, is said to be an *Einstein manifold* if its Ricci tensor S satisfies the condition $S = \frac{r}{n}g$, where r denotes the scalar curvature of M. The notion of a quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity in [2]. A non-flat Riemannian manifold (M, g), $(n \geq 2)$, is defined to be a *quasi-Einstein manifold* if the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) \tag{1}$$

is fulfilled on M, where α and β are scalars of which $\beta \neq 0$ and A is a non-zero 1-form such that

$$g(X, \xi) = A(X), \tag{2}$$

for every vector field X ; ξ being a unit vector field. If $\beta = 0$, then the manifold reduces to an Einstein manifold.

The relation (1) can be written as follows

$$Q = \alpha I + \beta A \otimes \xi,$$

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where Q is the Ricci operator and I is the identity function.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker space-times are quasi-Einstein manifolds. For more information about quasi-Einstein manifolds see [7], [8] and [9].

A non-flat Riemannian manifold is called a *generalized quasi-Einstein manifold* (see [6]), if its Ricci tensor S satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \tag{3}$$

where α, β and γ are certain non-zero scalars and A, B are two non-zero 1-forms. The unit vector fields ξ_1 and ξ_2 corresponding to the 1-forms A and B are defined by

$$g(X, \xi_1) = A(X) \ , \ g(X, \xi_2) = B(X), \tag{4}$$

respectively, and the vector fields ξ_1 and ξ_2 are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$. If $\gamma = 0$, then the manifold reduces to a quasi-Einstein manifold.

The generalized quasi-Einstein condition (3) can be also written as

$$Q = \alpha I + \beta A \otimes \xi_1 + \gamma B \otimes \xi_2.$$

In [6], U. C. De and G. C. Ghosh showed that a 2-quasi umbilical hypersurface of an Euclidean space is a generalized quasi-Einstein manifold. In [11], the present authors generalized the result of De and Ghosh and they proved that a 2-quasi umbilical hypersurface of a Riemannian space of constant curvature $\widetilde{M}^{n+1}(c)$ is a generalized quasi-Einstein manifold.

Let M be an m-dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u \wedge v)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, where $\{u, v\}$ is an orthonormal basis of π. For any n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is denoted by

$$\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where $\{e_1, ..., e_n\}$ is any orthonormal basis of L [4]. When $L = T_pM$, the scalar curvature $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p.

The well-known characterization of 4-dimensional Einstein spaces was given by I. M. Singer and J. A. Thorpe in [12] as follows:

Theorem 1.1. *A Riemannian 4-manifold M is an Einstein space if and only if $K(\pi) = K(\pi^{\perp})$ for any plane section $\pi \subseteq T_pM$, where $\pi^{\perp}$ denotes the orthogonal complement of π in T_pM.*

As a generalization of the Theorem 1.1, in [4], B.Y. Chen, F. Dillen, L.Verstraelen and L.Vrancken gave the following result:

Theorem 1.2. *A Riemannian $2n$-manifold M is an Einstein space if and only if $\tau(L) = \tau(L^{\perp})$ for any n-plane section $L \subseteq T_pM$, where $L^{\perp}$ denotes the orthogonal complement of L in T_pM, at $p \in M$.*

On the other hand, in [10] D. Dumitru obtained the following result for odd dimensional Einstein spaces:

Theorem 1.3. *A Riemannian $(2n+1)$-manifold M is an Einstein space if and only if $\tau(L) + \frac{\lambda}{2} = \tau(L^{\perp})$ for any n-plane section $L \subseteq T_pM$, where $L^{\perp}$ denotes the orthogonal complement of L in T_pM, at $p \in M$.*

Theorem 1.2 and Theorem 1.3 were generalized by C.L. Bejan in [1] as follows:

Theorem 1.4. *Let (M, g) be a Riemannian $(2n+1)$-manifold, with $n \geq 2$. Then M is quasi-Einstein if and only if the Ricci operator Q has an eigenvector field ξ such that at any $p \in M$, there exist two real numbers a, b satisfying $\tau(P) + a = \tau(P^{\perp})$ and $\tau(N) + b = \tau(N^{\perp})$, for any n-plane section P and $(n+1)$-plane section N, both orthogonal to ξ in T_pM, where $P^{\perp}$ and $N^{\perp}$ denote respectively the orthogonal complements of P and N in T_pM.*

