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The Gâteaux derivative and orthogonality in C_∞

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Abstract

The general problem in this paper is minimizing the $C_\infty-$ norm of suitable affine mappings from $B(H)$ to C_∞, using convex and differential analysis (Gateaux derivative) as well as input from operator theory. The mappings considered generalize the so-called elementary operators and in particular the generalized derivations, which are of great interest by themselves. The main results obtained characterize global minima in terms of (Banach space) orthogonality.

1 Introduction

The general problem in this paper is minimizing the $C_\infty-$ norm of suitable affine mappings from $B(H)$ to C_∞, using convex and differential analysis (Gateaux derivative) as well as input from operator theory. The mappings considered generalize the so-called elementary operators and in particular the generalized derivations, which are of great interest by themselves. The main results obtained characterize global minima in terms of (Banach space) orthogonality, and constitute an interesting combination of infinite-dimensional differential analysis, operator theory and duality. This leads us to characterize the orthogonality in the sense of Birkhoof in C_∞. Let E be a complex Banach space. We first define orthogonality in E. We say that $b \in E$ is orthogonal to $a \in E$ if for all complex λ there holds

$$\|a + \lambda b\| \geq \|a\| . \tag{1.1}$$

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This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is orthogonal to the open ball $K(0, \|a\|)$, i.e., if and only if this complex line is a tangent one. Note that if b is orthogonal to a, then a need not be orthogonal to b. If E is a Hilbert space, then from (1.1) follows $\langle a, b\rangle = 0$, i.e., orthogonality in the usual sense. Recall [1] that the norm $||.||$ of the Banach space V is said to be Gateaux differentiable at a non-zero element $x \in V$ if

$$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t} = ReD(x, y)$$

for all $y \in V$ and $t \in \mathbb{R}$. Here $\mathbb{R}$ denotes the set of reals, Re denotes the real part and $D(x$ is the unique support functional (in the dual space V^*) such that $||D(x)|| = 1$ and $D(x, x) = ||x||$ [4, 7]. It is well known (see [8] and the references therein) that for $1 < p < \infty$, the von Neumann-Schatten class C_p is a uniformly convex Banach space. Therefore every non-zero $T \in C_p$ is a smooth point and in this case the support functional of T is given by

$$D(T, X) = tr\left[\frac{|T|^{p-1} UX^*}{\|T\|_p^{p-1}}\right], \tag{1.2}$$

for all $X \in C_p$, where $T = U|T|$ is the polar decomposition of T. The first result concerning the orthogonality in a Banach space was given by Anderson[2] showing that if A is a normal operator on a Hilbert space H, then $AS = SA$ implies that for any bounded linear operator X there holds

$$\|S + AX - XA\| \geq \|S\|. \tag{1.3}$$

This means that the range of the derivation $\delta_A : B(H) \to B(H)$ defined by $\delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized in two directions: by extending to the class of elementary mappings

$$E : B(H) \to B(H); \quad E(X) = \sum_{i=1}^{n} A_i X B_i$$

and

$$\widetilde{E}: B(H) \to B(H); \quad \widetilde{E}(X) = \sum_{i=1}^{n} A_i X B_i - X,$$

where $(A_1, A_2, ...A_n)$ and $(B_1, B_2, ...B_n)$ are $n-$ tuples of bounded operators on H, and by extending the inequality (1.3) to C_p-classes with $1 < p < \infty$ see [9], [12]. The Gâteaux derivative concept was used in [6, 9, 10, 15] and [11]. In all of the above results A was not arbitrary; in fact, certain normality-like assumptions have been imposed on A. A characterization of $T \in C_p$ for

$1 < p < \infty$, which are orthogonal to $R(\delta_A \mid C_p)$ (the range of $\delta_A \mid C_p$) for a general operator A has been carried out by F. Kittaneh [8], who showed that, if T has the polar decomposition $T = U\,|T|$, then

$$\|T + \delta_A(X)\|_p \geq \|T\|_p \tag{1.4}$$

for all $X \in C_p$ ($1 < p < \infty$), if and only if, $|T|^{p-1}\,U^* \in \ker \delta_A$.

Let C_∞ be the class of compact operators with $||T|| = \sup_{||f||=1} ||Tf||$ denoting the usual operator norm. In order to characterize those operators which are orthogonal to the range of a derivation in C_∞. First we characterize the global minimum of the map

$$X \mapsto \|S + \phi(X)\|_{C_\infty},\ \phi \text{ is a linear map in } B(H),$$

in C_∞ by using the Gateaux derivative. These results are then applied to characterize the operators $S \in C_\infty$ which are orthogonal to the range of elementary operators.

