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An iterative method for finding common solutions of system of equilibrium problems and fixed point problems in Hilbert spaces

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Abstract

In this paper, we introduce an iterative algorithm for finding a common element of the set of solutions of a system of equilibrium problems and of the set of fixed points of a nonexpansive mapping in a Hilbert space. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem. Our results extend and generalize related work.

1 Introduction

Let C be a closed convex subset of a Hilbert space H. Then, a mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T.

Let C be a nonempty closed convex subset of H, Γ be an arbitrary index set, and $\{F_k\}_{k\in\Gamma}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$. Combettes and Hirstoaga [5] considered the following system of equilibrium problems:

$$\text{Finding} \quad x \in C \quad \text{such that} \quad F_k(x, y) \geq 0, \quad \forall k \in \Gamma, \quad \forall y \in C. \tag{1.1}$$

The formulation (1.1) covers, as special cases, monotone inclusion problems, saddle point problems, minimization problems, optimization problems, variational inequality problems, Nash equilibria in noncooperative games and various forms of feasibility problems (see [5, 2, 6] and the references therein). If Γ

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is a singleton, then problem (1.1) becomes the following equilibrium problem:

$$\text{Finding} \quad x \in C \quad \text{such that} \quad F(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$. It is shown [5] that under suitable hypotheses on F (to be stated precisely in Section 2), the mapping $T_r^F : H \to C$ defined by

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C \right\}$$

is single-valued and firmly nonexpansive and satisfies $F(T_r^F) = EP(F)$.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings with $\cap_{i=1}^N F(T_i) \neq \emptyset$, Atsushiba and Takahashi [1], and Yao [14] defined the mapping W_n as follows:

$$\begin{aligned} &U_{n,1} = \lambda_{n,1}T_1 + (1-\lambda_{n,1})I,\\ &U_{n,2} = \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})I,\\ &\vdots\\ &U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})I,\\ &W_n := U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})I. \end{aligned} \tag{1.3}$$

Recently, the mapping W_n has been intensively studied and applied to develop various iterative algorithms for finding common solutions of fixed points of a finite family of nonexpansive mappings and of other problems (see [1, 4, 10, 14, 15]). Since, for $\{F_k\}_{k=1}^N$ satisfying suitable hypotheses, mappings $\{T_{r_{k,n}}^{F_k}\}_{k=1}^N$, $n > 0$, are nonexpansive, inspiring by Atsushiba and Takahashi [1], and Yao [14], we define the new mapping W_n as follows:

$$\begin{aligned} &U_{n,1} = \lambda_{n,1}T_{r_{1,n}}^{F_1} + (1-\lambda_{n,1})I,\\ &U_{n,2} = \lambda_{n,2}T_{r_{2,n}}^{F_2}U_{n,1} + (1-\lambda_{n,2})I,\\ &\vdots\\ &U_{n,N-1} = \lambda_{n,N-1}T_{r_{N-1,n}}^{F_{N-1}}U_{n,N-2} + (1-\lambda_{n,N-1})I,\\ &W_n := U_{n,N} = \lambda_{n,N}T_{r_{N,n}}^{F_N}U_{n,N-1} + (1-\lambda_{n,N})I. \end{aligned} \tag{1.4}$$

Such a mapping W_n is called the W_n-mapping generated by $\{T_{r_{k,n}}^{F_k}\}_{k=1}^N$ and $\{\lambda_{n,k}\}_{k=1}^N$. Nonexpansivity of $T_{r_{k,n}}^{F_k}$ yields the nonexpansivity of W_n.

For system of equilibrium problems $\{F_k\}_{k=1}^N$, Saeidi [10] proposed the following scheme with respect to a semigroup $\{T(t), t \in S\}$ of nonexpansive mappings and a finite family of nonexpansive mappings $\{T_i\}_{i=1}^M$:

$$x_{n+1} = \epsilon_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)T(\mu_n)W_nT_{r_{N,n}}^{F_N}\ldots T_{r_{2,n}}^{F_2}T_{r_{1,n}}^{F_1}x_n \tag{1.5}$$

where $T(\mu_n) \in \{T(t), t \in S\}$ and W_n is generated by $T_1, \dots, T_M$ and $\lambda_{n,1}, \dots, \lambda_{n,M}$. He proved that under some hypotheses, both sequences $\{x_n\}$ and $\{T^{F_k}_{r_{k,n}}\}_{k=1}^N$ converge strongly to a point $x \in F = F(\{T(t), t \in S\}) \cap (\cap_{i=1}^M F(T_i)) \cap (\cap_{i=1}^N EP(F_i))$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad \forall x \in F.$$

Very recently, Peng and Yao [9] introduced the following scheme for system of equilibrium problems $\{F_k\}_{k=1}^N$, the monotone mapping A and an infinite family of nonexpansive mappings $\{S_i\}_{i=1}^\infty$:

$$\begin{cases} x_1 = x \in C \\ u_n = T^{F_N}_{r_{N,n}} T^{F_{N-1}}_{r_{N-1,n}} \dots T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}} x_n \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n f(S_n x_n) + \beta_n x_n + \gamma_n S_n P_C(u_n - \lambda_n A y_n), \end{cases}$$

and showed that sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to the same point $w \in \Omega = (\cap_{i=1}^\infty S_i) \cap VI(C, A) \cap (\cap_{k=1}^N EP(F_k))$ where $w = P_\Omega f(w)$.

