



## Approximate Cycles of the Second Kind in Hilbert space for a Generalized Barbashin-Ezeilo Problem

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### Abstract

In this work we show that the Volterra integral operator defined on the space of absolutely stable functions induces an asymptotically pseudocontractive operator. We, then, show that Afuwape's [1] generalization of the Barbashin-Ezeilo problem is solvable in a Banach space (but not in Hilbert space  $L_2[0, \infty)$ ). However applying Osilike-Akuchu[10] theorem and recent results (in Hilbert space) of Igbokwe and Udo-utun[8] we formulate conditions for finding approximate cycles of the second kind (in the Hilbert space  $W_0^{2,2}[0, \infty)$ ) to this problem given in the form  $x''' + \alpha x'' + g(x') + \varphi(x) = 0$ .

### 1 Introduction

In 2006 using the ideas of nonlocal reduction method, Afuwape[1] generalized and modified the Barbashin-Ezeilo problem by asking: *When will a general third-order equation of the form*

$$x''' + \alpha x'' + g(x') + \varphi(x) = 0 \quad (1.1)$$

*have a cycle of the second kind where  $\alpha$  is an arbitrary constant  $g(y)$  a continuous bounded function and  $\varphi(x)$  is a  $2\pi$ -periodic odd function having zeros  $0, x_0$  in  $[0, 2\pi)$  and at any point  $x \in [0, 2\pi)$  satisfies*

$$\varphi^2(x) + [\varphi'(x)]^2 \neq 0. \quad (1.2)$$

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Transforming (1.1) as

$$\begin{cases} \frac{dz}{dt} = Az + b\varphi(\sigma) \\ \frac{d\sigma}{dt} = c^*z \end{cases} \quad (1.3)$$

where  $A = \begin{pmatrix} -\alpha & -\beta \\ 1 & 0 \end{pmatrix}$ ;  $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ;  $c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\varphi(\sigma) = [\hat{g}(\dot{\sigma}) + \varphi(\sigma)]$ ,  $g(y) = \beta y + \hat{g}(y)$ , with  $0 \leq \hat{g}(y) \leq \mu$ , the following theorem was proved:

**Theorem 1.1.** (Afuwape[1])

Suppose there exists positive numbers  $\beta$  and  $\mu$  such that  $\beta \leq g(y) \leq \mu + \beta$ , with  $\alpha^2 > 4\beta$  and  $\mu < \lambda_1(\lambda_1^2 - \alpha\lambda_1 + \beta)$  for some constant parameter  $\lambda_1$ .

Suppose also that for some  $\lambda_2 > 0$ ,

$$\alpha - \sqrt{\alpha^2 - 4\beta} < 2\lambda_2 < \alpha + \sqrt{\alpha^2 - 4\beta} \quad (1.4)$$

and  $\varphi'(0) > \alpha\beta$ . Then (1.1) is Bakaev-stable and has a nontrivial periodic solution in every strip  $\Pi = \{[2k\pi, x_0 + 2k\pi]; k\mathbb{Z}\}$ .

In this paper our objective is to transform the system (1.3) into equivalent Volterra integral equation form and verify that the resulting operator is an asymptotically pseudocontractive map. This will enable us to apply the Osilike-Akuchu Theorem in [10] and the results of Igbokwe and Udo-utun Theorem in [8] to show that the resulting feedback problem (1.1) has a fixed point in Banach space for all  $\alpha$  (and in Hilbert space for  $\alpha > 0$ ).

## 2 Preliminaries

**Definition 2.1.** The solution  $(x(t), \sigma(t))$  of (1.1) is said to be circular if there exists a number  $\epsilon > 0$ , and a time  $\tilde{t} \geq 0$  such that  $\frac{d\sigma(t)}{dt} \geq \epsilon$  for all  $t \geq \tilde{t}$ .

**Definition 2.2.** The solution  $(x(t), \sigma(t))$  of (1.1) is said to be cycle of the second kind if there exists an integer  $j \neq 0$ , and a moment  $\tilde{t} > 0$  such that for all  $x(\tilde{t}) = 0$ ,  $\sigma(\tilde{t}) - \sigma(0) = j\tau$  where  $\varphi(\sigma + \tau) = \varphi(\sigma)$ .

Let  $C_0$  denote the space of absolutely stable solutions  $\sigma(t)$  of the one-dimensional feedback system

$$\sigma(t) = \eta\Phi(t) + \int_0^t \Phi(t-s)g(\sigma(s))ds$$

then each element  $\sigma \in C_0$  satisfies  $\lim_{t \rightarrow \infty} \sigma(t) = 0$ .