2 Preliminaries

Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space H and let $T \in B(H)$ be compact, and let $s_1(T) \geq s_2(T) \geq ... \geq 0$ denote the singular values of T , i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to be belong to the Schatten p-classes C_p if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_i(T)^p\right]^{\frac{1}{p}} = [tr(T)^p]^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

where tr denotes the trace functional. Hence C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ corresponds to the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$

denoting the usual operator norm. For the general theory of the Schatten p-classes the reader is referred to [16]. We state the following theorem which we will use in proving our main result Theorem 3.2. Recall that the polar decomposition of $A \in B(H)$ is $A = U|A|$, where U is a partial isometry, $\ker U = \ker(A^*A)$ and $A^*A = |A|$. This decomposition is unique.

Theorem 2.1 ([7])**.** *Let $X, Y \in C_\infty$. Then, there holds*

$$D(X;Y) = \max_{\substack{f \in \Gamma \\ \|f\|=1}} \{Re\langle U^*Yf, f\rangle\},$$

where $X = U|X|$ is the polar decomposition of X and Γ is the subspace in which $X \in C_\infty$ attains its norm.

Theorem 2.2. *[7] Let $(V, \|\cdot\|)$ be an arbitrary Banach space and $F : V \to \mathbb{R}$. If F has a global minimum at $v \in V$, then*

$$DF(v; y) \geq 0,$$

for all $y \in V$.

3 Main Results

Let X be a Banach space, ϕ a linear map $X \to X$, and $\psi(x) = \phi(x) + s$ for some element $s \in X$. Use the notation

$$D(x; y) = \lim_{t \to 0^+} \frac{1}{t}(\|x + ty\| - \|x\|).$$

Recall that the rank one operator $f \otimes g$ is defined by $f \otimes g : x \mapsto \langle x, f\rangle g$ and $tr[T(f \otimes g)] = \langle Tg, f\rangle$. The following theorem is a simple consequence of the known result in convex analysis (the necessary and sufficient condition for optimality)

Theorem 3.1. *The map $(F_\psi)(x) = \|\psi(x)\|$ has a global minimum at $x \in X$ if and only if*

$$D(\psi(x); \phi(y)) \geq 0, \ \forall\, y \in X. \tag{3.1}$$

Now we are ready to prove our first result in C_∞-classes. It gives a necessary and sufficient optimality condition for minimizing F_ψ.

Let $\phi : B(H) \to B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha\phi(X) + \beta\phi(Y)$, for all α, β, X, Y, and let $S \in C_\infty$. Put

$$\mathcal{U} = \{X \in B(H) : \ \phi(X) \in C_\infty\}.$$

Let $\psi : \mathcal{U} \to C_\infty$ defined by

$$\psi(X) = S + \phi(X).$$

Define the function $F_\psi : \mathcal{U} \to \mathbb{R}^+$ by $F_\psi(X) = \|\psi(X)\|_{C_\infty}$.

In the following theorem we characterize the global minimum of F_ψ on C_∞, when ϕ is a linear map satisfying

$$tr(X\phi(Y)) = tr(\phi^*(X)Y), \text{ for all } X, Y \in C_\infty, \tag{3.2}$$

where ϕ^* is an appropriate conjugate of the linear map ϕ. Recall that (3.2) is the definition of the adjoint mapping ϕ^* of ϕ.

An example of ϕ and ϕ^* which satisfy condition (3.2) is given by:
The elementary operator $E : \mathcal{I} \to \mathcal{I}$ defined by

$$E(X) = \sum_{i=1}^{n} A_i X B_i,$$

where $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ are $n-$tuples of bounded Hilbert space operators and $\mathcal{I}$ is a separable ideal of compact operators associated with some unitarily invariant norm. In [7], Keckic showed that the conjugate operator $E^* : \mathcal{I}^* \to \mathcal{I}^*$ of E has the form

$$E^*(X) = \sum_{i=1}^{n} B_i X A_i,$$

and that E and E^* satisfy condition (3.2).

Theorem 3.2. *Let $V \in C_\infty$ be a smooth point and let $\psi(V)$ have the polar decomposition $\psi(V) = U\,|\psi(V)|$ and let $f \in \Gamma$. Then F_ψ has a global minimum on C_∞ at V if and only if $(f \otimes Uf) \in \ker \phi^*$.*

Proof. Let $V \in C_\infty$ be a smooth point and let $\psi(V)$ have the polar decomposition $\psi(V) = U\,|\psi(V)|$.

