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Green's Relations on $Hyp_G(2)$

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Abstract

A generalized hypersubstitution of type $\tau = (2)$ is a mapping which maps the binary operation symbol f to a term $\sigma(f)$ which does not necessarily preserve the arity. Any such σ can be inductively extended to a map $\hat{\sigma}$ on the set of all terms of type $\tau = (2)$, and any two such extensions can be composed in a natural way. Thus, the set $Hyp_G(2)$ of all generalized hypersubstitutions of type $\tau = (2)$ forms a monoid. Green's relations on the monoid of all hypersubstitutions of type $\tau = (2)$ were studied by K. Denecke and Sh.L. Wismath. In this paper we describe the classes of generalized hypersubstitutions of type $\tau = (2)$ under Green's relations.

1 Introduction

The concept of generalized hypersubstitutions was introduced by S. Leeratanavalee and K. Denecke [11]. We use it as a tool to study strong hyperidentities and use strong hyperidentities to classify varieties into collections called *strong hypervarieties*. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called *strongly solid*.

A generalized hypersubstitution of type $\tau = (n_i)_{i \in I}$, or simply, a generalized hypersubstitution is a mapping σ which maps each n_i-ary operation symbol of type τ to the set $W_\tau(X)$ of all terms of type τ built up by operation symbols from $\{f_i | i \in I\}$ where f_i is n_i-ary and variables from a countably infinite alphabet of variables $X := \{x_1, x_2, x_3, \ldots\}$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of

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type τ by $Hyp_G(\tau)$. First, we define inductively the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, m < j \in I\!N$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

We extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i supposed that $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. We proved the following propositions.

Proposition 1.1. ([11]) *For arbitrary terms* $t, t_1, \ldots, t_n \in W_\tau(X)$ *and for arbitrary generalized hypersubstitutions* $\sigma, \sigma_1, \sigma_2$ *we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 1.2. ([11]) $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ *is a monoid and the set of all hypersubstitutions of type* τ *forms a submonoid of* $\underline{Hyp_G(\tau)}$.

In this paper we describe the classes of generalized hypersubstitutions of type $\tau = (2)$ under Green's relations.

2 Green's relations on Semigroups

Let S be a semigroup and $1 \notin S$. We extend the binary operation on S to $S \cup \{1\}$ by define $x1 = 1x = x$ for all $x \in S \cup \{1\}$. Then $S \cup \{1\}$ is a semigroup with identity 1.

Let S be a semigroup. Then we define,

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Let S be a semigroup and $\emptyset \neq A \subseteq S$. We now set

$$\begin{aligned} (A)_l &= \cap\{L|L \text{ is a left ideal of } S \text{ containing } A\}, \\ (A)_r &= \cap\{R|R \text{ is a right ideal of } S \text{ containing } A\}, \\ (A)_i &= \cap\{I|I \text{ is an ideal of } S \text{ containing } A\}. \end{aligned}$$

Then $(A)_l$,$(A)_r$ and $(A)_i$ are left ideal, right ideal and ideal of S, respectively. We call $(A)_l$ $((A)_r, (A)_i)$ the *left ideal* (*right ideal, ideal*) *of* S *generated by* A.

It is easy to see that

$$\begin{aligned} (A)_l &= S^1A = SA \cup A, \\ (A)_r &= AS^1 = A \cup SA, \\ (A)_i &= S^1AS^1 = SAS \cup SA \cup AS \cup A. \end{aligned}$$

For $a_1, a_2, \ldots, a_n \in S$, we write $(a_1, a_2, \ldots, a_n)_l$ instead of $(\{a_1, a_2, \ldots, a_n\})_l$ and call it the *left ideal of* S *generated by* $a_1, a_2, \ldots, a_n$. Similarly, we write $(a_1, a_2, \ldots, a_n)_r$ and $(a_1, a_2, \ldots, a_n)_i$ for the right ideal and the ideal of S generated by $a_1, a_2, \ldots, a_n$, respectively. If A is a left ideal of S and $A = (a)_l$ for some $a \in S$, we then call A the *principal left ideal generated by* a. We can define the concept of a principal right ideal and a principal ideal in the same manner.

Let S be a semigroup. We define the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on S as follows:

$$\begin{aligned} a\mathcal{L}b &\Leftrightarrow (a)_l = (b)_l, \\ a\mathcal{R}b &\Leftrightarrow (a)_r = (b)_r, \\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R}, \\ \mathcal{D} &= \mathcal{L} \circ \mathcal{R}, \\ a\mathcal{J}b &\Leftrightarrow (a)_i = (b)_i. \end{aligned}$$

Then we have, for all $a, b \in S$

$$
\begin{aligned}
a\mathcal{L}b &\Leftrightarrow Sa \cup \{a\} = Sb \cup \{b\} \\
&\Leftrightarrow S^1a = S^1b \\
&\Leftrightarrow a = xb \text{ and } b = ya \text{ for some } x, y \in S^1. \\
a\mathcal{R}b &\Leftrightarrow aS \cup \{a\} = bS \cup \{b\} \\
&\Leftrightarrow aS^1 = bS^1 \\
&\Leftrightarrow a = bx \text{ and } b = ay \text{ for some } x, y \in S^1. \\
a\mathcal{H}b &\Leftrightarrow a\mathcal{L}b \text{ and } a\mathcal{R}b. \\
a\mathcal{D}b &\Leftrightarrow (a, c) \in \mathcal{L} \text{ and } (c, b) \in \mathcal{R} \text{ for some } c \in S. \\
a\mathcal{J}b &\Leftrightarrow SaS \cup Sa \cup aS \cup \{a\} = SbS \cup Sb \cup bS \cup \{b\} \\
&\Leftrightarrow S^1aS^1 = S^1bS^1 \\
&\Leftrightarrow a = xby \text{ and } b = zau \text{ for some } x, y, z, u \in S^1.
\end{aligned}
$$

Remark 2.1. *Let S be a semigroup. Then the following statements hold.*

1. *$\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are equivalence relations.*
2. *$\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$.*

We call the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ the *Green's relations on S*. For each $a \in S$, we denote $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

For more details on Green's relations see [7].

3 Green's relations on $Hyp_G(2)$

Let $\tau = (2)$ be a type with the binary operation symbol f. The generalized hypersubstitution σ of type $\tau = (2)$ which maps f to the term t in $W_{(2)}(X)$ is denoted by σ_t. In this section we want to study Green's relations on $Hyp_G(2)$. First, we introduce some notations.

For $s, f(c, d) \in W_{(2)}(X), S \subseteq W_{(2)}(X) \setminus X, H \subseteq Hyp_G(2) \setminus P_G(2), x_i, x_j \in X, i, j \in \mathbb{N}$ we denote :

$vb(s)$:= the total number of variables occurring in the term s,
$leftmost(s)$:= the first variable (from the left) that occurs in s,
$rightmost(s)$:= the last variable that occurs in s,
$W^G_{(2)}(\{x_1\}) := \{t \in W_{(2)}(X) | x_1 \in var(t), x_2 \notin var(t)\}$,
$W^G_{(2)}(\{x_2\}) := \{t \in W_{(2)}(X) | x_2 \in var(t), x_1 \notin var(t)\}$,
$W(\{x_1\}) := W^G_{(2)}(\{x_1\}) \setminus \{x_1\}$,
$W(\{x_2\}) := W^G_{(2)}(\{x_2\}) \setminus \{x_2\}$,