In this paper, motivated by Yao [14], Saeidi [10] and Peng and Yao [9], we introduce the following iterative algorithm for finding a common element of the set of solutions of a system of equilibrium problems $\{F_k\}_{k=1}^N$ and of the set of fixed points of a nonexpansive mapping S:

$$x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1 - \beta)I - \epsilon_n A) S W_n x_n$$

where W_n is defined by (1.4). We prove that under certain appropriate assumptions on parameters, the sequences $\{x_n\}$ and $\{W_n x_n\}$ converge strongly to $x \in \Omega = F(S) \cap (\cap_{k=1}^M EP(F_k))$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad \forall x \in \Omega. \tag{1.6}$$

We extend and generalize results of Saeidi [10] and Peng and Yao [9] from $T^{F_N}_{r_{N,n}} T^{F_{N-1}}_{r_{N-1,n}} \dots T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}}$ to general W_n mapping defined by (1.4).

2 Preliminaries

Let C be a closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1. (*[12]*) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2. (*[11]*) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n$ and $\limsup_{n\to\infty} \beta_n < 1$. Suppose

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

for all integers $n \geq 0$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.3. (*[13]*) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \beta_n, \quad n \geq 0$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\beta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = +\infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n \beta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. (*[8]*) Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.5. (*[5]*) Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ satisfy following conditions:

(A1) $F(x, x) = 0$, $\forall x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;

(A3) $\limsup_{t\to 0^+} F(tz + (1 - t)x, y) \leq F(x, y)$, $\forall x, y, z \in C$;

(A4) for each $x \in C$, $F(x, \cdot)$ is convex and lower semicontinuous.

For $x \in C$ and $r > 0$, set $T_r^F : H \to C$ to be

$$T_r^F(x) = \Big\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \Big\}.$$

Then T_r^F is well defined and the following hold:

1. T_r^F is single-valued;
2. T_r^F is firmly nonexpansive [7], i.e., for any $x, y \in E$,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

3. $F(T_r^F) = EP(F)$;
4. $EP(F)$ is closed and convex.

By the proof of Lemma 5 in [3], we have following lemma.

Lemma 2.6. *Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction. Let $x \in C$ and $r_1, r_2 \in (0, \infty)$. Then*

$$\|T_{r_1}^F x - T_{r_2}^F x\| \leq \left|1 - \frac{r_2}{r_1}\right| (\|T_{r_1}^F x\| + \|x\|). \tag{2.1}$$

From the definition 2.6 given by Colao, Marino and Xu [4], we can introduce following definition.

Definition 2.7. *Let C be a nonempty convex subset of a Banach space. Let F_i, $i \in \{1, 2, \ldots, N\}$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and $\lambda_1, \cdots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for $i = 1, 2, \cdots, N$. We define a mapping W of C into itself as follows:*

$$\begin{aligned} &U_1 = \lambda_1 T_{r_1}^{F_1} + (1 - \lambda_1)I, \\ &U_2 = \lambda_{n,2} T_{r_2}^{F_2} U_{n,1} + (1 - \lambda_{n,2})I, \\ &\vdots \\ &U_{N-1} = \lambda_{N-1} T_{r_{N-1}}^{F_{N-1}} U_{M-2} + (1 - \lambda_{N-1})I, \\ &W := U_N = \lambda_N T_{r_N}^{F_N} U_{N-1} + (1 - \lambda_N)I, \end{aligned} \tag{2.2}$$

Such a mapping W is called the W-mapping generated by $T_{r_1}^{F_1}, \ldots, T_{r_N}^{F_N}$ and $\lambda_1, \ldots, \lambda_N$.

Following the proof presented by Atsushiba and Takahashi [1] and using $F(T_{r_i}^{F_i}) = EP(F_i)$, $i \in \{1, \ldots, N\}$, we have following lemma.

