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On some properties of submanifolds of a Riemannian manifold endowed with a semi-symmetric non-metric connection

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Abstract

We study submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection. We prove that the induced connection is also a semi-symmetric non-metric connection. We consider the total geodesicness, total umbilicity and the minimality of a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection. We have obtained the Gauss, Codazzi and Ricci equations with respect to the semi-symmetric non-metric connection. The relation between the sectional curvatures of the Levi-Civita connection and the semi-symmetric non-metric connection is also given.

1 Introduction

The notion of a semi-symmetric metric connection on a Riemannian manifold was introduced by H.A. Hayden in [6]. In [13], K. Yano studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In [7] and [8], T. Imai found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Submanifolds of a Riemannian manifold with a semi-symmetric metric connection were studied by Z. Nakao in [9]. The notion of a semi-symmetric non-metric connection was introduced by N. S. Agashe and M. R. Chafle in [1]. Later in [2], the same authors studied submanifolds of a Riemannian manifold with the semi-symmetric non-metric connection. In [12], J. Sengupta, U. C.

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De and T. Q. Binh defined a type of semi-symmetric non-metric connection. In [10], C. Özgür studied submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection in the sense of [12]. In [11], J. Sengupta and U. C. De defined another type of semi-symmetric non-metric connection. They also considered a hypersurface of a Riemannian manifold with semi-symmetric non-metric connection in their sense.

In the present paper, we study submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection defined in [11]. The paper is organized as follows: In Section 2, we give some properties of the semi-symmetric non-metric connection. In Section 3, some necessary informations about a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection is given and we prove that the induced connection is also a semi-symmetric non-metric connection. We also consider the total geodesicness, total umbilicity and the minimality of a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection. In Section 4, we have obtained the Gauss, Codazzi and Ricci equations with respect to the semi-symmetric non-metric connection. The relation between the sectional curvatures of the Levi-Civita connection and the semi-symmetric non-metric connection is also found.

2 Preliminaries

Let $\widetilde{M}$ be an $(n+d)$-dimensional Riemannian manifold with a Riemannian metric g and $\widetilde{\nabla}$ be the Levi-Civita connection on $\widetilde{M}$. In [11], J. Sengupta and U. C. De defined a linear connection $\overset{*}{\widetilde{\nabla}}$ on $\widetilde{M}$ by

$$\overset{*}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} + \omega(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})\widetilde{U} - \eta(\widetilde{X})\widetilde{Y} - \eta(\widetilde{Y})\widetilde{X}, \tag{1}$$

where $\widetilde{U}$ is a vector field associated with the 1-form ω defined by

$$\omega(\widetilde{X}) = g(\widetilde{X},\widetilde{U}) \tag{2}$$

and E is a vector field associated with the 1-form η as

$$\eta(\widetilde{X}) = g(\widetilde{X},\widetilde{E}). \tag{3}$$

Using (1), the torsion tensor T of $\widetilde{M}$ with respect to the connection $\overset{*}{\widetilde{\nabla}}$ is given by

$$T(\widetilde{X},\widetilde{Y}) = \overset{*}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} - \overset{*}{\widetilde{\nabla}}_{\widetilde{Y}}\widetilde{X} - \left[\widetilde{X},\widetilde{Y}\right] = \omega(\widetilde{Y})\widetilde{X} - \omega(\widetilde{X})\widetilde{Y}. \tag{4}$$

Using (1) we have

$$\left(\overset{*}{\widetilde{\nabla}}_{\widetilde{X}} g\right)\left(\widetilde{Y},\widetilde{Z}\right)=2\eta(\widetilde{X})g(\widetilde{Y},\widetilde{Z})+\eta(\widetilde{Y})g(\widetilde{X},\widetilde{Z})+\eta(\widetilde{Z})g(\widetilde{X},\widetilde{Y}). \tag{5}$$

Hence the connection $\overset{*}{\widetilde{\nabla}}$ is not a metric connection. Because of this reason this connection is called a *semi-symmetric non-metric connection* (for more details see [11]).