$W^G_{(2)}(\{x_1, x_2\}) := \{t \in W_{(2)}(X) | x_1, x_2 \in var(t)\}$,
$P_G(2)$:=$\{\sigma_{x_i} \in Hyp_G(2) | i \in \mathbb{N}, x_i \in X\}$,
$E^G(\{x_1\})$:=$\{\sigma_t \in Hyp_G(2) | t \in W(\{x_1\})\}$,
$E^G(\{x_2\})$:=$\{\sigma_t \in Hyp_G(2) | t \in W(\{x_2\})\}$,
$E^G(\{x_1, x_2\})$:=$\{\sigma_t \in Hyp_G(2) | t \in W^G_{(2)}(\{x_1, x_2\})\}$,
$E^G_{x_1}$:=$\{\sigma_{f(x_1,t)} \in Hyp_G(2) | t \in W_{(2)}(X), x_2 \notin var(t)\}$,
$E^G_{x_2}$:=$\{\sigma_{f(t,x_2)} \in Hyp_G(2) | t \in W_{(2)}(X), x_1 \notin var(t)\}$,
$W^G := \{t \in W_{(2)}(X) | t \notin X, x_1, x_2 \notin var(t)\}$,
G:=$\{\sigma_t \in Hyp_G(2) | t \in W_{(2)}(X) \setminus X, x_1, x_2 \notin var(t)\}$,

$\overline{f(c,d)}$:= the term obtained from $f(c,d)$ by interchanging all occurrences of the letters x_1 and x_2, i.e. $\overline{f(c,d)} = S^2(f(c,d), x_2, x_1)$ and $f(c,d) = S^2(\overline{f(c,d)}, x_2, x_1)$,

$f(c,d)'$:= the term defined inductively by $x_i' = x_i$ and $f(c,d)' = f(d', c')$,

${}_{x_i}C[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_1 by x_i i.e. ${}_{x_i}C[f(c,d)] = S^2(f(c,d), x_i, x_2)$,

$C_{x_i}[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_2 by x_i i.e. $C_{x_i}[f(c,d)] = S^2(f(c,d), x_1, x_i)$,

${}_{x_i}C_{x_j}[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_1 by x_i and the letter x_2 by x_j i.e. ${}_{x_i}C_{x_j}[f(c,d)] = S^2(f(c,d), x_i, x_j)$,

$\overline{S} := \{\overline{s} | s \in S\}$,
$S' := \{s' | s \in S\}$,
$\overline{H} := \{\sigma_{\overline{t}} | \sigma_t \in H\}$,
$H' := \{\sigma_{t'} | \sigma_t \in H\}$.

Then we have for any $t \in W_{(2)}(X) \setminus X$, $(t')' = t, \overline{\overline{t}} = t$, $\overline{t'} = \overline{t}'$, $\sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_{t'}$ and $\sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\overline{t}}$.

Lemma 3.1. ([12]) *Let $f(c,d), f(u,v) \in W_{(2)}(X)$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$. Then $vb(w) > vb(f(c,d))$ unless $f(c,d)$ and $f(u,v)$ match one of the following 16 possibilities:*

E(1) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(c,d)}$ *where* $\sigma_{f(c,d)} \in G$.

E(2) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{C_{x_1}[f(c,d)]}$.

E(3) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_2)} = \sigma_{{}_{x_2}C[f(c,d)]}$.

E(4) $\sigma_{f(c,d)} \circ_G \sigma_{id} = \sigma_{f(c,d)}$.

E(5) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ *where* $x_i \in X$, $i > 2$.

E(6) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\overline{f(c,d)}}$.

E(7) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_i)} = \sigma_{{}_{x_2}C_{x_i}[f(c,d)]}$ *where* $x_i \in X$, $i > 2$.

E(8) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_1)} = \sigma_{{}_{x_i}C_{x_1}[f(c,d)]}$ *where* $x_i \in X$, $i > 2$.

E(9) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_2)} = \sigma_{{}_{x_i}C[f(c,d)]}$ *where* $x_i \in X$, $i > 2$.

E(10) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_j)} = \sigma_{{}_{x_i}C_{x_j}[f(c,d)]}$ *where* $x_i, x_j \in X$, $i, j > 2$.

E(11) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,v)} = \sigma_{f(c,d)}$ *where* $v \notin X$, $f(c,d) \in W^G_{(2)}(\{x_1\})$.

E(12) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,v)} = \sigma_{\overline{f(c,d)}}$ *where* $v \notin X$, $f(c,d) \in W^G_{(2)}(\{x_1\})$.

E(13) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,v)} = \sigma_{{}_{x_i}C[f(c,d)]}$ *where* $x_i \in X$,$i > 2$, $v \notin X$, $f(c,d) \in W^G_{(2)}(\{x_1\})$.

E(14) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_1)} = \sigma_{\overline{f(c,d)}}$ *where* $u \notin X, f(c,d) \in W^G_{(2)}(\{x_2\})$.

E(15) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_2)} = \sigma_{f(c,d)}$ *where* $u \notin X, f(c,d) \in W^G_{(2)}(\{x_2\})$.

E(16) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ *where* $x_i \in X, i > 2, u \notin X, f(c,d) \in W^G_{(2)}(\{x_2\})$.

Proposition 3.2. ([12]) *$P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup \{\sigma_{id}\} \cup G$ is the set of all idempotent elements in $Hyp_G(2)$.*

Lemma 3.3. *Let $f(c,d) \in W_{(2)}(X) \setminus X$, $\sigma_{x_i} \in P_G(2)$, $\sigma_s \in Hyp_G(2)$ and $\sigma_t \in G$. Then the following statements hold:*

(i) $\sigma_s \circ_G \sigma_{x_i} = \sigma_{x_i}$,

(ii) $\sigma_{x_i} \circ_G \sigma_s \in P_G(2)$,

(iii) $\sigma_t \circ_G \sigma_{f(c,d)} = \sigma_t$.

Proof. (i) Consider $(\sigma_s \circ_G \sigma_{x_i})(f) = (\hat{\sigma}_s \circ \sigma_{x_i})(f) = \hat{\sigma}_s[\sigma_{x_i}(f)] = \hat{\sigma}_s[x_i] = x_i = \sigma_{x_i}(f)$. So $\sigma_s \circ_G \sigma_{x_i} = \sigma_{x_i}$.

(ii) If $s \in X$, then by (i) we get $\sigma_{x_i} \circ_G \sigma_s = \sigma_s \in P_G(2)$. Assume that $s = f(u,v)$ where $u, v \in W_{(2)}(X)$ and $\sigma_{x_i} \circ_G \sigma_u, \sigma_{x_i} \circ_G \sigma_v \in P_G(2)$. Thus $\hat{\sigma}_{x_i}[u], \hat{\sigma}_{x_i}[v] \in X$. Consider $(\sigma_{x_i} \circ_G \sigma_s)(f) = (\sigma_{x_i} \circ_G \sigma_{f(u,v)})(f) = S^2(x_i, \hat{\sigma}_{x_i}[u], \hat{\sigma}_{x_i}[v])$. If $x_i = x_1$, then $(\sigma_{x_i} \circ_G \sigma_s)(f) = \hat{\sigma}_{x_i}[u] \in X$. If $x_i = x_2$, then $(\sigma_{x_i} \circ_G \sigma_s)(f) = \hat{\sigma}_{x_i}[v] \in X$. If $i > 2$, then $(\sigma_{x_i} \circ_G \sigma_s)(f) = x_i \in X$. So $\sigma_{x_i} \circ_G \sigma_s \in P_G(2)$.

(iii) Since $x_1, x_2 \notin var(t)$, thus $(\sigma_t \circ_G \sigma_{f(c,d)})(f) = S^2(t, \hat{\sigma}_t[c], \hat{\sigma}_t[d]) = t = \sigma_t(f)$. So $\sigma_t \circ_G \sigma_{f(c,d)} = \sigma_t$.