Lemma 2.8. *Let C be a nonempty closed convex set of a strictly convex Banach space and F_i, $i \in \{1, 2, \dots, N\}$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and $\cap_{i=1}^N EP(F_i) \neq \emptyset$. Let $\lambda_1, \dots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for $i = 1, \dots, N-1$ and $0 < \lambda_N \leq 1$. Let W be the W-mapping of C into itself generated by $T_{r_1}^{F_1}, \dots, T_{r_N}^{F_N}$ and $\lambda_1, \dots, \lambda_N$. Then $F(W) = \cap_{i=1}^N EP(F_i)$.*

Lemma 2.9. *Let C be a nonempty convex set of a Banach space and F_i, $i \in \{1, 2, \dots, N\}$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $\{\lambda_{n,i}\}_{i=1}^N$ be sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$ and $\{r_{i,n}\}$ be sequences in $(0,\infty)$ such that $r_{i,n} \to r_i$, $r_i \in (0,\infty)$ $(i = 1, \dots, N)$. Moreover for every $n \in \mathbb{N}$, let W be the W-mappings generated by $T_{r_1}^{F_1}, \dots, T_{r_N}^{F_N}$ and $\lambda_1, \dots, \lambda_N$ and W_n be the W_n-mappings generated by $T_{r_{1,n}}^{F_1}, \dots, T_{r_{N,n}}^{F_N}$ and $\lambda_{n,1}, \dots, \lambda_{n,N}$. Then for every $x \in C$, it follows that*

$$\lim_{n\to\infty} \|W_n x - Wx\| = 0. \tag{2.3}$$

Proof. Let $x \in C$. U_k and $U_{n,k}$ be generated by $T_{r_1}^{F_1}, \dots, T_{r_N}^{F_N}$ and $\lambda_1, \dots, \lambda_N$ and $T_{r_{1,n}}^{F_1}, \dots, T_{r_{N,n}}^{F_N}$ and $\lambda_{n,1}, \dots, \lambda_{n,N}$ respectively, as in Definition 2.7. From Lemma 2.6, we have

$$\begin{aligned}
\|U_{n,1}x - U_1 x\| &= \|\lambda_{n,1} T_{r_{1,n}}^{F_1} x + (1-\lambda_{n,1})x - \lambda_1 T_{r_1}^{F_1} x - (1-\lambda_1)x\| \\
&= \|\lambda_{n,1}(T_{r_{1,n}}^{F_1} x - T_{r_1}^{F_1} x) + (\lambda_{n,1} - \lambda_1)(T_{r_1}^{F_1} x - x)\| \\
&\leq \lambda_{n,1}\|T_{r_{1,n}}^{F_1} x - T_{r_1}^{F_1} x\| + |\lambda_{n,1} - \lambda_1|\|T_{r_1}^{F_1} x - x\| \\
&\leq \left|1 - \frac{r_{1,n}}{r_1}\right| (\|T_{r_1}^{F_1} x\| + \|x\|) + |\lambda_{n,1} - \lambda_1|(\|T_{r_1}^{F_1} x\| + \|x\|) \\
&\leq \left(\left|1 - \frac{r_{1,n}}{r_1}\right| + |\lambda_{n,1} - \lambda_1|\right)(\|T_{r_1}^{F_1} x\| + \|x\|).
\end{aligned}$$

Similarly, we get, for $k \in \{2, ..., N\}$,

$$\begin{aligned}
\|U_{n,k}x - U_k x\| &= \|\lambda_{n,k} T_{r_{k,n}}^{F_k} U_{n,k-1}x + (1-\lambda_{n,k})x - \lambda_k T_{r_k}^{F_k} U_{k-1}x - (1-\lambda_k)x\| \\
&= \|\lambda_{n,k}(T_{r_{k,n}}^{F_k} U_{n,k-1}x - T_{r_{k,n}}^{F_k} U_{k-1}x) + \lambda_{n,k}(T_{r_{k,n}}^{F_k} U_{k-1}x - T_{r_k}^{F_k} U_{k-1}x) \\
&\quad + (\lambda_{n,k} - \lambda_k)(T_{r_k}^{F_k} U_{k-1}x - x)\| \\
&\leq \lambda_{n,k}\|T_{r_{k,n}}^{F_k} U_{n,k-1}x - T_{r_{k,n}}^{F_k} U_{k-1}x\| + \lambda_{n,k}\|T_{r_{k,n}}^{F_k} U_{k-1}x - T_{r_k}^{F_k} U_{k-1}x\| \\
&\quad + |\lambda_{n,k} - \lambda_k|(\|T_{r_k}^{F_k} U_{k-1}x\| + \|x\|) \\
&\leq \|U_{n,k-1}x - U_{k-1}x\| + \left|1 - \frac{r_{k,n}}{r_k}\right| (\|T_{r_k}^{F_k} U_{k-1}x\| + \|U_{k-1}x\|) \\
&\quad + |\lambda_{n,k} - \lambda_k|(\|T_{r_k}^{F_k} U_{k-1}x\| + \|x\|).
\end{aligned}$$