We denote the curvature tensor of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection $\overset{*}{\widetilde{\nabla}}$ by

$$\begin{aligned}\overset{*}{\widetilde{R}}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z} &= \overset{*}{\widetilde{\nabla}}_{\widetilde{X}}\overset{*}{\widetilde{\nabla}}_{\widetilde{Y}}\widetilde{Z}-\overset{*}{\widetilde{\nabla}}_{\widetilde{Y}}\overset{*}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Z}-\overset{*}{\widetilde{\nabla}}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z} \\ &= \widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z}-\alpha\left(\widetilde{Y},\widetilde{Z}\right)\widetilde{X}+\alpha\left(\widetilde{X},\widetilde{Z}\right)\widetilde{Y} \\ &-g\left(\widetilde{Y},\widetilde{Z}\right)Q\widetilde{X}+g\left(\widetilde{X},\widetilde{Z}\right)Q\widetilde{Y}+\beta\left(\widetilde{Y},\widetilde{X}\right)\widetilde{Z}-\beta\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z} \\ &+\beta\left(\widetilde{Y},\widetilde{Z}\right)\widetilde{X}-\beta\left(\widetilde{X},\widetilde{Z}\right)\widetilde{Y},\end{aligned} \tag{6}$$

where

$$\widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z}=\widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z}-\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z}-\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

is the curvature tensor of the manifold with respect to the Levi-Civita connection $\widetilde{\nabla}$ and α and β are $(0,2)$-tensor field defined by

$$\alpha\left(\widetilde{X},\widetilde{Y}\right)=\left(\widetilde{\nabla}_{\widetilde{X}}\omega\right)\widetilde{Y}-\omega(\widetilde{X})\omega(\widetilde{Y})+\frac{1}{2}\omega(\widetilde{U})g\left(\widetilde{X},\widetilde{Y}\right), \tag{7}$$

$$Q\widetilde{X}=\widetilde{\nabla}_{\widetilde{X}}\widetilde{U}-\omega(\widetilde{X})\widetilde{U}+\frac{1}{2}\omega(\widetilde{U})\widetilde{X} \tag{8}$$

and

$$\begin{aligned}\beta\left(\widetilde{X},\widetilde{Y}\right) &= (\widetilde{\nabla}_{\widetilde{X}}\eta)(\widetilde{Y})-\eta(\widetilde{X})\omega\left(\widetilde{Y}\right)+\eta(\widetilde{X})\eta\left(\widetilde{Y}\right) \\ &-\omega(\widetilde{X})\eta\left(\widetilde{Y}\right)+\eta(\widetilde{U})g\left(\widetilde{X},\widetilde{Y}\right),\end{aligned} \tag{9}$$

(see [11]). The Riemannian Christoffel tensors of the connections $\overset{*}{\widetilde{\nabla}}$ and $\widetilde{\nabla}$ are defined by

$$\overset{*}{\widetilde{R}}\left(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}\right)=g\left(\overset{*}{\widetilde{R}}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z},\widetilde{W}\right)$$

and

$$\widetilde{R}\left(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}\right)=g\left(\widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z},\widetilde{W}\right).$$

respectively.

3 Submanifolds

Let M be an n-dimensional submanifold of an $(n+d)$-dimensional Riemannian manifold $\widetilde{M}$ with a semi-symmetric non-metric connection $\overset{*}{\widetilde{\nabla}}$. Decomposing the vector fields $\widetilde{U}$ and $\widetilde{E}$ on M uniquely into their tangent and normal components $U^T, U^\perp$ and $E^T, E^\perp$ respectively, we have

$$\widetilde{U} = U^T + U^\perp, \tag{10}$$

$$\widetilde{E} = E^T + E^\perp. \tag{11}$$

The Gauss formula for a submanifold M of a Riemannian manifold $\widetilde{M}$ with respect to the Riemannian connection $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{12}$$

where $X, Y \in TM$ and h is the second fundamental form of M in $\widetilde{M}$. If $h = 0$ then M is called *totally geodesic*. $H = \frac{1}{n} traceh$ is called the *mean curvature vector* of the submanifold. If $H = 0$ then M is called minimal. If $h(X,Y) = g(X,Y)H$ for any X, Y tangent to M then M is called *totally umbilical*. For the second fundamental form h, the covariant derivative of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^\perp_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{13}$$

for any vector fields X, Y, Z tangent to M. Then $\overline{\nabla} h$ is a normal bundle valued tensor of type $(0,3)$ and is called the *third fundamental form* of M. $\overline{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M, i.e., $\overline{\nabla}$ is the connection in $TM \oplus T^\perp M$ built with ∇ and $\nabla^\perp$[4].