Proposition 3.4. *For any $\sigma_t \in Hyp_G(2) \setminus P_G(2)$, we have $\sigma_t \mathcal{R} \sigma_{\bar{t}}$, $\sigma_t \mathcal{L} \sigma_{t'}$ and $\sigma_t \mathcal{D} \sigma_{\bar{t}} \mathcal{D} \sigma_{t'} \mathcal{D} \sigma_{\overline{t'}}$.*

Proof. Let $\sigma_t \in Hyp_G(2) \setminus P_G(2)$. Then $\sigma_{\bar{t}} \circ_G \sigma_{f(x_2,x_1)} = \sigma_t$, $\sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\bar{t}}$, $\sigma_{f(x_2,x_1)} \circ_G \sigma_{t'} = \sigma_t$ and $\sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_{t'}$. So $\sigma_t \mathcal{R} \sigma_{\bar{t}}$ and $\sigma_t \mathcal{L} \sigma_{t'}$. Therefore $\sigma_t \mathcal{D} \sigma_{\bar{t}} \mathcal{D} \sigma_{t'} \mathcal{D} \sigma_{\overline{t'}}$.

Proposition 3.5. *Any $\sigma_{x_i} \in P_G(2)$ is $\mathcal{L}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $P_G(2)$, and not related to any other generalized hypersubstitutions. Moreover, the set $P_G(2)$ forms a complete $\mathcal{R}$-, $\mathcal{D}$- and $\mathcal{J}$- class.*

Proof. By Lemma 3.3, we get for any $\sigma_{x_i} \in P_G(2)$, $\sigma \circ_G \sigma_{x_i} = \sigma_{x_i}$ for all $\sigma \in Hyp_G(2)$. This shows that any $\sigma_{x_i} \in P_G(2)$ can be $\mathcal{L}$-related only to itself. Since $\sigma_{x_i} \circ_G \sigma_{x_j} = \sigma_{x_j}$ for all $\sigma_{x_i}, \sigma_{x_j} \in P_G(2)$, so any two elements in $P_G(2)$ are $\mathcal{R}$-related. From $\mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$, we obtain that any two elements in $P_G(2)$ are $\mathcal{D}-$ and $\mathcal{J}-$ related. Moreover by Lemma 3.3, we get $\sigma_s \circ_G \sigma_{x_i} \circ_G \sigma_t \in P_G(2)$ for all $\sigma_s, \sigma_t \in Hyp_G(2), \sigma_{x_i} \in P_G(2)$. This implies if $\sigma \notin P_G(2)$, then σ cannot be $\mathcal{J}$-related to every element in $P_G(2)$. So $P_G(2)$ is the $\mathcal{J}$-class of its elements. Since any two elements in $P_G(2)$ are $\mathcal{R}-$ and $\mathcal{D}-$ related, $\mathcal{R} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and $P_G(2)$ is the $\mathcal{J}$-class of its elements, thus $P_G(2)$ forms a complete $\mathcal{R}$-, $\mathcal{D}$-class.

Lemma 3.6. *Let $\sigma_s, \sigma_t \in Hyp_G(2)$. Then the following statements hold:*

(i) If $\sigma_s \circ_G \sigma_t = \sigma_{id}$, then either $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{f(x_2,x_1)}$.

(ii) If $\sigma_s \circ_G \sigma_t = \sigma_{f(x_2,x_1)}$, then either $\sigma_s = \sigma_{id}, \sigma_t = \sigma_{f(x_2,x_1)}$ or $\sigma_s = \sigma_{f(x_2,x_1)}, \sigma_t = \sigma_{id}$.

Proof. (i) Assume that $\sigma_s \circ_G \sigma_t = \sigma_{id}$. Since $f(x_1, x_2) \notin X$, by Lemma 3.3 we get $s, t \notin X$ and thus $s = f(a, b), t = f(c, d)$ for some $a, b, c, d \in W_{(2)}(X)$. From $\sigma_s \circ_G \sigma_t = \sigma_{id}$, we obtain that $S^2(f(a, b), \hat{\sigma}_{f(a,b)}[c], \hat{\sigma}_{f(a,b)}[d]) = f(x_1, x_2)$. So $a = c = x_1$ or $a = x_2, d = x_1$ and $b = d = x_2$ or $b = x_1, c = x_2$. This implies $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{f(x_2,x_1)}$.

The proof of (ii) is similar to the proof of (i).

Proposition 3.7. *All of $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-classes of σ_{id} are equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.*

Proof. By Proposition 3.4, we get σ_{id} and $\sigma_{f(x_2,x_1)}$ are $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-related. This implies the $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-class of σ_{id} contain at least $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$. Let $\sigma_t \in Hyp_G(2)$ where $\sigma_t \mathcal{D} \sigma_{id}$. So $\sigma_t \mathcal{L} \sigma_s$ and $\sigma_s \mathcal{R} \sigma_{id}$ for some $\sigma_s \in Hyp_G(2)$. Then there exist $\sigma_u, \sigma_v, \sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_t = \sigma_p \circ_G \sigma_s$, $\sigma_s = \sigma_q \circ_G \sigma_t$, $\sigma_s = \sigma_{id} \circ_G \sigma_u$ and $\sigma_{id} = \sigma_s \circ_G \sigma_v$. From $\sigma_{id} = \sigma_s \circ_G \sigma_v$, by Lemma 3.6 we get $\sigma_s = \sigma_{id}$ or $\sigma_s = \sigma_{f(x_2,x_1)}$. From $\sigma_s = \sigma_{id}$ or $\sigma_s = \sigma_{f(x_2,x_1)}$

and $\sigma_s = \sigma_q \circ_G \sigma_t$, by Lemma 3.6 we get $\sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_{f(x_2,x_1)}$. So the $\mathcal{D}$-class of σ_{id} is equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$. From $\mathcal{R} \subseteq \mathcal{D}, \mathcal{L} \subseteq \mathcal{D}$, we obtain that the $\mathcal{R}$- and the $\mathcal{L}$-class of σ_{id} are equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

Proposition 3.8. *$(\sigma_{id})_i = Hyp_G(2) = (\sigma_{f(x_2,x_1)})_i$, and if $\sigma \in Hyp_G(2)$ and $(\sigma)_i = Hyp_G(2)$, then σ is one of σ_{id} or $\sigma_{f(x_2,x_1)}$. Moreover, the $\mathcal{J}$-class of σ_{id} is equal to its $\mathcal{D}$-class, $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.*

Proof. Let $\sigma \in Hyp_G(2)$. Then $\sigma \circ_G \sigma_{id} \circ_G \sigma_{id} = \sigma$ and $\sigma \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_1)} = \sigma$. So $(\sigma_{id})_i = Hyp_G(2) = (\sigma_{f(x_2,x_1)})_i$. This implies $\sigma_{id}\mathcal{J}\sigma_{f(x_2,x_1)}$. Assume that $(\sigma)_i = Hyp_G(2)$. Then $\sigma\mathcal{J}\sigma_{id}$ and thus there exist $\delta, \rho \in Hyp_G(2)$ such that $\delta \circ_G \sigma \circ_G \rho = \sigma_{id}$. By Lemma 3.6, we get $\sigma \circ_G \rho = \sigma_{id}$ or $\sigma \circ_G \rho = \sigma_{f(x_2,x_1)}$. Again by Lemma 3.6, we get $\sigma = \sigma_{id}$ or $\sigma = \sigma_{f(x_2,x_1)}$.