Hence,

$$\begin{aligned}\|W_nx - Wx\| &= \|U_{n,N}x - U_Nx\| \\ &\leq \sum_{k=2}^{N}\left(\left|1-\frac{r_{k,n}}{r_k}\right|(\|T_{r_k}^{F_k}U_{k-1}x\| + \|U_{k-1}x\|) + |\lambda_{n,k}-\lambda_k|(\|T_{r_k}^{F_k}U_{k-1}x\| + \|x\|)\right) \\ &\quad + \left(\left|1-\frac{r_{1,n}}{r_1}\right| + |\lambda_{n,1}-\lambda_1|\right)(\|T_{r_1}^{F_1}x\| + \|x\|).\end{aligned}$$

Since for every $k \in \{1,\dots,N\}$, $\lim_{n\to\infty}|\lambda_{n,k}-\lambda_k| = 0$ and $\lim_{n\to\infty}|r_{k,n} - r_k| = 0$, the result follows. □

Lemma 2.10. *For all $x, y \in H$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

3 Main result

Theorem 3.1. *Let C be nonempty closed convex subset of a Hilbert space H. Let S be a nonexpansive mapping from H into itself, and F_i, $i \in \{1, 2, \dots, N\}$ be bifunctions from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and $F(S)\cap(\cap_{k=1}^{N}EP(F_k)) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and f be an α-contraction on H for some $0 < \alpha < 1$. Moreover, let $\{\epsilon_n\}$ be a sequence in $(0, 1)$, $\{\lambda_{n,i}\}_{i=1}^{N}$ a sequence in $[a, b]$ with $0 < a \leq b < 1$, $\{r_n\}$ a sequence in $(0,\infty)$ and γ and β two real numbers such that $0 < \beta < 1$ and $0 < \gamma < \bar{\gamma}/\alpha$. Assume*

(B1) $\lim_n \epsilon_n = 0$;

(B2) $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(C1) $\liminf_n r_{j,n} > 0$, *for every* $j \in \{1,\dots,N\}$;

(C2) $\lim_n r_{j,n+1}/r_{j,n} = 1$, *for every* $j \in \{1,\dots,N\}$;

(D1) $\lim_n |\lambda_{n,j} - \lambda_{n-1,j}| = 0$, *for every* $j \in \{1,\dots,N\}$.

If $\{x_n\}$ is the sequence generated by $x_1 \in H$ and $\forall n \geq 1$,

$$x_{n+1} = \epsilon_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n A)SW_nx_n, \tag{3.1}$$

then $\{x_n\}$ and $\{W_nx_n\}$ converge strongly to $x^* \in \Omega = F(S)\cap(\cap_{k=1}^{N}EP(F_k))$ which is the unique solution of the variational inequality (1.6). Equivalently, we have $P_\Omega(I - A + \gamma f)x^* = x^*$.

Proof. Since A is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$, $\frac{A}{1-\beta}$ is a strongly positive bounded linear operator with coefficient $\frac{\bar{\gamma}}{1-\beta}$. By $\epsilon_n \to 0$, we may assume, with no loss of generality, that $\epsilon_n \le (1-\beta)\|A\|^{-1}$. From Lemma 2.4, we know that

$$\|(1-\beta)I - \epsilon_n A\| = (1-\beta)\|I - \frac{\epsilon_n A}{1-\beta}\| \le (1-\beta)(1 - \frac{\epsilon_n \bar{\gamma}}{1-\beta}) = 1 - \beta - \epsilon_n \bar{\gamma}.$$

We shall divide the proof into several steps.
Step 1. The sequence $\{x_n\}$ is bounded.
Proof of Step 1. Put $p \in \Omega$. Then noting (1.4), nonexpansivity of W_n and $p = T_{r_{i,n}}^{F_i} p$, $i \in \{1, \ldots, N\}$, we derive that

$$\|W_n x_n - p\| \le \|x_n - p\|.$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\epsilon_n(\gamma f(x_n) - Ap) + \beta(x_n - p) + ((1-\beta)I - \epsilon_n A)(SW_n x_n - p)\| \\ &\le (1 - \epsilon_n(\bar{\gamma} - \alpha\gamma))\|x_n - p\| + \epsilon_n(\bar{\gamma} - \alpha\gamma)\frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \alpha\gamma},\end{aligned}$$

which implies that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \alpha\gamma}\right\}, \quad \forall n \ge 1.$$

Step 2. Let $\{w_n\}$ be a bounded sequence in H. Then

$$\lim_{n\to\infty} \|W_{n+1} w_n - W_n w_n\| = 0. \tag{3.2}$$

Proof of Step 2. Let $j \in \{0, \ldots, N-2\}$ and set

$$M := \sup_{n\in\mathbb{N}} \left\{ \|w_n\| + \|T_{r_{1,n}}^{F_1} w_n\| + \sum_{j=2}^{N} (\|T_{r_{j,n}}^{F_j} U_{n,j-1} w_n\| + \|U_{n,j-1} w_n\|) \right\} < \infty.$$