Let $\overset{*}{\nabla}$ be the induced connection from the semi-symmetric non-metric connection. We define

$$\overset{*}{\widetilde{\nabla}}_X Y = \overset{*}{\nabla}_X Y + \overset{*}{h}(X,Y), \quad X, Y \in TM. \tag{14}$$

The equation (14) may be called the Gauss equation with respect to the semi-symmetric non-metric connection $\overset{*}{\widetilde{\nabla}}$ (see [2]). Hence using (1), (12) and (14) we have

$$\begin{aligned} \overset{*}{\nabla}_X Y + \overset{*}{h}(X,Y) &= \nabla_X Y + h(X,Y) + \omega(Y)X - g(X,Y)U^T \\ &\quad - g(X,Y)U^\perp - \eta(X)Y - \eta(Y)X. \end{aligned} \tag{15}$$

So comparing the tangential and normal parts of (15) we obtain

$$\overset{*}{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X,Y)U^T - \eta(X)Y - \eta(Y)X \tag{16}$$

and

$$\overset{*}{h}(X,Y) = h(X,Y) - g(X,Y)U^{\perp}. \tag{17}$$

If $\overset{*}{h} = 0$ then M is called totally geodesic with respect to the semi-symmetric non-metric connection.

From (16), we have

$$\overset{*}{T}(X,Y) = \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X - [X,Y] = \omega(Y)X - \omega(X)Y, \tag{18}$$

where $\overset{*}{T}$ is the torsion tensor of M with respect to $\overset{*}{\nabla}$ and $X, Y \in TM$. Moreover using (16) we have

$$\begin{aligned} \left(\overset{*}{\nabla}_X g\right)(Y,Z) &= \overset{*}{\nabla}_X g(Y,Z) - g(\overset{*}{\nabla}_X Y, Z) - g(Y, \overset{*}{\nabla}_X Z) \\ &= 2g(Y,Z)\eta(X) + g(X,Z)\eta(Y) + g(X,Y)\eta(Z), \quad (19) \end{aligned}$$

for $X, Y, Z \in TM$. In view of (1), (16), (18) and (19) we can state the following theorem:

Theorem 1. *The induced connection $\overset{*}{\nabla}$ on a submanifold of a Riemannian manifold admitting the semi-symmetric non-metric connection in the sense of [11] is also a semi-symmetric non-metric connection.*

In Theorem 5.1 of [11], J. Sengupta and U. C. De proved the above theorem for a hypersurface of a Riemannian manifold admitting the semi-symmetric non-metric connection. So the above theorem is a generalization of their theorem.

Let $\{E_1, E_2, ..., E_n\}$ be an orthonormal basis of the tangent space of M. We define the mean curvature vector $\overset{*}{H}$ of M with respect to the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$ by

$$\overset{*}{H} = \frac{1}{n}\sum_{i=1}^{n} \overset{*}{h}(E_i, E_i).$$

So from (17) we find

$$\overset{*}{H} = H - U^{\perp}.$$

If $\overset{*}{H} = 0$ then M is called minimal with respect to the semi-symmetric non-metric connection.

So we have the following result:

Theorem 2. *Let M be a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection in the sense of [11] and*

i) Let the vector field $\widetilde{U}$ be tangent to M. Then M is totally geodesic with respect to the Levi-Civita connection if and only if it is totally geodesic with respect to the semi-symmetric non-metric connection.

ii) M is totally umbilical with respect to the Levi-Civita connection if and only if M is totally umbilical with respect to the semi-symmetric non-metric connection.

iii) Let the vector field $\widetilde{U}$ be tangent to M. Then the mean curvature normal of M with respect to the Levi-Civita connection and with respect to the semi-symmetric non-metric connection coincide. Hence M is minimal with respect to the Levi-Civita connection if and only if it is minimal with respect to the semi-symmetric non-metric connection.