Lemma 3.9. *Let $u \in W_{(2)}(X)$, $\sigma_t \in Hyp_G(2)$ and $x = x_1$ or $x = x_2$. If $x \notin var(u)$, then $x \notin var(\hat{\sigma}_t[u])$ (x is not a variable occurring in the term $(\sigma_t \circ_G \sigma_u)(f)$).*

Proof. If $u \in X$, then $\hat{\sigma}_t[u] = u$ and so $x \notin var(\hat{\sigma}_t[u])$. Assume that $u = f(u_1, u_2)$ where $u_1, u_2 \in W_{(2)}(X)$, $x \notin var(\hat{\sigma}_t[u_1])$ and $x \notin var(\hat{\sigma}_t[u_2])$. Since $x \notin var(\hat{\sigma}_t[u_1])$, $x \notin var(\hat{\sigma}_t[u_2])$ and $\hat{\sigma}_t[u] = \hat{\sigma}_t[f(u_1, u_2)] = S^2(t, \hat{\sigma}_t[u_1], \hat{\sigma}_t[u_2])$, thus $x \notin var(\hat{\sigma}_t[u])$.

Proposition 3.10. *Any $\sigma_t \in G$ is $\mathcal{R}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of G, and not related to any other generalized hypersubstitutions. Moreover, the set G forms a complete $\mathcal{L}$-, $\mathcal{D}$- and $\mathcal{J}$- class.*

Proof. Let $\sigma_t \in G$. Assume that $\sigma_s \in Hyp_G(2)$ where $\sigma_s\mathcal{R}\sigma_t$. By Proposition 3.5, we get $s \notin X$. Then there exists $\sigma_p \in Hyp_G(2)$ such that $\sigma_s = \sigma_t \circ_G \sigma_p$. Since $s \notin X$ and $\sigma_s = \sigma_t \circ_G \sigma_p$, by Lemma 3.3 we get $p \notin X$. Since $\sigma_t \in G$ and $p \notin X$, by Lemma 3.3 we get $\sigma_t \circ_G \sigma_p = \sigma_t$. So $\sigma_s = \sigma_t$. Thus σ_t is $\mathcal{R}$-related only to itself. Let $\sigma_s, \sigma_t \in G$. By Lemma 3.3, we get $\sigma_s \circ_G \sigma_t = \sigma_s$ and $\sigma_t \circ_G \sigma_s = \sigma_t$. Thus $\sigma_s\mathcal{L}\sigma_t$. So any two elements in G are $\mathcal{L}$-related. Since $\mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$, thus any two elements in G are $\mathcal{D}-$ and $\mathcal{J}-$ related. Assume that $\sigma_t \in G$ and $\sigma_s \in Hyp_G(2)$ where $\sigma_s\mathcal{J}\sigma_t$. By Proposition 3.5, we get $s \notin X$. Then there exist $\sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_p \circ_G \sigma_t \circ_G \sigma_q = \sigma_s$. Since $s \notin X$ and $\sigma_p \circ_G \sigma_t \circ_G \sigma_q = \sigma_s$, thus by Lemma 3.3 we get $p, q \notin X$. Since $\sigma_t \in G$ and $q \notin X$, by Lemma 3.3 we get $\sigma_t \circ_G \sigma_q = \sigma_t$. Since $x_1, x_2 \notin var(t)$, by Lemma 3.9 we get x_1, x_2 are not variables occurring in the term $(\sigma_p \circ_G \sigma_t)(f) = (\sigma_p \circ_G \sigma_t \circ_G \sigma_q)(f)$. Thus $x_1, x_2 \notin var(s)$ and so $\sigma_s \in G$. So G is the $\mathcal{J}$-class of its elements. Since any two elements in G are $\mathcal{L}-$ and $\mathcal{D}-$ related, $\mathcal{L} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and G is the $\mathcal{J}$-class of its elements, thus G forms a complete $\mathcal{L}$-, $\mathcal{D}$-class.

Proposition 3.11. *Let $\tau = (n_i)_{i\in I}$ be a type and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then $\sigma_1 \mathcal{R} \sigma_2$ if and only if $Im\hat{\sigma}_1 = Im\hat{\sigma}_2$.*

Proof. Assume that $\sigma_1 \mathcal{R} \sigma_2$. Then $\sigma_1 = \sigma_2 \circ_G \sigma_3$ and $\sigma_2 = \sigma_1 \circ_G \sigma_4$ for some $\sigma_3, \sigma_4 \in Hyp_G(\tau)$. So $\hat{\sigma}_1 = (\sigma_2 \circ_G \sigma_3)\hat{} = \hat{\sigma}_2 \circ \hat{\sigma}_3$ and $\hat{\sigma}_2 = (\sigma_1 \circ_G \sigma_4)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_4$. Thus $Im\hat{\sigma}_1 = \hat{\sigma}_1[W_\tau(X)] = (\hat{\sigma}_2 \circ \hat{\sigma}_3)[W_\tau(X)] = \hat{\sigma}_2[\hat{\sigma}_3[W_\tau(X)]] \subseteq \hat{\sigma}_2[W_\tau(X)] = Im\hat{\sigma}_2$. By the same way we can show that $Im\hat{\sigma}_2 \subseteq Im\hat{\sigma}_1$. Conversely, assume that $Im\hat{\sigma}_1 = Im\hat{\sigma}_2$. For each $i \in I$, we have $\sigma_1(f_i) = S^{n_i}(\sigma_1(f_i), x_1, \ldots, x_{n_i}) = \hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})] \in Im\hat{\sigma}_1 = Im\hat{\sigma}_2$. So $\sigma_1(f_i) = \hat{\sigma}_2[t_i]$ for some $t_i \in W_\tau(X)$. We define $\gamma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$ by $\gamma(f_i) = t_i$ for all $i \in I$. Let $i \in I$. Then $(\sigma_2 \circ_G \gamma)(f_i) = \hat{\sigma}_2[\gamma(f_i)] = \hat{\sigma}_2[t_i] = \sigma_1(f_i)$. So $\sigma_1 = \sigma_2 \circ_G \gamma$. By the same way we can show that $\sigma_2 = \sigma_1 \circ_G \beta$ for some $\beta \in W_\tau(X)$.

Proposition 3.12. *For any $\sigma_s, \sigma_t \in Hyp_G(2) \setminus P_G(2)$, $\sigma_s \mathcal{R} \sigma_t$ if and only if $s = t$ or $s = \bar{t}$.*

Proof. Assume that $\sigma_s \mathcal{R} \sigma_t$. Then there exist $\sigma_u, \sigma_v \in Hyp_G(2)$ such that $\sigma_s = \sigma_t \circ_G \sigma_u$ and $\sigma_t = \sigma_s \circ_G \sigma_v$. By Lemma 3.3, we get $u, v \notin X$. Then $u = f(u_1, u_2)$ and $v = f(v_1, v_2)$ for some $u_1, u_2, v_1, v_2 \in W_{(2)}(X)$. Then we have two equations

$$s = S^2(t, \hat{\sigma}_t[u_1], \hat{\sigma}_t[u_2]) \cdots (1)$$
$$t = S^2(s, \hat{\sigma}_s[v_1], \hat{\sigma}_s[v_2]) \cdots (2).$$

From (1) and (2), we get $vb(s) = vb(t)$. We consider four cases:

Case 1: $t \in W^G$. From (1), we get $s = t$.