Assume that F_ψ has a global minimum on C_∞ at V. Then

$$D(\psi(V); \phi(Y)) \geq 0, \tag{3.3}$$

for all $Y \in C_\infty$. Then $\forall\, Y \in C_\infty$ we get

$$\max_{\substack{f\in\Gamma \\ \|f\|=1}} \{Re[\langle U^*\phi(Y)f, f\rangle]\} \geq 0.$$

or

$$Re(tr(f \otimes Uf)\phi(Y)) \geq 0, \text{ for all}\, Y \in C_\infty. \tag{3.4}$$

Since the map ϕ satisfies (3.2), one has

$$tr((f \otimes Uf)\phi(Y)) = tr(\phi^*(f \otimes Uf)Y).$$

Then (3.4) is equivalent to

$$tr(\phi^*(f \otimes Uf)Y) \geq 0, \text{ for all}\, Y \in C_\infty.$$

Equivalently, on taking $Y = h \otimes g$ we get,

$$Re\langle \phi^*(f \otimes Uf)g, h\rangle \geq 0, \text{ for all}\, g, h \in H.$$

As h, g are arbitrary we can easily check that

$$Re\langle\phi^*(f \otimes Uf)g, h\rangle = 0, \text{ for all } g, h \in H.$$

Thus $\phi^*(f \otimes Uf) = 0$, i.e., $f \otimes Uf \in \ker\phi^*$. Conversely, let $f \otimes Uf \in \ker\phi^*$, then $tr((f \otimes Uf)\phi(Y)) = 0$, for all $Y \in C_\infty$. Or $tr(\phi^*(f \otimes Uf)Y) = 0$, for all $Y \in C_\infty$. By taking $Y = h \otimes g$ we get, $Retr\langle\phi^*(f \otimes Uf)g, h\rangle = 0$, for all $h, g \in H$. As h, g are arbitrary we can easily check that $Retr\langle\phi^*(f \otimes Uf)g, h\rangle \geq 0$, for all $h, g \in H$. Equivalently $Retr(\phi^*(f \otimes Uf)Y = Retr((f \otimes Uf)\phi(Y)) \geq 0$ for all $Y \in C_\infty$. Now as Y is taken arbitrary, we get (3.3), which completes the proof of the second part of the theorem. □

We state our first corollary of Theorem 3.2. Let $\phi = \delta_{A,B}$, where $\delta_{A,B}$: $B(H) \to B(H)$ is the generalized derivation defined by $\delta_{A,B}(X) = AX - XB$.

Corollary 3.1. *Let $V \in C_\infty$, $\psi(V)$ has the polar decomposition $\psi(V) = U\,|\psi(V)|$ and let $f \in \Gamma$. Then F_ψ has a global minimum on C_∞ at V, if and only if $(f \otimes Uf) \in \ker\delta_{B^*,A^*}$.*

Proof. It is easily seen that $f \otimes Uf \in ker\phi^*$ is equivalent to $tr((f \otimes Uf)\phi(Y)) = 0$ □

This result may be reformulated in the following form where the global minimum V does not appear. It characterizes the operators V in C_∞ which are orthogonal to the range of the derivation $\delta_{A,B}$. Let Γ be the subspace in which the operator $S \in C_\infty$ attains its norm

Theorem 3.3. *Let $S \in C_\infty$, $\psi(S)$ has the polar decomposition $\psi(S) = U\,|\psi(S)|$ and let $f \in \Gamma$. Then*

$$\|S + (AX - XB)\|_{C_\infty} \geq \|\psi(S)\|_{C_\infty},$$

$(f \otimes Uf) \in \ker\delta_{B^*,A^*}$, for all $X \in C_\infty$.

As a corollary of this theorem we have

Corollary 3.2. *Let $S \in C_\infty \cap \ker\delta_{A,B}$, $\psi(S)$ has the polar decomposition $\psi(S) = U\,|\psi(S)|$ and let $f \in \Gamma$. Then the two following assertions are equivalent:*

1.
$$\|S + (AX - XB)\|_{C_\infty} \geq \|S\|_{C_\infty}, \text{ for all } X \in C_\infty.$$

2. $(f \otimes Uf) \in \ker\delta_{B^*,A*}$.