Theorem 1.5. *Let (M, g) be a Riemannian $2n$-manifold, with $n \geq 2$. Then M is quasi-Einstein if and only if the Ricci operator Q has an eigenvector field ξ such that at any $p \in M$, there exist two real numbers a, b satisfying $\tau(P) + c = \tau(P^{\perp})$, for any n-plane section P orthogonal to ξ in T_pM, where $P^{\perp}$ denotes the orthogonal complement of P in T_pM.*

Motivated by the above studies, as generalizations of quasi-Einstein manifolds, we give characterizations of generalized quasi-Einstein manifolds for both even and odd dimensions.

2 Characterizations of Generalized Quasi-Einstein Manifolds

Now, we consider two results which characterize generalized quasi-Einstein spaces in even and odd dimensions, by generalizing the characterizations of quasi-Einstein spaces given in [1] :

Theorem 2.1. *Let (M, g) be a Riemannian $(2n+1)$-manifold, with $n \geq 2$. Then M is generalized quasi-Einstein if and only if the Ricci operator Q has* *eigenvector fields ξ_1 and ξ_2 such that at any $p \in M$, there exist three real numbers a, b and c satisfying*

$$\tau(P) + a = \tau(P^{\perp}); \quad \xi_1, \xi_2 \in T_p P^{\perp}$$

$$\tau(N) + b = \tau(N^{\perp}); \quad \xi_1 \in T_p N, \xi_2 \in T_p N^{\perp}$$

and

$$\tau(R) + c = \tau(R^{\perp}); \quad \xi_1 \in T_p R, \xi_2 \in T_p R^{\perp}$$

for any n-plane sections P , N and $(n+1)$-plane section R, where $P^{\perp}$, $N^{\perp}$ and $R^{\perp}$ denote the orthogonal complements of P, N and R in T_pM, respectively, and $a = \frac{(\alpha+\beta+\gamma)}{2}$, $b = \frac{(\alpha-\beta+\gamma)}{2}$, $c = \frac{(\gamma-\alpha-\beta)}{2}$.

Proof. Assume that M is a $(2n+1)$-dimensional generalized quasi-Einstein manifold, such that

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \tag{5}$$

for any vector fields X, Y holds on M, where A and B are defined by

$$g(X, \xi_1) = A(X) \ , \ g(X, \xi_2) = B(X).$$

The equation (5) shows that ξ_1 and ξ_2 are eigenvector fields of Q.

Let $P \subseteq T_pM$ be an n-plane orthogonal to ξ_1 and ξ_2 and let $\{e_1, ..., e_n\}$ be an orthonormal basis of it. Since ξ_1 and ξ_2 are orthogonal to P, we can take an orthonormal basis $\{e_{n+1}, ..., e_{2n}, e_{2n+1}\}$ of $P^{\perp}$ such that $e_{2n} = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n}, e_{2n+1}\}$ is an orthonormal basis of T_pM. Then taking $X = Y = e_i$ in (5), we can write

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \left\{ \begin{array}{c} \alpha, \quad 1 \leq i \leq 2n-1 \\ \alpha+\beta, \quad i = 2n \\ \alpha+\gamma, \quad i = 2n+1 \end{array} \right\}.$$

By the use of (5) for any $1 \leq i \leq 2n+1$, we can write

$$S(e_1, e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + ... + K(e_1 \wedge e_{2n-1}) + K(e_1 \wedge \xi_1) + K(e_1 \wedge \xi_2) = \alpha,$$

$$S(e_2, e_2) = K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + ... + K(e_2 \wedge e_{2n-1}) + K(e_2 \wedge \xi_1) + K(e_2 \wedge \xi_2) = \alpha,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$S(e_{2n-1}, e_{2n-1}) = K(e_{2n-1} \wedge e_1) + K(e_{2n-1} \wedge e_2) + ... + K(e_{2n-1} \wedge \xi_1) + K(e_{2n-1} \wedge \xi_2) = \alpha,$$

$$S(\xi_1, \xi_1) = K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + ... + K(\xi_1 \wedge e_{2n-1}) + K(\xi_1 \wedge \xi_2) = \alpha + \beta,$$

$$S(\xi_2, \xi_2) = K(\xi_2 \wedge e_1) + K(\xi_2 \wedge e_2) + ... + K(\xi_2 \wedge e_{2n-1}) + K(\xi_2 \wedge \xi_1) = \alpha + \gamma.$$