It follows from (1.4) and Lemma 2.6 that

$$\begin{aligned}
\|U_{n+1,N-j}w_n - U_{n,N-j}w_n\| &= \|\lambda_{n+1,N-j}T^{F_{N-j}}_{r_{N-j,n+1}}U_{n+1,N-j-1}w_n + (1-\lambda_{n+1,N-j})w_n \\
&\quad - \lambda_{n,N-j}T^{F_{N-j}}_{r_{N-j,n}}U_{n,N-j-1}w_n - (1-\lambda_{n,N-j})w_n\| \\
&\le \lambda_{n+1,N-j}\|T^{F_{N-j}}_{r_{N-j,n+1}}U_{n+1,N-j-1}w_n - T^{F_{N-j}}_{r_{N-j,n+1}}U_{n,N-j-1}w_n\| \\
&\quad + \lambda_{n+1,N-j}\|T^{F_{N-j}}_{r_{N-j,n+1}}U_{n,N-j-1}w_n - T^{F_{N-j}}_{r_{N-j,n}}U_{n,N-j-1}w_n\| \\
&\quad + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\|T^{F_{N-j}}_{r_{N-j,n}}U_{n,N-j-1}w_n - w_n\| \\
&\le \|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| + \left|1 - \frac{r_{N-j,n+1}}{r_{N-j,n}}\right|(\|T^{F_{N-j}}_{r_{N-j,n}}U_{n,N-j-1}w_n\| \\
&\quad + \|U_{n,N-j-1}w_n\|) + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|(\|T^{F_{N-j}}_{r_{N-j,n}}U_{n,N-j-1}w_n\| + \|w_n\|) \\
&\le \|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| + M\left(\left|1 - \frac{r_{N-j,n+1}}{r_{N-j,n}}\right| + |\lambda_{n+1,N-j} - \lambda_{n,N-j}|\right).
\end{aligned}$$

Thus, repeatedly using the above recursive inequalities, we deduce

$$\begin{aligned}
\|W_{n+1}w_n - W_nw_n\| &= \|U_{n+1,N}w_n - U_{n,N}w_n\| \\
&\le M\sum_{j=2}^{N}\left(\left|1 - \frac{r_{j,n+1}}{r_{j,n}}\right| + |\lambda_{n+1,j} - \lambda_{n,j}|\right) \\
&\quad + \left(\left|1 - \frac{r_{1,n+1}}{r_{1,n}}\right| + |\lambda_{n+1,1} - \lambda_{n,1}|\right)(\|T^{F_1}_{r_{1,n}}w_n\| + \|x_n\|) \\
&\le M\sum_{j=1}^{N}\left(\left|1 - \frac{r_{j,n+1}}{r_{j,n}}\right| + |\lambda_{n+1,j} - \lambda_{n,j}|\right).
\end{aligned} \tag{3.3}$$

Now by condition (C2), (D1) and using (3.3), we obtain (3.2) and Step 2 is proven.

Step 3. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Proof of Step 3. Rewrite the iterative process (3.1) as follows:

$$\begin{aligned}
x_{n+1} &= \epsilon_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_nA)SW_nx_n \\
&= \beta x_n + (1-\beta)\frac{\epsilon_n\gamma f(x_n) + ((1-\beta)I - \epsilon_nA)SW_nx_n}{1-\beta} \\
&= \beta x_n + (1-\beta)v_n,
\end{aligned}$$

where

$$v_n = \frac{\epsilon_n\gamma f(x_n) + ((1-\beta)I - \epsilon_nA)SW_nx_n}{1-\beta}. \tag{3.4}$$

Since $\{x_n\}$ is bounded, we have, for some big enough constant $M > 0$,

$$\begin{aligned}\|v_{n+1} - v_n\| &= \bigg\| \frac{\epsilon_{n+1}\gamma f(x_{n+1}) - \gamma\epsilon_n f(x_n)}{1-\beta} + (SW_{n+1}x_{n+1} - SW_n x_n) \\ &\quad - \frac{\epsilon_{n+1}ASW_{n+1}x_{n+1} - \epsilon_n ASW_n x_n}{1-\beta} \bigg\| \\ &\leq \frac{\gamma}{1-\beta}\left(\epsilon_{n+1}\|f(x_{n+1})\| + \epsilon_n\|f(x_n)\|\right) + \|W_{n+1}x_{n+1} - W_n x_n\| \\ &\quad + \frac{1}{1-\beta}\left(\epsilon_{n+1}\|ASW_{n+1}x_{n+1}\| + \epsilon_n\|ASW_n x_n\|\right) \\ &\leq \|W_{n+1}x_{n+1} - W_n x_{n+1}\| + \|W_n x_{n+1} - W_n x_n\| + M(\epsilon_{n+1} + \epsilon_n) \\ &\leq \|W_{n+1}x_{n+1} - W_n x_{n+1}\| + \|x_{n+1} - x_n\| + M(\epsilon_{n+1} + \epsilon_n).\end{aligned} \tag{3.5}$$