Case 2: $t \in W^G_{(2)}(\{x_1, x_2\})$. Suppose that $u_1 \notin X$ or $u_2 \notin X$. Then $\hat{\sigma}_t[u_1] \notin X$ or $\hat{\sigma}_t[u_2] \notin X$. From (1) and $x_1, x_2 \in var(t)$, we obtain that $vb(s) > vb(t)$ and it is a contradiction. So $u_1, u_2 \in X$. Suppose that $u_1 = u_2 = x_1$. Then $\hat{\sigma}_t[u_1] = \hat{\sigma}_t[u_2] = x_1$. From (1), we get $s \in W(\{x_1\})$. Suppose that $v_1 \notin X$. Then $\hat{\sigma}_s[v_1] \notin X$. From (2) and $x_1 \in var(s)$, we obtain that $vb(t) > vb(s)$ and it is a contradiction. So $v_1 \in X$ and thus $\hat{\sigma}_s[v_1] = v_1$. Since $s \in W(\{x_1\})$ and $\hat{\sigma}_s[v_1] = v_1$, from (2) we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$ which contradicts to $t \in W^G_{(2)}(\{x_1, x_2\})$. If $u_1 = x_1, u_2 = x_2$, then $\hat{\sigma}_t[u_1] = x_1, \hat{\sigma}_t[u_2] = x_2$. From (1), we get $s = t$. If $u_1 = x_1, u_2 = x_i$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_2, u_2 = x_1$, then $\hat{\sigma}_t[u_1] = x_2, \hat{\sigma}_t[u_2] = x_1$. From (1), we get $s = \bar{t}$. If $u_1 = x_2, u_2 = x_2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_2, u_2 = x_i$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_i, u_2 = x_1$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_i, u_2 = x_2$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. Suppose that

$u_1 = x_i, u_2 = x_j$ where $i, j > 2$. Then $\hat{\sigma}_t[u_1] = x_i, \hat{\sigma}_t[u_2] = x_j$. From (1), we get $s \in W^G$. Since $x_1, x_2 \notin var(s)$, from (2) we get $s = t$. So $x_1, x_2 \notin var(t)$ and it is a contradiction.

Case 3: $t \in W(\{x_1\})$. Suppose that $u_1 \notin X$. Then $\hat{\sigma}_t[u_1] \notin X$. From (1), $x_1 \in var(t)$ and $\hat{\sigma}_t[u_1] \notin X$, we obtain that $vb(s) > vb(t)$ and it is a contradiction. So $u_1 \in X$ and thus $\hat{\sigma}_s[u_1] = u_1$. If $u_1 = x_1$, then by (1) we get $s = t$. If $u_1 = x_2$, then by (1) we get $s = \bar{t}$. Suppose that $u_1 = x_i$ where $i > 2$. From (1), we get $s \in W^G$. Since $x_1, x_2 \notin var(s)$, from (2) we get $s = t$. So $x_1 \notin var(t)$ and it is a contradiction.

Case 4: $t \in W(\{x_2\})$. By the same proof as the case $t \in W(\{x_1\})$ we get $s = t$ or $s = \bar{t}$.

Conversely, assume that $s = t$ or $s = \bar{t}$. By Proposition 3.4, we get $\sigma_s \mathcal{R} \sigma_t$.

Lemma 3.13. *Let $\sigma_{f(c,d)} \in Hyp_G(2) \setminus \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$ and $u \in W_{(2)}(X) \setminus X$. If $\sigma_{f(c,d)} \in E^G(\{x_1, x_2\})$, then $vb((\sigma_{f(c,d)} \circ_G \sigma_u)(f)) > vb(u)$.*

Proof. Since $x_1, x_2 \in var(f(c,d))$ and $f(c,d) \neq f(x_1,x_2), f(x_2,x_1)$, thus $c \notin X$ or $d \notin X$ and $vb(f(c,d)) \geq 3$. Let $vb(u) = 2$. Then $u = f(x_i, x_j)$ for some $x_i, x_j \in X$. So $vb(w) = vb((\sigma_{f(c,d)} \circ_G \sigma_u)(f)) = vb((\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_j)})(f)) = vb(S^2(f(c,d), x_i, x_j)) \geq 3 > vb(u)$. Let $u = f(s,t)$ where $s \in X$ and $t \notin X$. Then $\hat{\sigma}_{f(c,d)}[s] = s \in X$. Assume that $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$. Since $x_1, x_2 \in var(f(c,d))$ and $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$, thus $vb(w) = vb((\sigma_{f(c,d)} \circ_G \sigma_u)(f)) = vb((\sigma_{f(c,d)} \circ_G \sigma_{f(s,t)})(f)) = vb(S^2(f(c,d), s, \hat{\sigma}_{f(c,d)}[t])) > vb(f(s,t)) = vb(u)$. Let $u = f(s,t)$ where $s, t \notin X$. Assume that $vb(\hat{\sigma}_{f(c,d)}[s]) > vb(s)$ and $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$. Since $x_1, x_2 \in var(f(c,d))$ and $vb(\hat{\sigma}_{f(c,d)}[s]) > vb(s)$, $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$, thus $vb(w) = vb((\sigma_{f(c,d)} \circ_G \sigma_u)(f)) = vb((\sigma_{f(c,d)} \circ_G \sigma_{f(s,t)})(f)) = vb(S^2(f(c,d), \hat{\sigma}_{f(c,d)}[s], \hat{\sigma}_{f(c,d)}[t])) > vb(f(s,t)) = vb(u)$.

Lemma 3.14. *If $f(c,d) \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$ ($x_1 \notin var(f(c,d))$ or $x_2 \notin var(f(c,d))$), then for any $u, v \in W_{(2)}(X)$ the term w corresponding to $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)}$ is in $W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$.*

Proof. Assume that $f(c,d) \in W(\{x_1\})$. We have to consider the letters used in the term $w = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$. If $u \in X$, then $\hat{\sigma}_{f(c,d)}[u] = u \in X$. Since $f(c,d) \in W(\{x_1\})$, $\hat{\sigma}_{f(c,d)}[u] \in X$ and $w = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$, thus $w \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. Assume that $u = f(p,q)$ where $p, q \in W_{(2)}(X)$ and $\hat{\sigma}_{f(c,d)}[p] \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. So $\hat{\sigma}_{f(c,d)}[u] = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[p], \hat{\sigma}_{f(c,d)}[q]) \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. Since $f(c,d) \in W(\{x_1\})$, $\hat{\sigma}_{f(c,d)}[u] \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$ and $w = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$, thus $w \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. By the same way we can show that if $f(c,d) \in W(\{x_2\})$, then $w \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. If $f(c,d) \in W^G$, then $w = f(c,d) \in W^G$.

Lemma 3.15. *$E^G_{x_1}$ is a left zero band.*

Proof. Let $\sigma_{f(x_1,s)}, \sigma_{f(x_1,t)} \in E^G_{x_1}$. Since $x_2 \notin var(s)$, thus $(\sigma_{f(x_1,s)} \circ_G \sigma_{f(x_1,t)})(f) = S^2(f(x_1,s), x_1, \hat{\sigma}_{f(x_1,s)}[t]) = f(x_1,s)$. So $\sigma_{f(x_1,s)} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,s)}$. Thus every element in $E^G_{x_1}$ is left zero. So $E^G_{x_1}$ is a left zero band.

Proposition 3.16. *The $\mathcal{L}$-class of the element $\sigma_{f(x_1,x_1)}$ is precisely the set $E^G_{x_1} \cup \overline{E^G_{x_2}}$.*

Proof. For any two idempotent elements e and f in a semigroup S, $e\mathcal{L}f$ if and only if $ef = e$ and $fe = f$. Since $E^G_{x_1}$ is a left zero band, it follows that $\sigma_{f(x_1,x_1)}$ is $\mathcal{L}$-related to any element of $E^G_{x_1}$. By Proposition 3.4, we get $\sigma_{f(x_1,x_1)}$ is $\mathcal{L}$-related to any element of $(E^G_{x_1})' = \overline{E^G_{x_2}}$. Thus the $\mathcal{L}$-class of $\sigma_{f(x_1,x_1)}$ contains at least $E^G_{x_1} \cup \overline{E^G_{x_2}}$. For the opposite inclusion, assume that $\sigma_t \in Hyp_G(2)$ where $\sigma_t \mathcal{L} \sigma_{f(x_1,x_1)}$. By Proposition 3.5, we get $t \notin X$. Then $t = f(u,v)$ for some $u, v \in W_{(2)}(X)$. From $\sigma_t \mathcal{L} \sigma_{f(x_1,x_1)}$, then there exist $\sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_p \circ_G \sigma_{f(x_1,x_1)} = \sigma_t$ and $\sigma_q \circ_G \sigma_t = \sigma_{f(x_1,x_1)}$. Since $t, f(x_1,x_1) \notin X$, by Lemma 3.3 we get $p, q \notin X$. Then there exist $a, b, c, d \in W_{(2)}(X)$ such that $p = f(a,b)$ and $q = f(c,d)$. Thus we have $\sigma_{f(a,b)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{f(u,v)}$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(x_1,x_1)}$. From $\sigma_{f(a,b)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{f(u,v)}$, by Lemma 3.9 we get $x_2 \notin var(f(u,v))$. From $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(x_1,x_1)}$, we obtain that $S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v]) = f(x_1,x_1)$. Suppose that $u, v \neq x_1$. Thus $\hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v] \neq x_1$. This implies $S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v]) \neq f(x_1,x_1)$, which is a contradiction. So $u = x_1$ or $v = x_1$. Since $x_2 \notin var(f(u,v))$ and $u = x_1$ or $v = x_1$, thus $\sigma_t = \sigma_{f(u,v)} \in E^G_{x_1} \cup \overline{E^G_{x_2}}$.