Now, by summing up the first n-equations, we get

$$2\tau(P) + \sum_{1\leq i\leq n<j\leq 2n+1} K(e_i \wedge e_j) = n\alpha. \tag{6}$$

By summing up the last $(n+1)$-equations, we also get

$$2\tau(P^{\perp}) + \sum_{1\leq j\leq n+1<i\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta + \gamma. \tag{7}$$

Then, by substracting the equation (6) from (7), we obtain

$$\tau(P^{\perp}) - \tau(P) = \frac{(\alpha+\beta+\gamma)}{2}. \tag{8}$$

Similarly, let $N \subseteq T_pM$ be an n-plane orthogonal to ξ_2 and let $\{e_1, ..., e_{n-1}, e_n\}$ be an orthonormal basis of it. Since ξ_2 is orthogonal to N, we can take an orthonormal basis $\{e_{n+1}, ..., e_{2n}, e_{2n+1}\}$ of $N^{\perp}$ orthogonal to ξ_1, such that $e_n = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n}, e_{2n+1}\}$ is an orthonormal basis of T_pM. By making use of the above $(2n+1)$ equations for $S(e_i, e_i)$, $1 \leq i \leq 2n+1$, from the sum of the first n-equations we obtain

$$2\tau(N) + \sum_{1\leq i\leq n<j\leq 2n+1} K(e_i \wedge e_j) = n\alpha + \beta, \tag{9}$$

and from the sum of the last $(n+1)$-equations, we have

$$2\tau(N^{\perp}) + \sum_{1\leq j\leq n+1<i\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \gamma. \tag{10}$$

By substracting the equation (9) from (10), we find

$$\tau(N^{\perp}) - \tau(N) = \frac{(\alpha-\beta+\gamma)}{2}.$$

Analogously, let $R \subseteq T_pM$ be an $(n+1)$-plane orthogonal to ξ_2 and let $\{e_1, ..., e_n, e_{n+1}\}$ be an orthonormal basis of it. Since ξ_2 is orthogonal to R, we can take an orthonormal basis $\{e_{n+2}, ..., e_{2n}, e_{2n+1}\}$ of $R^{\perp}$ orthogonal to ξ_1, such that $e_{n+1} = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n}, e_{2n+1}\}$ is an orthonormal basis of T_pM. Similarly writing again the above $(2n+1)$-equations for $S(e_i, e_i)$, $1 \leq i \leq 2n+1$, from the sum of the first $(n+1)$-equations we get

$$2\tau(R) + \sum_{1\leq i\leq n+1<j\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta, \tag{11}$$

and from the sum of the last n-equations, we have

$$2\tau(R^{\perp}) + \sum_{1\leq j\leq n<i\leq 2n+1} K(e_i \wedge e_j) = n\alpha + \gamma. \tag{12}$$

Again by substracting (11) from (12), it follows that

$$\tau(R^{\perp}) - \tau(R) = \frac{(\gamma - \alpha - \beta)}{2}.$$

Therefore the direct statement is satisfied for

$$a = \tfrac{(\alpha+\beta+\gamma)}{2}, \quad b = \tfrac{(\alpha-\beta+\gamma)}{2} \quad \text{and} \quad c = \tfrac{(\gamma-\alpha-\beta)}{2}.$$