By conditions on $\{\epsilon_n\}$, and Steps 2, we immediately conclude from (3.5)

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|) \leq \limsup_{n\to\infty}(\|W_{n+1}x_{n+1}-W_n x_{n+1}\|+M(\epsilon_{n+1}+\epsilon_n)) = 0.$$

By Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_n - v_n\| = 0,$$

which implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta)\|x_n - v_n\| = 0.$$

Step 4. $\lim_{n\to\infty}\|x_n - SW_n x_n\| = 0$.
Proof of Step 4. We have

$$\begin{aligned}\|x_n - SW_n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - SW_n x_n\| \\ &= \|x_n - x_{n+1}\| + \|\epsilon_n(\gamma f(x_n) - ASW_n x_n) + \beta(x_n - SW_n x_n)\| \\ &\leq \|x_n - x_{n+1}\| + \epsilon_n\|\gamma f(x_n) - ASW_n x_n\| + \beta\|x_n - SW_n x_n\|.\end{aligned}$$

It follows from Step 3 that

$$\|x_n - SW_n x_n\| \leq \frac{1}{1-\beta}(\|x_n - x_{n+1}\| + \epsilon_n\|\gamma f(x_n) - ASW_n x_n\|) \to 0.$$

Step 5. $\lim_{n\to\infty}\|T_{r_{k,n}}^{F_k}U_{n,k-1}x_n - U_{n,k-1}x_n\| = 0$, $\lim_{n\to\infty}\|T_{r_{k,n}}^{F_k}U_{n,k-1}x_n - x_n\| = 0,\ k \in \{1,\dots,N\}$.

Proof of Step 5. Set $U_{n,0} = I$, then $U_{n,1} = \lambda_{n,1} T^{F_1}_{r_{1,n}} U_{n,0} + (1 - \lambda_{n,1}) I$. Take $v \in \Omega$, then we have, for $k \in \{0, 1, \dots, N-1\}$,

$$\begin{aligned}\|v - U_{n,k+1}x_n\|^2 &= \|\lambda_{n,k+1} T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} v + (1 - \lambda_{n,k+1}) v - \lambda_{n,k+1} T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n \\ &\quad - (1 - \lambda_{n,k+1}) x_n\|^2 \\ &\leq \lambda_{n,k+1} \|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} v - T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n\|^2 + (1 - \lambda_{n,k+1}) \|x_n - v\|^2.\end{aligned}$$

Since $T^{F_{k+1}}_{r_{k+1,n}}$ is firmly nonexpansive, we obtain

$$\begin{aligned}\|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} v - T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n\|^2 &\leq \langle T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - v, U_{n,k} x_n - v\rangle \\ &= \frac{1}{2}(\|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - v\|^2 + \|U_{n,k} x_n - v\|^2 - \|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - U_{n,k} x_n\|^2) \\ &\leq \|U_{n,k} x_n - v\|^2 - \|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - U_{n,k} x_n\|^2.\end{aligned}$$

It follows that

$$\|U_{n,k+1}x_n - v\|^2 \leq \lambda_{n,k+1} \|U_{n,k} x_n - v\|^2 + (1 - \lambda_{n,k+1}) \|x_n - v\|^2 - \lambda_{n,k+1} \|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - U_{n,k} x_n\|^2,$$

which implies

$$\begin{aligned}\|U_{n,N} x_n - v\|^2 &\leq \lambda_{n,N} \|U_{n,N-1} x_n - v\|^2 + (1 - \lambda_{n,N}) \|x_n - v\|^2 \\ &\leq \Pi^N_{i=k+2} \lambda_{n,i} \|U_{n,k+1} x_n - v\|^2 + (1 - \Pi^N_{i=k+2} \lambda_{n,i}) \|x_n - v\|^2 \\ &\leq \Pi^N_{i=k+1} \lambda_{n,i} \|U_{n,k} x_n - v\|^2 + (1 - \Pi^N_{i=k+1} \lambda_{n,i}) \|x_n - v\|^2 \\ &\quad - \Pi^N_{i=k+1} \lambda_{n,i} \|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - U_{n,k} x_n\|^2 \\ &\leq \|x_n - v\|^2 - a^{N-k} \|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - U_{n,k} x_n\|^2.\end{aligned} \tag{3.6}$$

Set $z_n = \gamma f(x_n) - ASW_n x_n$ and let $\lambda > 0$ be a constant such that

$$\lambda > \sup_{n,k} \{\|z_n\|, \|x_k - v\|\}.$$

Using Lemma 2.10 and noting that $\|\cdot\|^2$ is convex, we derive, using (3.6)