Remark 3.1. *We point out that, thanks to our general results given previously with more general linear maps ϕ, Theorem 3.3 and its Corollary 3.2 are still true for more general classes of operators than $\delta_{A,B}$ such as the elementary operators $E(X)$ and $\widetilde{E}(X)$. Note that Theorem 3.2 and Corollary 3.1 generalize the results given in [8]*

Remark 3.2. *Since C_∞ contains $C_p\,(0 < p < \infty)$ and if $I \neq \{0\}$, then $C_\infty \supset I \supset F(H)$, where $F(H)$ is the set of all finite rank operators and I is a bilateral ideal of $B(H)$. These show that our results in C_∞ generalize some results in the literature concerning the orthogonality in the sense of Birkhoff (see [8], [9]).*

Now we will present an other characterization of the orthogonality in the sense of Birkhoof.

Theorem 3.4. *Let $S, Y \in C_\infty$ and $f \in \Gamma$, where $S = U|S|$ is a smooth point in C_∞. The following conditions are mutually equivalent.*

(i) The map F_ψ has a global minimum on C_∞ at S;

(ii) $\max_{f\in\Gamma,\,\|f\|=1} Re\,\langle \phi(Y)f, Uf\rangle \geq 0$;

(iii) $tr((f \otimes Uf)\phi(Y)) = 0$ *for all* $Y \in C_\infty$*;*

(iv) $\phi(Y)f \perp Sf$.

Proof. (i) $\Leftrightarrow$ (ii). Applying Theorem 3.2 by taking into account Theorem 3.1.

(ii) $\Leftrightarrow$ (iii). (See the proof of Theorem 3.2)

(ii) $\Leftrightarrow$ (iv) Let Γ be the subspace where S attains its norm. Note that the set

$$\{\big\langle X^*\phi(Y)f, f\big\rangle \mid f \in \Gamma :\ \|f\| = 1\},$$

is the numerical range of $X^*\phi(Y)$ on the subspace Γ, which has in the complex plane, such a position that it contains at least one value with positive real part, under all rotation around the origin. By Toeplitz -Haussdorf Theorem the numerical range is a closed convex set. Hence the condition (ii) is equivalent to the condition that the numerical range of the operator $X^*\phi(Y)$ contains the origin. Since the vectors Uf and Sf always have the same direction. Thus (iv) is equivalent to (ii). Notice that for $\varphi \in \Gamma$ there holds $Sf = \|S\|Uf$. $\square$

As consequences of the above theorem we obtain.

Corollary 3.3. *Let $\phi(Y) = AY - YB$, $S, Y \in C_\infty$ and let $f \in \Gamma$, where $S = U|S|$ is a smooth point in C_∞. Then the following conditions are equivalent.*

(i) The map $\|S + AY - YB\|$ has a global minimum on C_∞ at S;

(ii) $\max_{f\in\Gamma,\,\|f\|=1} Re\,\langle (AY - YB)f, Uf\rangle \geq 0$, $\forall Y \in C_\infty$;

(iii) $tr((f \otimes Uf)AY - YB) = 0$, $\forall Y \in C_\infty$*;*

(iv) $(AY - YB)f \perp Sf$, $\forall Y \in C_\infty$.

If we assume that $S \in ker\delta_{A,B}$ we obtain.

Corollary 3.4. *Let $\phi(Y) = AY - YB$, where $S, Y \in C_\infty$, $S = U|S|$ is a smooth point in C_∞ and let $f \in \Gamma$. Then the following conditions are equivalent.*
(i) $||S + AY - YB|| \geq ||S||$, $\forall S \in \ker \delta_{A,B}$;
(ii) $\max_{f\in\Gamma,\,||f||=1} Re\,\langle (AY - YB)f, Uf\rangle \geq 0$, $\forall Y \in C_\infty$;
(iii) $tr((f \otimes Uf)AY - YB) = 0$, $\forall Y \in C_\infty$;
(iv) $(AY - YB)f \perp Sf$, $\forall Y \in C_\infty$

If we take $\phi(Y) = Y$, we obtain the following theorem which characterize the orthogonality in the sense of Birkhoof of two operators in C_∞.

Corollary 3.5. *Let $S, Y \in C_\infty$, where S is a smooth point in C_∞ and let $\varphi \in \Gamma$. Then the following conditions are mutually equivalent.*
(i) $Y \perp S$ *in the sense of Birkhoof;*
(ii) $\max_{f\in\Gamma,\,||f||=1} Re\,\langle Yf, Uf\rangle \geq 0$, $\forall Y \in C_\infty$;
(iii) $tr((f \otimes Uf)Y) = 0$, $\forall Y \in C_\infty$;
(iv)$Yf \perp Sf$, $\forall Y \in C_\infty$.

Remark 3.3. *Note that a related result to Corollary 3.5 has been given by L. Gajek et al [5, Theorem 2.1].*

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