Conversely, let v be an arbitrary unit vector of T_pM, at $p \in M$, orthogonal to ξ_1 and ξ_2. We take an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n}, e_{2n+1}\}$ of T_pM such that $v = e_1$, $e_{n+1} = \xi_1$ and $e_{2n+1} = \xi_2$. We consider n-plane section N and $(n+1)$-plane section R in T_pM as follows

$$N = span\{e_2, ..., e_{n+1}\}$$

and

$$R = span\{e_1, ..., e_{n+1}\},$$

respectively. Then we have

$$N^{\perp} = span\{e_1, e_{n+2}, ..., e_{2n}, e_{2n+1}\}$$

and

$$R^{\perp} = span\{e_{n+2}, ..., e_{2n}, e_{2n+1}\}.$$

After some calculations we get

$$\begin{aligned} S(v,v) &= [K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + ... + K(e_1 \wedge e_{n+1})] \\ &\quad +[K(e_1 \wedge e_{n+2}) + ... + K(e_1 \wedge e_{2n}) + K(e_1 \wedge e_{2n+1})] \\ &= [\tau(R) - \sum_{2\leq i<j\leq n+1} K(e_i \wedge e_j)] + [\tau(N^{\perp}) - \sum_{n+2\leq i<j\leq 2n+1} K(e_i \wedge e_j)] \\ &= [\tau(R^{\perp}) - c - \tau(N)] + [\tau(N) + b - \tau(R^{\perp})] = b - c. \end{aligned}$$

Therefore $S(v,v) = b - c$, for any unit vector $v \in T_pM$, ortohogonal to ξ_1 and ξ_2. Then we can write for any $1 \leq i \leq 2n+1$,

$$S(e_i, e_i) = b - c.$$

Since $S(v, v) = (b - c)g(v, v)$ for any unit vector $v \in T_pM$ orthogonal to ξ_1 and ξ_2, it follows that

$$S(X, X) = (b - c)g(X, X) + (a - b)A(X)A(X) \tag{13}$$

and

$$S(Y, Y) = (b - c)g(Y, Y) + (a + c)B(Y)B(Y), \tag{14}$$

for any $X \in [span\{\xi_1\}]^\perp$ and $Y \in [span\{\xi_2\}]^\perp$, where A and B denote dual forms of ξ_1 and ξ_2 with respect to g, respectively.

In view of the equations (13) and (14), we get from their symmetry that S with tensors $(b-c)g+(a-b)A\otimes A$ and $(b-c)g+(a+c)B\otimes B$ must coincide on the complement of ξ_1 and ξ_2, respectively, that is,

$$S(X, Y) = (b - c)g(X, Y) + (a - b)A(X)A(Y) + (a + c)B(X)B(Y), \tag{15}$$

for any $X, Y \in [span\{\xi_1, \xi_2\}]^\perp$.

Since ξ_1 and ξ_2 are eigenvector fields of Q, we also have

$$S(X, \xi_1) = 0$$

and

$$S(Y, \xi_2) = 0,$$

for any $X, Y \in T_pM$ orthogonal to ξ_1 and ξ_2. Thus, we can extend the equation (15) to

$$S(X, Z) = (b - c)g(X, Z) + (a - b)A(X)A(Z) + (a + c)B(X)B(Z), \tag{16}$$

for any $X \in [span\{\xi_1, \xi_2\}]^\perp$ and $Z \in T_pM$.

Now, let consider the n-plane section P and $(n + 1)$-plane section R in T_pM as follows

$$P = span\{e_1, ..., e_n\}$$

and

$$R = span\{e_1, ..., e_n, \xi_1\},$$

respectively. Then we have

$$P^\perp = span\{\xi_1, e_{n+2}, ..., e_{2n+1}\}$$

and

$$R^\perp = span\{e_{n+2}, ..., e_{2n}, e_{2n+1}\}.$$

Similarly after some calculations we obtain

$$\begin{aligned} S(\xi_1,\xi_1) &= [K(\xi_1\wedge e_1)+K(\xi_1\wedge e_2)+...+K(\xi_1\wedge e_n)] \\ &\quad +[K(\xi_1\wedge e_{n+2})+...+K(\xi_1\wedge e_{2n})+K(\xi_1\wedge e_{2n+1})] \\ &= [\tau(R)-\sum_{1\leq i<j\leq n}K(e_i\wedge e_j)]+[\tau(P^{\perp})-\sum_{n+2\leq i<j\leq 2n+1}K(e_i\wedge e_j)] \\ &= [\tau(R^{\perp})-c-\tau(P)]+[\tau(P)+a-\tau(R^{\perp})]=a-c. \end{aligned}$$