In Theorem 5.2 and Theorem 5.3 of [11], J. Sengupta and U. C. De considered the cases (ii) and (iii) of the above theorem for a hypersurface of a Riemannian manifold admitting the semi-symmetric non-metric connection. So the above theorem generalizes their results.

Let ξ be a normal vector field on M. From (1) we have

$$\overset{*}{\widetilde{\nabla}}_X \xi = \widetilde{\nabla}_X \xi + \omega(\xi)X - \eta(X)\xi - \eta(\xi)X. \tag{20}$$

It is well-known that

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi, \tag{21}$$

which is the Weingarten formula for a submanifold of a Riemannian manifold, where A_ξ is the shape operator of M in the direction of ξ. So from (21) the equation (20) can be written as

$$\overset{*}{\widetilde{\nabla}}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi + \omega(\xi)X - \eta(X)\xi - \eta(\xi)X. \tag{22}$$

Now we define a $(1,1)$ tensor field $\overset{*}{A}$ on M by

$$\overset{*}{A}_\xi = \left(A_\xi - \omega(\xi) + \eta(\xi)\right) I. \tag{23}$$

Then the equation (22) turns into

$$\overset{*}{\widetilde{\nabla}}_X \xi = -\overset{*}{A}_\xi X + \nabla^{\perp}_X \xi - \eta(X)\xi. \tag{24}$$

Equation (24) may be called Weingarten's formula with respect to the semi-symmetric non-metric connection $\overset{*}{\widetilde{\nabla}}$. Since A_ξ is symmetric, it is easy to see that

$$g\left(\overset{*}{A}_\xi X, Y\right) = g\left(X, \overset{*}{A}_\xi Y\right)$$

and

$$g\left(\left[\overset{*}{A}_\xi, \overset{*}{A}_\upsilon\right] X, Y\right) = g\left(\left[A_\xi, A_\upsilon\right] X, Y\right), \tag{25}$$

where $g\left(\left[\overset{*}{A}_\xi, \overset{*}{A}_\upsilon\right] X, Y\right) = \overset{*}{A}_\xi \overset{*}{A}_\upsilon - \overset{*}{A}_\upsilon \overset{*}{A}_\xi$ and $g\left(\left[A_\xi, A_\upsilon\right] X, Y\right) = A_\xi A_\upsilon - A_\upsilon A_\xi$ and ξ, υ are unit normal vector fields on M.

Then by the similar proofs of Theorem 3.2 and Theorem 3.3 in [2] we have the following theorems:

Theorem 3. *Let M be a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection in the sense of [11]. Then the shape operators with respect to the semi-symmetric non-metric connection are simultaneously diagonalizable if and only if the shape operators with respect to the Levi-Civita connection are simultaneously diagonalizable.*

Theorem 4. *Principal directions of the unit normal vector field ξ with respect to the Levi-Civita connection and the semi-symmetric non-metric connection in the sense of [11] coincide and the principal curvatures are equal if and only if ξ is orthogonal to $U^\perp$ and $E^\perp$ or $U^\perp = E^\perp$.*

4 Gauss, Codazzi and Ricci equations with respect to semi-symmetric non-metric connection

We denote the curvature tensor of a submanifold M of a Riemannian manifold $\widetilde{M}$ with respect to the induced semi-symmetric non-metric connection $\overset{*}{\nabla}$ and the induced Riemannian connection ∇ by

$$\overset{*}{R}(X, Y) Z = \overset{*}{\nabla}_X \overset{*}{\nabla}_Y Z - \overset{*}{\nabla}_Y \overset{*}{\nabla}_X Z - \overset{*}{\nabla}_{[X,Y]} Z \tag{26}$$

and

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

respectively.