It has been shown by Corduneanu[3, 4] and Holtzman[7] that  $C_0 \subseteq L_2[0, \infty)$  which is a Hilbert space.

We shall show that the operator  $(Au)(t) = \int_0^t \Phi(t-s)u(s)ds$  is compact by applying Arzela-Ascoli theorem for suitable functions defined on  $[0, \eta]$  and that if  $A$  is self-adjoint then the operator  $T : C_0 \rightarrow C_0$  given by

$$(T\sigma)(t) = \eta\Phi(t) + \int_0^t \Phi(t-s)g(\sigma(s))ds = \eta\Phi(t) + (A(g\sigma))(t)$$

has a fixed point if some conditions given below are satisfied. Here,  $g$  is an odd function satisfying the sector condition

$$0 \leq \frac{g(\xi_1) - g(\xi_2)}{\xi_1 - \xi_2} \leq k, (\xi_1 \neq \xi_2), k > 0 \quad (2.1)$$

for all  $\xi_1, \xi_2 \in [0, \eta]$ .

Let  $E$  be an arbitrary real Banach space and let  $J$  denote normalized duality mapping from  $E$  into  $2^{E^*}$  given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2; \|x\|^2 = \|f\|^2\}$$

where  $E^*$  denotes the dual space of  $E$  and  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing. If  $E^*$  is strictly convex, then  $J$  is single-valued. In the sequel, we shall denote single-valued duality mappings by  $j$ . In Hilbert spaces,  $j$  is the identity.

Let us give some basic definition and remarks that will be used in this work.

**Definition 2.3.** *In Hilbert spaces  $H$ , a self-mapping  $T$  of a nonempty subset  $K$  of  $H$  is asymptotically pseudocontractive if it satisfies*

$$\|T^n x - T^n y\| \leq a_n \|x - y\|^2 + \|x - y - (T^n x - T^n y)\|^2 \quad (2.2)$$

for all  $x, y \in K$  and for some sequence  $\{a_n\} \subseteq [0, \infty)$  such that  $\lim_{n \rightarrow \infty} a_n = 1$ .

We remark that the class of asymptotically pseudocontractive mappings contains the important class of *asymptotically nonexpansive* mappings (that is mappings  $T : K \rightarrow K$  such that

$$\|T^n x - T^n y\| \leq a_n \|x - y\| \forall x, y \in K \quad (2.3)$$

for all  $x, y \in K$  and for some sequence  $\{a_n\} \subseteq [0, \infty)$  such that  $\lim_{n \rightarrow \infty} a_n = 1$ ).

**Definition 2.4.**  $T$  is called asymptotically quasi-nonexpansive if  $F(T) = \{x \in K : Tx = x\} \neq \emptyset$  and (2.3) is satisfied  $\forall x \in K$  and  $y \in F(T)$ .

**Definition 2.5.** If there exists  $L > 0$  such that  $\|T^n x - T^n y\| \leq L\|x - y\|$  for all  $n \geq 1$  and for all  $x, y \in K$ , then  $T$  is called uniformly Lipschitzian.

Let  $H$  be a Hilbert space,  $\rho, \bar{\rho} \in H$  be such that  $\bar{\rho} = g(\rho)$  (where  $g$  is a continuous function satisfying  $0 \leq \frac{g(\xi_1) - g(\xi_2)}{\xi_1 - \xi_2} \leq k, k > 0$ ) and  $T : H \rightarrow H$  be given by

$$\begin{aligned} T\rho(t) &= \eta\Phi(t) + \int_0^t \Phi(t-s)g(\rho(s))ds \\ &= \eta\Phi(t) + \int_0^t \Phi(t-s)\bar{\rho}(s)ds. \end{aligned} \quad (2.4)$$

### 3 Main Results

Our main result shall be the following theorem

**Theorem 3.1.** Let  $\alpha > 0$ ; and  $p_1, p_2 = -\frac{\alpha}{2} \mp \sqrt{\frac{\alpha^2}{4} - \beta}$ ; then a cycle of second kind of the generalized Barbashin-Ezeilo problem (1.1) is the limit in some Hilbert space of the iterative process

$$\left. \begin{aligned} x_n &= \alpha_n x_{n-1} + (1 - \alpha_n) T^k y_n \\ y_n &= \beta_n x_{n-1} + (1 - \beta_n) T^k x_n \end{aligned} \right\}, n \geq 1 \quad (3.1)$$

where  $n = k \geq 1$ ,  $T$  is  $L$ -Lipschitzian operator given by

$$(T\sigma)(t) = -p_1 \int_0^t \int_0^s e^{p_1(t-\xi)} [\hat{g}(\dot{\sigma}(\xi)) + \varphi(\sigma(\xi))] d\xi ds \quad (3.2)$$

and  $\alpha_n, \beta_n$  satisfy the conditions

$$\begin{aligned} (i) \quad & \sum_{n=1}^{\infty} (1 - \alpha_n) = \infty & (ii) \quad & \sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty \\ (iii) \quad & \sum_{n=1}^{\infty} (1 - \beta_n) < \infty & (iv) \quad & (1 - \beta_n)(1 - \alpha_n)L^2 < 1. \end{aligned}$$