Corollary 3.17. *The $\mathcal{D}$-class of the element $\sigma_{f(x_1,x_1)}$ is precisely the set $E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$.*

Proof. Assume that $\sigma_t \in Hyp_G(2)$ where $\sigma_t \mathcal{D} \sigma_{f(x_1,x_1)}$. Then there exists $\sigma_s \in Hyp_G(2)$ such that $\sigma_t \mathcal{R} \sigma_s$ and $\sigma_s \mathcal{L} \sigma_{f(x_1,x_1)}$. Since $\sigma_t \mathcal{R} \sigma_s$, by Proposition 3.12 we get $\sigma_t = \sigma_s$ or $\sigma_t = \sigma_{\overline{s}}$. Since $\sigma_s \mathcal{L} \sigma_{f(x_1,x_1)}$, by Proposition 3.16 we get $\sigma_s \in E^G_{x_1} \cup \overline{E^G_{x_2}}$. If $\sigma_s \in E^G_{x_1}$, then $\sigma_t \in E^G_{x_1} \cup \overline{E^G_{x_1}} \subseteq E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$. If $\sigma_s \in \overline{E^G_{x_2}}$, then $\sigma_t \in E^G_{x_2} \cup \overline{E^G_{x_2}} \subseteq E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$. For the opposite inclusion, assume that $\sigma_t \in E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$. If $\sigma_t \in E^G_{x_1} \cup \overline{E^G_{x_2}}$, then by Proposition 3.16 we get $\sigma_t \mathcal{L} \sigma_{f(x_1,x_1)}$. Since $\mathcal{L} \subseteq \mathcal{D}$, thus $\sigma_t \mathcal{D} \sigma_{f(x_1,x_1)}$. If $\sigma_t \in E^G_{x_2} \cup \overline{E^G_{x_1}}$, then $\sigma_{\overline{t}} \in E^G_{x_1} \cup \overline{E^G_{x_2}}$. By Proposition 3.16, we get $\sigma_{\overline{t}} \mathcal{L} \sigma_{f(x_1,x_1)}$. By Proposition 3.12, we get $\sigma_t \mathcal{R} \sigma_{\overline{t}}$. So $\sigma_t \mathcal{D} \sigma_{f(x_1,x_1)}$.

Proposition 3.18. *The following statements hold:*

(i) $(\sigma_{f(x_1,x_1)})_i = I := \{\sigma_t \in Hyp_G(2) | t \in W^G_{(2)}(\{x_1\}) \cup W^G_{(2)}(\{x_2\})$ *or* $x_1, x_2 \notin var(t)\}$.

(ii) If $\sigma \in I$ *where* $\sigma \notin E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G}$*, then* $(\sigma)_i \subsetneq I$.

(iii) The $\mathfrak{J}$*-class of* $\sigma_{f(x_1,x_1)}$ *is equal to its* $\mathfrak{D}$*-class,* $E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G}$.

Proof. (i) Assume that $\sigma_s \in (\sigma_{f(x_1,x_1)})_i$. Then there exist $\delta, \rho \in Hyp_G(2)$ such that $\delta \circ_G \sigma_{f(x_1,x_1)} \circ_G \rho = \sigma_s$. If δ or $\rho \in P_G(2)$, then by Lemma 3.3 we get $\sigma_s = \delta \circ_G \sigma_{f(x_1,x_1)} \circ_G \rho \in P_G(2) \subseteq I$. Assume that $\delta, \rho \notin P_G(2)$. By Lemma 3.14, we get $\sigma_{f(x_1,x_1)} \circ_G \rho \in I$. By Lemma 3.9, we get $\sigma_s = \delta \circ_G (\sigma_{f(x_1,x_1)} \circ_G \rho) \in I$. For the opposite inclusion, suppose that $\sigma_s \in I$. If $\sigma_s \in P_G(2)$, then by Lemma 3.3 we get $\sigma_s = \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_s \in (\sigma_{f(x_1,x_1)})_i$. Let $\sigma_s \notin P_G(2)$. If $x_1, x_2 \notin var(s)$, then by Lemma 3.3 we get $\sigma_s = \sigma_s \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_s \in (\sigma_{f(x_1,x_1)})_i$. If $s \in W(\{x_1\})$, then $\sigma_s = \sigma_s \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_1,v)} \in (\sigma_{f(x_1,x_1)})_i$ for some $v \in W_{(2)}(X)$. If $s \in W(\{x_2\})$, then $\sigma_s = \sigma_s \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_2,v)} \in (\sigma_{f(x_1,x_1)})_i$ for some $v \in W_{(2)}(X)$.