From (14) and (22) we get

$$\begin{aligned} \overset{*}{\widetilde{\nabla}}_X \overset{*}{\widetilde{\nabla}}_Y Z &= \overset{*}{\nabla}_X \overset{*}{\nabla}_Y Z + \overset{*}{h}\left(X, \overset{*}{\nabla}_Y Z\right) - A_{\overset{*}{h}(Y,Z)} X + \\ &+ \nabla^\perp_X \overset{*}{h}(Y, Z) + \omega\left(\overset{*}{h}(Y, Z)\right) X - \eta(X) \overset{*}{h}(Y, Z) - \eta(\overset{*}{h}(Y, Z)) X, \end{aligned} \tag{27}$$

$$\overset{*}{\widetilde{\nabla}}_Y \overset{*}{\widetilde{\nabla}}_X Z = \overset{*}{\nabla}_Y \overset{*}{\nabla}_X Z + \overset{*}{h}\left(Y, \overset{*}{\nabla}_X Z\right) - A_{\overset{*}{h}(X,Z)} Y + \tag{28}$$
$$+\nabla^{\perp}_Y \overset{*}{h}(X,Z) + \omega\left(\overset{*}{h}(X,Z)\right) Y - \eta(Y)\overset{*}{h}(X,Z) - \eta(\overset{*}{h}(X,Z))Y$$

and

$$\overset{*}{\widetilde{\nabla}}_{[X,Y]} Z = \overset{*}{\nabla}_{[X,Y]} Z + \overset{*}{h}([X,Y], Z). \tag{29}$$

Hence in view of (6) and (26), from (27)-(29), we have

$$\begin{aligned} \overset{*}{\widetilde{R}}(X,Y)Z &= \overset{*}{R}(X,Y)Z + \overset{*}{h}\left(X, \overset{*}{\nabla}_Y Z\right) - \overset{*}{h}\left(Y, \overset{*}{\nabla}_X Z\right) - \overset{*}{h}([X,Y], Z) \\ &\quad - A_{\overset{*}{h}(Y,Z)} X + A_{\overset{*}{h}(X,Z)} Y + \nabla^{\perp}_X \overset{*}{h}(Y,Z) - \nabla^{\perp}_Y \overset{*}{h}(X,Z) \\ &\quad + \omega\left(\overset{*}{h}(Y,Z)\right) X - \omega\left(\overset{*}{h}(X,Z)\right) Y - \eta(X)\overset{*}{h}(Y,Z) \\ &\quad - \eta(\overset{*}{h}(Y,Z))X + \eta(Y)\overset{*}{h}(X,Z) + \eta(\overset{*}{h}(X,Z))Y. \end{aligned} \tag{30}$$

Since $g(A_\xi X, Y) = g(h(X,Y), \xi)$, using (17) we find

$$\begin{aligned} \overset{*}{\widetilde{R}}(X,Y,Z,W) &= \overset{*}{R}(X,Y,Z,W) - g(h(Y,Z), h(X,W)) + g(h(X,Z), h(Y,W)) \\ &\quad + g(Y,Z)\,\omega(h(X,W)) - g(X,Z)\,\omega(h(Y,W)) \\ &\quad + g(X,W)[\omega(h(Y,Z)) - \eta(h(Y,Z))] \\ &\quad + g(Y,W)[\eta(h(X,Z)) - \omega(h(X,Z))] \\ &\quad + \omega\left(U^{\perp}\right)[g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)] \\ &\quad + \eta\left(U^{\perp}\right)[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)]. \end{aligned} \tag{31}$$

From (30), the normal component of $\overset{*}{\widetilde{R}}(X,Y)Z$ is given by

$$\begin{aligned} \left(\overset{*}{\widetilde{R}}(X,Y)Z\right)^{\perp} &= \overset{*}{h}\left(X, \overset{*}{\nabla}_Y Z\right) - \overset{*}{h}\left(Y, \overset{*}{\nabla}_X Z\right) - \overset{*}{h}([X,Y], Z) + \nabla^{\perp}_X \overset{*}{h}(Y,Z) \\ &\quad - \nabla^{\perp}_Y \overset{*}{h}(X,Z) - \eta(X)\overset{*}{h}(Y,Z) + \eta(Y)\overset{*}{h}(X,Z), \end{aligned}$$