Before proving this main result we shall, first, establish the existence and validity of the operator  $T$ .

**Proposition 3.1.** Let the operator  $A(g(\sigma))(t) = \int_0^t \Phi(t-s)g(\sigma(s))ds$  be self-adjoint with eigenfunctions  $g(\sigma_j) \in C_0$  (whenever  $\sigma_j \in C_0$ ) corresponding to eigenvalues  $\lambda_j$  ( $j \geq 1$ ). The operator  $T$  has at least one fixed point if  $k^n \|A\| \leq (1 + \nu_n)$  for some sequence  $\{\nu_n\}$ .

## PROOF

Any bounded sequence of functions  $\{\sigma_n\} \subseteq C_0$  is equibounded and equicontinuous on  $[0, \eta)$  and by Arzela-Ascoli theorem such sequences are relatively compact. Now,  $A$  being a bounded linear operator is continuous so it maps relatively compact sets into relatively compact sets, therefore we infer that  $A$  is a compact operator so  $T$  maps every closed convex set  $K$  into itself. Further,

$$\begin{aligned} \|T\sigma_i - T\sigma_j\| &= \left\| \int_0^t \Phi(t-s) (g(\sigma_i(s)) - g(\sigma_j(s))) ds \right\| \\ &= \|A(g(\sigma_i) - g(\sigma_j))\| = \|\lambda_i g(\sigma_i) - \lambda_j g(\sigma_j)\| \\ &\leq \max\{|\lambda_i|, |\lambda_j|\} \|k(\sigma_i - \sigma_j)\| \\ &= k \max\{|\lambda_i|, |\lambda_j|\} \|\sigma_i - \sigma_j\|. \end{aligned} \quad (3.3)$$

Therefore

$$\|T^n \sigma_i - T^n \sigma_j\| \leq k^n \max\{|\lambda_i|, |\lambda_j|\} \|\sigma_i - \sigma_j\| \quad (3.4)$$

verifying that  $T$  is an asymptotically nonexpansive map if there exists a sequence  $\{\nu_n\}$  with  $\nu_n \rightarrow 0$  such that  $k^n \max\{|\lambda_i|, |\lambda_j|\} \leq (1 + \nu_n)$  since (3.4) yields  $\|T^n \sigma_i - T^n \sigma_j\| \leq a_n \|\sigma_i - \sigma_j\|$  (where  $a_n = (1 + \nu_n)$ ).

Inequality (3.4) follows from condition (2.1) and since  $A$  is a compact operator (3.4) becomes

$$\|T^n \sigma_i - T^n \sigma_j\| \leq k^n \|A\| \|\sigma_i - \sigma_j\|. \quad (3.5)$$

then applying the results of [10], we conclude that the modified implicit process

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1$$

as given by Sun[12] converges to the fixed point of (2.4).

**Theorem 3.2.** *Let the operator  $A(g(\sigma))(t) = \int_0^t \Phi(t-s)g(\sigma(s))ds$  be self-adjoint with eigenfunctions  $g(\sigma_j) \in C_0$  (whenever  $\sigma_j \in C_0$ ) corresponding to eigenvalues  $\lambda_j$  ( $j \geq 1$ ). Then the averaging composite implicit iteration process (3.1)*

$$\left. \begin{aligned} \sigma_{n+1} &= \alpha_n \sigma_n + (1 - \alpha_n) T_i^l \rho_{n+1} \\ \rho_{n+1} &= \beta_n \sigma_n + (1 - \beta_n) T_i^k \sigma_{n+1} \end{aligned} \right\} n \geq 1$$

converges to at least one fixed point of (2.4) provided  $k^n \|A\| \leq (1 + \nu_n)$  for some sequence  $\{\nu_n\}$  satisfying  $\nu_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## PROOF

The argument is the same as in the proof of Proposition 3.1 so we only need to prove that  $T$  is asymptotically pseudocontractive.