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \|(1-\beta)(SW_n x_n - v) + \beta(x_n - v) + \epsilon_n z_n\|^2 \\ &\leq \|(1-\beta)(SW_n x_n - v) + \beta(x_n - v)\|^2 + 2\epsilon_n \langle z_n, x_{n+1} - v\rangle \\ &\leq (1-\beta)\|U_{n,N} x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n \\ &\leq (1-\beta)(\|x - v\|^2 - a^{N-k} \|T^{F_{k+1}}_{r_{k+1,n}} U_{n,k} x_n - U_{n,k} x_n\|^2) \\ &\quad + \beta\|x_n - v\|^2 + 2\lambda^2 \epsilon_n \\ &= \|x_n - v\|^2 - (1-\beta) a^{N-k} \|T^{F_k}_{r_{k,n}} U_{n,k-1} x_n - U_{n,k-1} x_n\|^2 + 2\lambda^2 \epsilon_n.\end{aligned}$$

It follows, by Step 3 and condition (B1), that

$$\begin{aligned}\|T^{F_k}_{r_{k,n}}U_{n,k-1}x_n - U_{n,k-1}x_n\|^2 &\le \frac{1}{(1-\beta)a^{N-k}}(\|x_n - v\|^2 - \|x_{n+1} - v\|^2 + 2\lambda^2\epsilon_n)\\ &\le \frac{1}{(1-\beta)a^{N-k}}(2\lambda\|x_n - x_{n+1}\| + 2\lambda^2\epsilon_n) \to 0.\end{aligned} \tag{3.7}$$

So we have, from $U_{n,0} = I$,

$$\begin{aligned}\|T^{F_k}_{r_{k,n}}U_{n,k-1}x_n - x_n\| &\le \|T^{F_k}_{r_{k,n}}U_{n,k-1}x_n - U_{n,k-1}x_n\| + \|U_{n,k-1}x_n - x_n\|\\ &\le \|T^{F_k}_{r_{k,n}}U_{n,k-1}x_n - U_{n,k-1}x_n\| + \|\lambda_{n,k-1}T^{F_{k-1}}_{r_{k-1,n}}U_{n,k-2}x_n\\ &\quad + (1-\lambda_{n,k-1})x_n - x_n\|\\ &\le \|T^{F_k}_{r_{k,n}}U_{n,k-1}x_n - U_{n,k-1}x_n\| + \|T^{F_{k-1}}_{r_{k-1,n}}U_{n,k-2}x_n - x_n\|\\ &\le \cdots\\ &\le \|T^{F_1}_{r_{1,n}}U_{n,0}x_n - x_n\| + \sum_{i=2}^{k}\|T^{F_i}_{r_{i,n}}U_{n,i-1}x_n - U_{n,i-1}x_n\|\\ &= \sum_{i=1}^{k}\|T^{F_i}_{r_{i,n}}U_{n,i-1}x_n - U_{n,i-1}x_n\|.\end{aligned}$$

Combining with (3.7), we obtain

$$\lim_{n\to\infty}\|T^{F_k}_{r_{k,n}}U_{n,k-1}x_n - x_n\| = 0. \tag{3.8}$$

Thus we get the results.
Step 6. The weak ω-limit set of $\{x_n\}$, $\omega(x_n)$, is a subset of Ω.
Proof of Step 6. Let $z \in \omega(x_n)$ and $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z. From Step 5, it follows

$$T^{F_k}_{r_{k,n_m}}U_{n_m,k-1}x_{n_m} \rightharpoonup z, \quad \forall k \in \{1,\dots,N\}.$$

We need to show that $z \in \Omega$. At first, note that by (A2) and given $y \in C$, for $k \in \{1,2,\dots,N\}$, we have

$$\frac{1}{r_{k,n_m}}\langle y - T^{F_k}_{r_{k,n_m}}U_{n_m,k-1}x_{n_m}, T^{F_k}_{r_{k,n_m}}U_{n_m,k-1}x_{n_m} - U_{n_m,k-1}x_{n_m}\rangle \ge F_k(y, T^{F_k}_{r_{k,n_m}}U_{n_m,k-1}x_{n_m}).$$

Thus

$$\left\langle y - T^{F_k}_{r_{k,n_m}}U_{n_m,k-1}x_{n_m}, \frac{T^{F_k}_{r_{k,n_m}}U_{n_m,k-1}x_{n_m} - U_{n_m,k-1}x_{n_m}}{r_{k,n_m}}\right\rangle \ge F_k(y, T^{F_k}_{r_{k,n_m}}U_{n_m,k-1}x_{n_m}). \tag{3.9}$$

Step 5 and condition (C1) imply

$$\frac{T^{F_k}_{r_{k,n_m}} U_{n_m,k-1} x_{n_m} - U_{n_m,k-1} x_{n_m}}{r_{k,n_m}} \to 0,$$

in norm. By condition (A4), $F(y, \cdot)$ is lower semicontinuous and convex, and thus weakly semicontinuous. Therefore, letting $m \to \infty$ in (3.8) yields

$$F_k(y, z) \leq \lim_{m\to\infty} F_k(y, T^{F_k}_{r_{k,n_m}} U_{n_m,k-1} x_{n_m}) \leq 0,$$

for all $y \in C$ and $k \in \{1, \dots, N\}$. Replacing y with $y_t := ty + (1-t)z$ with $t \in (0, 1)$ and using (A1) and (A4), we obtain

$$0 = F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, z) \leq tF_k(y_t, y).$$