Then, we can write

$$S(\xi_1,\xi_1)=(b-c)g(\xi_1,\xi_1)+(a-b)A(\xi_1)A(\xi_1). \tag{17}$$

Analogously, let consider n-plane sections P and N in T_pM as follows

$$P=span\{e_1,...,e_n\}$$

and

$$N=span\{e_{n+1},...,e_{2n}\},$$

respectively. Therefore we have

$$P^{\perp}=span\{e_{n+1},...,e_{2n},\xi_2\}$$

and

$$N^{\perp}=span\{e_1,...,e_n,\xi_2\}.$$

Similarly after some calculations we get

$$\begin{aligned} S(\xi_2,\xi_2) &= [K(\xi_2\wedge e_1)+K(\xi_2\wedge e_2)+...+K(\xi_2\wedge e_n)] \\ &\quad +[K(\xi_2\wedge e_{n+1})+...+K(\xi_2\wedge e_{2n})] \\ &= [\tau(N^{\perp})-\sum_{1\leq i<j\leq n}K(e_i\wedge e_j)]+[\tau(P^{\perp})-\sum_{n+1\leq i<j\leq 2n}K(e_i\wedge e_j)] \\ &= [\tau(N)+b-\tau(P)]+[\tau(P)+a-\tau(N)]=a+b. \end{aligned}$$

Then we may write

$$S(\xi_2,\xi_2)=(b-c)g(\xi_2,\xi_2)+(a+c)B(\xi_2)B(\xi_2). \tag{18}$$

By making use of the equations (16), (17) and (18), we obtain from the symmetry of the Ricci tensor S

$$S(X,Y)=(b-c)g(X,Y)+(a-b)A(X)A(Y)+(a+c)B(X)B(Y),$$

for any $X,Y\in T_pM$. Thus, M is a generalized quasi-Einstein manifold for $\alpha=b-c$, $\beta=a-b$ and $\gamma=a+c$, which finishes the proof of the theorem. $\square$

Similar to the proof of Theorem 2.1, we can give the following theorem for an even dimensional generalized quasi-Einstein manifold:

Theorem 2.2. *Let (M, g) be a Riemannian $2n$-manifold, with $n \geq 2$. Then M is generalized quasi-Einstein if and only if the Ricci operator Q has eigenvector fields ξ_1 and ξ_2 such that at any $p \in M$, there exist three real numbers a, b and c satisfying*

$$\tau(P) + a = \tau(P^{\perp}); \quad \xi_1, \xi_2 \in T_pP^{\perp}$$

$$\tau(N) + b = \tau(N^{\perp}); \quad \xi_1, \xi_2 \in T_pN^{\perp}$$

and

$$\tau(R) + c = \tau(R^{\perp}); \quad \xi_1 \in T_pR, \xi_2 \in T_pR^{\perp}$$

for any n-plane sections P , R and $(n-1)$-plane section N, where $P^{\perp}$, $N^{\perp}$ and $R^{\perp}$ denote the orthogonal complements of P, N and R in T_pM, respectively and $a = \frac{(\beta+\gamma)}{2}$, $b = \frac{(2\alpha+\beta+\gamma)}{2}$, $c = \frac{(\gamma-\beta)}{2}$.

Proof. Let P and R be n-plane sections and N be an $(n-1)$-plane section such that

$$P = span\{e_1, ..., e_n\}$$

$$R = span\{e_{n+1}, ..., e_{2n}\},$$

and

$$N = span\{e_2, ..., e_n\},$$

respectively. Therefore the orthogonal complements of these sections can be written as

$$P^{\perp} = span\{e_{n+1}, ..., e_{2n}\}$$

$$R^{\perp} = span\{e_1, ..., e_n\},$$

and

$$N^{\perp} = span\{e_1, e_{n+1} ..., e_{2n}\}.$$

Then the proof is similar to the proof of Theorem 2.1. □

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