$$\begin{aligned}
\langle T\sigma_i - T\sigma_j, j(\sigma_i - \sigma_j) \rangle &= \langle A(g(\sigma_i) - g(\sigma_j)), j(\sigma_i - \sigma_j) \rangle \\
&= \langle \lambda_i g(\sigma_i) - \lambda_j g(\sigma_j), j(\sigma_i - \sigma_j) \rangle \\
&\leq \max\{|\lambda_i|, |\lambda_j|\} \langle k(\sigma_i - \sigma_j), j(\sigma_i - \sigma_j) \rangle \\
&= k \max\{|\lambda_i|, |\lambda_j|\} \|\sigma_i - \sigma_j\|^2.
\end{aligned}$$

Therefore, iteratively, we have

$$\langle T^n \sigma_i - T^n \sigma_j, j(\sigma_i - \sigma_j) \rangle \leq k^n \max\{|\lambda_i|, |\lambda_j|\} \|\sigma_i - \sigma_j\|^2.$$

This implies that  $T$  is asymptotically pseudocontractive, and thus we can apply the results of [8] and [9]. This is the desired result.

## 4 PROOF OF THEOREM 3.1

We now give the proof of Theorem 3.1 by rewriting (1.1) as

$$\begin{cases} \dot{x}_1 = -\alpha x_1 - \beta x_2 - \hat{g}(\dot{\sigma}) - \varphi(\sigma) \\ \dot{x}_2 = x_1 \\ \dot{\sigma} = x_2 \end{cases}$$

This clearly implies that  $x_1 = \ddot{\sigma}$ ,  $x_2 = \dot{\sigma}$  and  $-\hat{g}(\dot{\sigma}) + \varphi(\sigma) = \ddot{\sigma} + \alpha \ddot{\sigma} + \beta \dot{\sigma}$ .

Observing that a fundamental matrix solution  $X$  is given by

$$X(t) = \begin{pmatrix} e^{p_1 t} & e^{p_2 t} \\ p_1 e^{p_1 t} & p_2 e^{p_2 t} \end{pmatrix},$$

it follows that that the desired cycle of the second kind is given by

$$\sigma(t) = -p_1 \int_0^t \int_0^s e^{p_1(s-\xi)} [\hat{g}(\dot{\sigma}(\xi)) + \varphi(\sigma(\xi))] d\xi ds \quad (4.1)$$

where

$$\begin{aligned}
p_1 &= -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta} \\
p_2 &= -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta}
\end{aligned}$$

But the output  $\dot{\sigma}(t)$  is given by

$$\dot{\sigma}(t) = -p_1 \int_0^t e^{p_1(t-s)} [\hat{g}(\sigma(s)) + \varphi(\sigma(s))] ds \quad (4.2)$$

which yields

$$\begin{aligned} \dot{\sigma}(t) &= p_1 [\ddot{\sigma}(t) + \alpha \dot{\sigma}(t) + \beta \sigma(t)] - p_1 [\ddot{\sigma}(0) + \alpha \dot{\sigma}(0) + \beta \sigma(0)] e^{p_1 t} \\ &\quad - p_1^2 \int_0^t e^{p_1(t-s)} [\ddot{\sigma}(s) + \alpha \dot{\sigma}(s) + \beta \sigma(s)] ds. \end{aligned}$$

This gives

$$\begin{aligned} \beta \sigma(t) &= -\ddot{\sigma}(t) + \frac{1 - \alpha p_1}{p_1} \dot{\sigma}(t) + [\ddot{\sigma}(0) + \alpha \dot{\sigma}(0) + \beta \sigma(0)] e^{p_1 t} \\ &\quad + p_1 \int_0^t e^{p_1(t-s)} [\ddot{\sigma}(s) + \alpha \dot{\sigma}(s) + \beta \sigma(s)] ds \\ \sigma(t) &= \frac{p_1}{\beta} \int_0^t e^{p_1(t-s)} [\ddot{\sigma}(s) + \alpha \dot{\sigma}(s) + \beta \sigma(s)] ds + \frac{1}{\beta} [\ddot{\sigma}(0) + \alpha \dot{\sigma}(0) + \beta \sigma(0)] e^{p_1 t} \\ &\quad - \frac{1}{\beta} \ddot{\sigma}(t) + \frac{1 - \alpha p_1}{\beta p_1} \dot{\sigma}(t) \\ &= p_1 \int_0^t e^{p_1(t-s)} \sigma(s) ds + \left[ \frac{-\ddot{\sigma}(0) + (\alpha - 1) \dot{\sigma}(0)}{p_1(p_1 - \alpha) + \beta} + \sigma(0) \right] e^{p_1 t} \\ &\quad - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}(t) + \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}(t) \end{aligned} \quad (4.3)$$