(ii) Assume that $\sigma \in I$ where $\sigma \notin E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G}$. If $\sigma \in P_G(2)$, then $(\sigma)_i = Hyp_G(2)\sigma Hyp_G(2) = P_G(2) \subsetneq I$. Assume that $\sigma \notin P_G(2)$ and $\sigma = \sigma_{f(u,v)}$ where $u, v \in W_{(2)}(X)$. Let $f(u,v) \in W(\{x_1\}) \cup W(\{x_2\})$. Suppose that $u, v \in X$. Since $f(u,v) \in W(\{x_1\}) \cup W(\{x_2\})$, thus $\sigma_{f(u,v)} \in E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G}$ and it is a contradiction. Suppose that $u \in X$ and $v \notin X$. If $u = x_1$ or $u = x_2$, then $\sigma_{f(u,v)} \in E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G}$ and it is a contradiction. So $u = x_i$ for some $i > 2$. Suppose that $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$. Since $f(x_1,x_1) \notin X$ and $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$, there exist $p, q, r, s \in W_{(2)}(X)$ such that $\sigma_{f(p,q)} \circ_G \sigma_{f(x_i,v)} \circ_G \sigma_{f(r,s)} = \sigma_{f(x_1,x_1)}$. Let w be the term $(\sigma_{f(x_i,v)} \circ_G \sigma_{f(r,s)})(f)$. So $w = f(x_i, k)$ for some $k \in W_{(2)}(X) \setminus X$. Then we have $\sigma_{f(p,q)} \circ_G \sigma_{f(x_i,k)} = \sigma_{f(x_1,x_1)}$. This implies $f(p,q) = f(x_2,x_2)$. Consider $(\sigma_{f(x_2,x_2)} \circ_G \sigma_{f(x_i,k)})(f) = S^2(f(x_2,x_2), x_i, \hat{\sigma}_{f(x_2,x_2)}[k]) = f(\hat{\sigma}_{f(x_2,x_2)}[k], \hat{\sigma}_{f(x_2,x_2)}[k]) \neq f(x_1,x_1)$, which is a contradiction. So $(\sigma)_i \subsetneq I$. By the same way we can show that if $u \notin X$ and $v \in X$, then $(\sigma)_i \subsetneq I$. Suppose that $u, v \notin X$. Then $vb(f(u,v)) \geq 4$. Suppose that $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$. Since $f(x_1,x_1) \notin X$ and $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$, there exist $p, q, r, s \in W_{(2)}(X)$ such that $\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)} \circ_G \sigma_{f(r,s)} = \sigma_{f(x_1,x_1)}$. Let w be the term $(\sigma_{f(u,v)} \circ_G \sigma_{f(r,s)})(f)$. Then $vb(w) \geq 4$. By Lemma 3.3, we get $x_1 \in var(f(p,q))$ or $x_2 \in var(f(p,q))$. Suppose that $f(p,q) \in W_{(2)}^G(\{x_1,x_2\})$. If $f(p,q) = f(x_1,x_2)$ or $f(p,q) = f(x_2,x_1)$, then $\sigma_w = \sigma_{f(x_1,x_1)}$ or $\sigma_{w'} = \sigma_{f(x_1,x_1)}$ and it is a contradiction. Suppose that $f(p,q) \neq f(x_1,x_2), f(x_2,x_1)$. By Lemma 3.13, we get $vb(f(x_1,x_1)) > vb(w)$, which is a contradiction. Suppose that $f(p,q) \in W(\{x_1\}) \cup W(\{x_2\})$. Then the equation $\sigma_{f(p,q)} \circ_G \sigma_w = \sigma_{f(x_1,x_1)}$ does not fit any of E(1) to E(16), so by Lemma 3.1 we must have $vb(f(x_1,x_1)) > vb(f(p,q))$ and it is a contradiction. So $(\sigma)_i \subsetneq I$. Let $f(u,v) \in W^G$. Suppose that $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$. Since $f(x_1,x_1) \notin X$ and $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$, there exist $p, q, r, s \in W_{(2)}(X)$ such that $\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)} \circ_G \sigma_{f(r,s)} = \sigma_{f(x_1,x_1)}$. By Lemma 3.3, we get $\sigma_{f(u,v)} \circ_G$

$\sigma_{f(r,s)} = \sigma_{f(u,v)}$. By Lemma 3.9, we get x_1, x_2 are not variables occurring in the term $(\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)})(f) = (\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)} \circ_G \sigma_{f(r,s)})(f)$, which is a contradiction. So $(\sigma)_i \subsetneq I$.

(*iii*) Since $\mathcal{D} \subseteq \mathcal{J}$, we must have $E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$ contained in the $\mathcal{J}$-class of $\sigma_{f(x_1,x_1)}$. Assume that $\sigma \in Hyp_G(2)$ where $\sigma \mathcal{J} \sigma_{f(x_1,x_1)}$. Then $(\sigma)_i = (\sigma_{f(x_1,x_1)})_i = I$. So $\sigma \in I$. By (ii), we get $\sigma \in E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$.

Proposition 3.19. *For any $\sigma_t \in E^G(\{x_1, x_2\})$, the elements which are $\mathcal{L}$-related to σ_t are only σ_t itself and $\sigma_{t'}$.*

Proof. Let $t = f(u, v)$ where $u, v \in W_{(2)}(X)$. Assume that $\sigma_s \in Hyp_G(2)$ where $\sigma_s \mathcal{L} \sigma_t$. By Proposition 3.5, we get $s \notin X$. Then $s = f(a, b)$ for some $a, b \in W_{(2)}(X)$. Since $s, t \notin X$ and $\sigma_s \mathcal{L} \sigma_t$, there exist $c, d, e, g \in W_{(2)}(X)$ such that $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(a,b)}$ and $\sigma_{f(e,g)} \circ_G \sigma_{f(a,b)} = \sigma_{f(u,v)}$. Suppose that $f(c, d), f(e, g) \notin \{f(x_1, x_2), f(x_2, x_1)\}$ and $f(c, d), f(e, g) \in W^G_{(2)}(\{x_1, x_2\})$. Then by Lemma 3.13, we get $vb(f(a, b)) > vb(f(u, v))$ and $vb(f(u, v)) > vb(f(a, b))$, which is a contradiction. Suppose that $f(c, d) \in W_{(2)}(X) \backslash W^G_{(2)}(\{x_1, x_2\})$. Then by Lemma 3.14, we get $x_1 \notin var(f(a, b))$ or $x_2 \notin var(f(a, b))$. Since $x_1 \notin var(f(a, b))$ or $x_2 \notin var(f(a, b))$, by Lemma 3.9 we get $x_1 \notin var(f(u, v))$ or $x_2 \notin var(f(u, v))$ which contradicts to $x_1, x_2 \in var(f(u, v))$. Suppose that $f(e, g) \in W_{(2)}(X) \setminus W^G_{(2)}(\{x_1, x_2\})$. Then by Lemma 3.14, we get $x_1 \notin var(f(u, v))$ or $x_2 \notin var(f(u, v))$ which contradicts to $x_1, x_2 \in var(f(u, v))$. So $f(c, d) \in \{f(x_1, x_2), f(x_2, x_1)\}$ or $f(e, g) \in \{f(x_1, x_2), f(x_2, x_1)\}$. This implies $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{t'}$.

Corollary 3.20. *For $\sigma_t \in E^G(\{x_1, x_2\})$, $D_{\sigma_t} = \{\sigma_t, \sigma_{t'}, \sigma_{\overline{t}}, \sigma_{\overline{t'}}\}$.*

Proof. By Proposition 3.12 and Proposition 3.19.

Proposition 3.21. *For $\sigma_t \in E^G(\{x_1, x_2\})$, the $\mathcal{J}$-class of σ_t is equal to its $\mathcal{D}$-class, $\{\sigma_t, \sigma_{t'}, \sigma_{\overline{t}}, \sigma_{\overline{t'}}\}$.*

Proof. If $\sigma_t = \sigma_{id}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$, then by Proposition 3.8 we get $D_{\sigma_{id}} = J_{\sigma_{id}}$. Let $\sigma_t \neq \sigma_{id}, \sigma_{f(x_2,x_1)}$ and $\sigma_s \in Hyp_G(2)$ where $\sigma_s \mathcal{J} \sigma_t$. By Proposition 3.5, we get $s \notin X$. Then there exist $\sigma_u, \sigma_v, \sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$. This implies $\sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_t$. Since $t \notin X$, by Lemma 3.3 we get $u, v, p, q \notin X$. Since $t \in W^G_{(2)}(\{x_1, x_2\})$, by Lemma 3.9 and Lemma 3.14 we get $u, v, p, q \in W^G_{(2)}(\{x_1, x_2\})$ and terms corresponding to the intermediate products are in $W^G_{(2)}(\{x_1, x_2\})$. We consider three cases.

Case 1: $\sigma_p \circ_G \sigma_u = \sigma_{id}$. Then by Lemma 3.6, we get $\sigma_p = \sigma_u = \sigma_{id}$ or $\sigma_p = \sigma_u = \sigma_{f(x_2,x_1)}$. If $\sigma_p = \sigma_u = \sigma_{id}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_s \circ_G \sigma_q = \sigma_t$. So $\sigma_s \mathcal{R} \sigma_t$. By

Proposition 3.12, we get $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{\bar{t}}$. If $\sigma_p = \sigma_u = \sigma_{f(x_2,x_1)}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_{t'} \circ_G \sigma_v = \sigma_s$ and $\sigma_s \circ_G \sigma_q = \sigma_{t'}$. So $\sigma_s \mathcal{R} \sigma_{t'}$. By Proposition 3.12, we get $\sigma_s = \sigma_{t'}$ or $\sigma_s = \sigma_{\overline{t'}}$.