Hence $F_k(ty + (1-t)z, y) \geq 0$, for all $t \in (0, 1)$ and $y \in C$. Letting $t \to 0^+$ and using (A3), we conclude $F_k(z, y) \geq 0$, for all $y \in C$ and $k \in \{1, \dots, N\}$. Therefore

$$z \in \cap_{k=1}^N EP(F_k).$$

Next show $z \in F(S)$. By $z \in \bigcap_{k=1}^N EP(F_k)$, we have $z \in F(W_n)$, i.e., $z = W_n z$, $\forall n \geq 1$. Assume that $z \notin F(S)$, then $z \neq SW_n z$. Since Step 4, and using Opials property of a Hilbert space, we have

$$\begin{aligned}
\liminf_m \|x_{n_m} - z\| &< \liminf_m \|x_{n_m} - SW_{n_m} z\| \\
&\leq \liminf_m (\|x_{n_m} - SW_{n_m} x_n\| + \|SW_{n_m} x_{n_m} - SW_{n_m} z\|) \\
&\leq \liminf_m \|x_{n_m} - z\|.
\end{aligned}$$

This is a contradiction. Therefore, z must belong to $F(S)$.

Step 7. Let x^* be the unique solution of the variational inequality (1.6). That is,

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \Omega. \tag{3.10}$$

Then

$$\limsup_n \langle (\gamma f - A)x^*, x_n - x^* \rangle \leq 0, \quad x \in \Omega. \tag{3.11}$$

Proof of Step 7. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_k \langle (\gamma f - A)x^*, x_{n_k} - x^* \rangle = \limsup_n \langle (\gamma f - A)x^*, x_n - x^* \rangle. \tag{3.12}$$

Without loss of generality, we can assume that $\{x_{n_k}\}$ weakly converges to some z in C. By Step 6, $z \in \Omega$. Thus combining (3.12) and (3.10), we get

$$\limsup_n \langle (\gamma f - A)x^*, x_n - x^* \rangle = \langle (\gamma f - A)x^*, z - x^* \rangle \leq 0,$$

as required.
Step 8. The sequences $\{x_n\}$ and $\{W_n x_n\}$ converge strongly to x^*.
Proof of Step 8. By the definition (3.1) of $\{x_n\}$ and using Lemmas 2.4 and 2.10, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|[((1-\beta)I-\epsilon_n A)(SW_n x_n - x^*)+\beta(x_n-x^*)]+\epsilon_n(\gamma f(x_n)-Ax^*)\|^2\\ &\le \|((1-\beta)I-\epsilon_n A)(SW_n x_n-x^*)+\beta(x_n-x^*)\|^2\\ &\quad + 2\epsilon_n\langle \gamma f(x_n)-Ax^*, x_{n+1}-x^*\rangle\\ &= \|(1-\beta)\frac{(1-\beta)I-\epsilon_n A}{1-\beta}(SW_n x_n-x^*)+\beta(x_n-x^*)\|^2\\ &\quad + 2\epsilon_n\gamma\langle f(x_n)-f(x^*), x_{n+1}-x^*\rangle + 2\epsilon_n\langle \gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle\\ &\le \frac{\|(1-\beta)I-\epsilon_n A\|^2}{1-\beta}\|SW_n x_n - x^*\|^2+\beta\|x_n-x^*\|^2\\ &\quad + \epsilon_n\gamma\alpha(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+2\epsilon_n\langle\gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle\\ &\le \left(\frac{(1-\beta-\bar{\gamma}\epsilon_n)^2}{1-\beta}+\beta+\epsilon_n\gamma\alpha\right)\|x_n-x^*\|^2+\epsilon_n\gamma\alpha\|x_{n+1}-x^*\|^2\\ &\quad + 2\epsilon_n\langle\gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \left(1-\frac{2(\bar{\gamma}-\alpha\gamma)\epsilon_n}{1-\alpha\gamma\epsilon_n}\right)\|x_n-x^*\|^2\\ &\quad + \frac{\epsilon_n}{1-\alpha\gamma\epsilon_n}\left(2\langle\gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle+\frac{\bar{\gamma}^2\epsilon_n}{1-\beta}\|x_n-x^*\|^2\right).\end{aligned}$$

Now, from conditions (B1) and (B2), Step 7 and Lemma 2.3, we get $\|x_n - x^*\| \to 0$. Namely, $x_n \to x^*$ in norm. Finally, noticing $\|W_n x_n - x^*\| \le \|x_n - x^*\|$, we also conclude that $W_n x_n \to x^*$ in norm.

□

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