So analysis of the nonlinear operator (4.1) is equivalent to analysis of the simpler nonlinear operator

$$\begin{aligned} (T\sigma)(t) &= p_1 \int_0^t e^{p_1(t-s)} \sigma(s) ds + \left[ \frac{-\ddot{\sigma}(0) + (\alpha - 1) \dot{\sigma}(0)}{p_1(p_1 - \alpha) + \beta} + \sigma(0) \right] e^{p_1 t} \\ &\quad - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}(t) + \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}(t) \end{aligned}$$

which anchors on the behavior of the linear operator (where  $\sigma$  is a fixed point of  $T$ .)

$$(A\sigma)(t) = p_1 \int_0^t e^{p_1(t-s)} \sigma(s) ds - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}(t) + \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}(t).$$

We observe that (3.2) implies  $\sigma(0) = 0$  while (4.2) gives  $\dot{\sigma}(0) = 0$  and so

(4.3) becomes

$$\begin{aligned} (T\sigma)(t) &= p_1 \int_0^t e^{p_1(t-s)} \sigma(s) ds - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}(t) \\ &+ \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}(t) \end{aligned} \quad (4.4)$$

therefore the operator  $T$  is equivalent to  $A$ .

Using condition (1.4) we observe that  $\alpha - \frac{3\alpha}{2} - \sqrt{\frac{\alpha^2}{2}} - \beta < \lambda_2$ . That is  $p_1 < \lambda_2$ .

Now, for  $\alpha > 0$  the operator  $T = A$  is defined on the Sobolev space  $W_0^{(2,2)}[0, \infty)$  which is a Hilbert space in  $L_2[0, \infty)$ -space. In this case we can apply Theorem 3.2 to the operator  $T$  choosing the iterates from  $C_0^\infty[0, \infty)$  where  $\sigma_n(t)$  are defined by (3.1).

Clearly, since  $\sigma_0 = \sigma(0) = 0$ , we have from (3.1) and (4.4) that

$$\begin{aligned} y_1 &= (1 - \beta_1) \left[ p_1 \int_0^t e^{p_1(t-s)} \sigma_1(s) ds - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}_1(t) \right. \\ &\quad \left. + \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}_1(t) \right] \\ \sigma_1 &= (1 - \alpha_1) T y_1 \\ &= (1 - \alpha_1)(1 - \beta_1) \left[ p_1 \int_0^t e^{p_1(t-s)} \sigma_1(s) ds - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}_1(t) \right. \\ &\quad \left. + \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}_1(t) \right] \quad (4.5) \\ \sigma_{n+1}(t) &= \alpha_{n+1} \sigma_n(t) + (1 - \alpha_{n+1}) \left\{ \beta_{n+1} \left[ p_1 \int_0^t e^{p_1(t-s)} \sigma_n(s) ds - \right. \right. \\ &\quad - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}_n(t) \\ &\quad \left. + \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}_n(t) \right] + (1 - \beta_{n+1}) \left[ p_1 \int_0^t e^{p_1(t-s)} \sigma_{n+1}(s) ds \right. \\ &\quad \left. - \frac{1}{p_1(p_1 - \alpha) + \beta} \ddot{\sigma}_{n+1}(t) + \frac{(1 - \alpha)p_1 + 1}{p_1(p_1 - \alpha) + \beta} \dot{\sigma}_{n+1}(t) \right] \left. \right\}. \quad (4.6) \end{aligned}$$

So when  $\alpha > 0$ , the equation (3.2) and (1.4) ensures that  $\sigma = T\sigma$  is asymptotically stable for  $p_1 < 0$ . This gives the required iterates to be asymptotically stable.

We shall complete the proof by showing that each iterate  $\sigma_n$  is a cycle of the second kind. This is obvious since from equation (4.5) if  $\sigma_1$  chosen as a



cycle of the the second kind then  $\sigma_n$  in (4.4) are cycles of the second kind for all  $n \in \mathbb{N}$ . This concludes the proof of theorem 3.1.

**Remark 4.1.** For  $\alpha < 0$  we have  $p_1 > 0$  then, the solutions are bounded cycles and the iterates  $\sigma_n$  are required to be bounded only but not asymptotically stable. This means that the problem is not solvable in Hilbert space since the iterates are not square integrable on  $[0, \infty)$ . Here we apply Proposition 3.1 and results from [10] to obtain some fixed point.

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