Case 2: $\sigma_p \circ_G \sigma_u = \sigma_{f(x_2,x_1)}$. Then by Lemma 3.6, we get $\sigma_p = \sigma_{id}, \sigma_u = \sigma_{f(x_2,x_1)}$ or $\sigma_p = \sigma_{f(x_2,x_1)}, \sigma_u = \sigma_{id}$. Then $\sigma_t = \sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_{f(x_2,x_1)} \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_{t'} \circ_G (\sigma_v \circ_G \sigma_q)$. By Lemma 3.1, we get $vb(t) > vb(t')$, unless the product $\sigma_{t'} \circ_G (\sigma_v \circ_G \sigma_q)$ fits one of $E(1)$ to $E(16)$. But $vb(t) = vb(t')$, thus the product $\sigma_{t'} \circ_G (\sigma_v \circ_G \sigma_q)$ fits one of $E(1)$ to $E(16)$. We see that the cases $E(1) - E(3)$, $E(5)$, $E(7) - E(16)$ are impossible. Assume that $E(4)$ holds. We have $\sigma_v \circ_G \sigma_q = \sigma_{id}$. By Lemma 3.6, we get $\sigma_v = \sigma_q = \sigma_{id}$ or $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$. If $\sigma_v = \sigma_q = \sigma_{id}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_t$. So $\sigma_s \mathcal{L} \sigma_t$. By Proposition 3.19, we get $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{t'}$. If $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_{f(x_2,x_1)} = \sigma_t$. This implies $\sigma_u \circ_G \sigma_{\bar{t}} = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_{\bar{t}}$. So $\sigma_s \mathcal{L} \sigma_{\bar{t}}$. By Proposition 3.19, we get $\sigma_s = \sigma_{\bar{t}}$ or $\sigma_s = \sigma_{\bar{t}'} = \sigma_{\overline{t'}}$. Assume that $E(6)$ holds. We have $\sigma_v \circ_G \sigma_q = \sigma_{f(x_2,x_1)}$. By Lemma 3.6, we get $\sigma_q = \sigma_{id}$ or $\sigma_q = \sigma_{f(x_2,x_1)}$. If $\sigma_p = \sigma_q = \sigma_{f(x_1,x_2)}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_t$. If $\sigma_p = \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_{\overline{t'}}$. If $\sigma_p = \sigma_{id}, \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_{\bar{t}}$. If $\sigma_p = \sigma_{f(x_2,x_1)}, \sigma_q = \sigma_{id}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_{t'}$.

Case 3: $\sigma_p \circ_G \sigma_u \neq \sigma_{id}, \sigma_{f(x_2,x_1)}$. Let $w = (\sigma_t \circ_G \sigma_v \circ_G \sigma_q)(f)$. By Lemma 3.13, we get $vb(t) > vb(w)$. By Lemma 3.1, we get $vb(w) > vb(t)$, unless the product $\sigma_t \circ_G (\sigma_v \circ_G \sigma_q)$ fits one of $E(1)$ to $E(16)$. But the case $vb(w) > vb(t)$ is impossible. We see that the cases $E(1) - E(3)$, $E(5)$, $E(7) - E(16)$ are impossible. Assume that $E(4)$ holds. We must have $\sigma_v \circ_G \sigma_q = \sigma_{id}$. By Lemma 3.6, we get $\sigma_v = \sigma_q = \sigma_{id}$ or $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$. If $\sigma_v = \sigma_q = \sigma_{id}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_t$. So $\sigma_s \mathcal{L} \sigma_t$. By Proposition 3.19, we get $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{t'}$. If $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_{f(x_2,x_1)} = \sigma_t$. This implies $\sigma_u \circ_G \sigma_{\bar{t}} = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_{\bar{t}}$. So $\sigma_s \mathcal{L} \sigma_{\bar{t}}$. By Proposition 3.19, we get $\sigma_s = \sigma_{\bar{t}}$ or $\sigma_s = \sigma_{\bar{t}'} = \sigma_{\overline{t'}}$. Assume that $E(6)$ holds. We must have $\sigma_v \circ_G \sigma_q = \sigma_{f(x_2,x_1)}$. Then $\sigma_t = \sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_{f(x_2,x_1)} = (\sigma_p \circ_G \sigma_u) \circ_G \sigma_{\bar{t}}$. Since $\sigma_p \circ_G \sigma_u \neq \sigma_{id}, \sigma_{f(x_2,x_1)}$, by Lemma 3.13 we get $vb(t) > vb(\bar{t})$ and it is a contradiction.

Proposition 3.22. *Let $t \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t)$ or $x_2 \in var(t)$. Then the following statements are equivalent:*

(i) σ_t *has an* $\mathcal{H}$*-class of size two,*

(ii) $t' = \bar{t}$,

(iii) $t = f(u, v)$ *for some* $u, v \in W_{(2)}(X)$ *with* $v = \overline{u'}$.

Proof. $(i) \Longrightarrow (ii)$ Assume that (i) holds. By Proposition 3.12, we get $R_{\sigma_t} = \{\sigma_t, \sigma_{\overline{t}}\}$. Since $H_{\sigma_t} \subseteq R_{\sigma_t}$ and $|H_{\sigma_t}| = 2$, thus $H_{\sigma_t} = \{\sigma_t, \sigma_{\overline{t}}\}$. So $\sigma_t \mathcal{L} \sigma_{\overline{t}}$. By Proposition 3.4, we get $\sigma_t \mathcal{L} \sigma_{t'}$. So $\sigma_{\overline{t}} \mathcal{L} \sigma_{t'}$. If $t \in W^G_{(2)}(\{x_1, x_2\})$, then by Proposition 3.19, we get $t' = \overline{t}$. If $t \in W(\{x_1\})$, then by Lemma 3.9, we get x_2 is not a variable occurring in the term $(\sigma \circ_G \sigma_t)(f)$ for all $\sigma \in Hyp_G(2)$. So $\sigma \circ_G \sigma_t \neq \sigma_{\overline{t}}$ for all $\sigma \in Hyp_G(2)$. Thus it is impossible that $\sigma_{\overline{t}}$ is $\mathcal{L}$-related to σ_t. By the same way we can show that if $t \in W(\{x_2\})$, then σ_t and $\sigma_{\overline{t}}$ are not related.

$(ii) \Longrightarrow (i)$ Assume that $t' = \overline{t}$. By Proposition 3.4, we get $\sigma_t \mathcal{L} \sigma_{\overline{t}}$. So $R_{\sigma_t} = \{\sigma_t, \sigma_{\overline{t}}\} \subseteq L_{\sigma_t}$. Thus $H_{\sigma_t} = L_{\sigma_t} \cap R_{\sigma_t} = R_{\sigma_t} = \{\sigma_t, \sigma_{\overline{t}}\}$. So $|H_{\sigma_t}| = 2$.

$(ii) \Longrightarrow (iii)$ Assume that $t = f(u, v)$ for some $u, v \in W_{(2)}(X)$ with $t' = \overline{t}$. So $\overline{f(u, v)} = f(u, v)'$

$$\begin{aligned} &\Rightarrow \quad f(\overline{u}, \overline{v}) = f(v', u') \\ &\Rightarrow \quad \overline{u} = v' \\ &\Rightarrow \quad v = (v')' = \overline{u}' = \overline{u'}. \end{aligned}$$

$(iii) \Longrightarrow (ii)$ Assume that $t = f(u, v)$ for some $u, v \in W_{(2)}(X)$ with $v = \overline{u'}$. So $t' = f(u, v)' = f(u, \overline{u'})' = f(\overline{u'}', u') = f(\overline{u}, u') = \overline{\overline{f(\overline{u}, u')}} = \overline{f(\overline{\overline{u}}, \overline{u'})} = \overline{f(u, v)} = \overline{t}$.

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