then

$$\left(\overset{*}{\widetilde{R}}(X,Y)Z\right)^{\perp} = \left(\overset{*}{\overline{\nabla}}_X \overset{*}{h}\right)(Y,Z) - \left(\overset{*}{\overline{\nabla}}_Y \overset{*}{h}\right)(X,Z) + \omega(Y)\overset{*}{h}(X,Z)$$

$$-\omega(X)\overset{*}{h}(Y,Z)+\eta(Y)\overset{*}{h}(X,Z)-\eta(X)\overset{*}{h}(Y,Z), \tag{32}$$

where

$$\left(\overset{*}{\overline{\nabla}}_X\overset{*}{h}\right)(Y,Z)=\nabla^{\perp}_X\overset{*}{h}(Y,Z)-\overset{*}{h}\left(\overset{*}{\nabla}_XY,Z\right)-\overset{*}{h}\left(Y,\overset{*}{\nabla}_XZ\right). \tag{33}$$

$\overset{*}{\overline{\nabla}}$ is the connection in $TM\oplus T^{\perp}M$ built with $\overset{*}{\nabla}$ and $\nabla^{\perp}$. It can be called the van der Waerden-Bortolotti connection with respect to the semi-symmetric non-metric connection. The equation (32) may be called the equation of Codazzi with respect to the semi-symmetric non-metric connection (see [2]).

From (24) and (14) we get

$$\begin{aligned}\overset{*}{\widetilde{\nabla}}_X\overset{*}{\widetilde{\nabla}}_Y\xi &= -\overset{*}{\nabla}_X\left(\overset{*}{A}_\xi Y\right)-\overset{*}{h}\left(X,\overset{*}{A}_\xi Y\right)-\overset{*}{A}_{\nabla^{\perp}_Y\xi}X+\nabla^{\perp}_X\nabla^{\perp}_Y\xi-\eta(X)\nabla^{\perp}_Y\xi\\&\quad -g(\widetilde{\nabla}_XY,E^T)\xi-g(Y,\widetilde{\nabla}_XE^T)\xi-\eta(Y)\widetilde{\nabla}_X\xi-\eta(Y)\omega(\xi)X\\&\quad +\eta(X)\eta(Y)\xi+\eta(Y)\eta(\xi)X,\end{aligned} \tag{34}$$

$$\begin{aligned}\overset{*}{\widetilde{\nabla}}_Y\overset{*}{\widetilde{\nabla}}_X\xi &= -\overset{*}{\nabla}_Y\left(\overset{*}{A}_\xi X\right)-\overset{*}{h}\left(Y,\overset{*}{A}_\xi X\right)-\overset{*}{A}_{\nabla^{\perp}_X\xi}Y+\nabla^{\perp}_Y\nabla^{\perp}_X\xi-\eta(Y)\nabla^{\perp}_X\xi\\&\quad -g(\widetilde{\nabla}_YX,E^T)\xi-g(X,\widetilde{\nabla}_YE^T)\xi-\eta(X)\widetilde{\nabla}_Y\xi-\eta(X)\omega(\xi)Y\\&\quad +\eta(Y)\eta(X)\xi+\eta(X)\eta(\xi)Y.\end{aligned} \tag{35}$$

and

$$\overset{*}{\widetilde{\nabla}}_{[X,Y]}\xi=-\overset{*}{A}_\xi\,[X,Y]+\nabla^{\perp}_{[X,Y]}\xi-\eta([X,Y])\xi. \tag{36}$$

So using (34)-(36) we get

$$\begin{aligned}\overset{*}{\widetilde{R}}(X,Y,\xi,\upsilon) &= R^{\perp}(X,Y,\xi,\upsilon)-g\left(\overset{*}{h}\left(X,\overset{*}{A}_\xi Y\right),\upsilon\right)+g\left(\overset{*}{h}\left(Y,\overset{*}{A}_\xi X\right),\upsilon\right)\\&\quad -\eta(X)g(\nabla^{\perp}_Y\xi,\upsilon)-g(Y,\widetilde{\nabla}_XE^T)g(\xi,\upsilon)-\eta(Y)g(\widetilde{\nabla}_X\xi,\upsilon)\\&\quad +\eta(Y)g(\nabla^{\perp}_X\xi,\upsilon)+g(X,\widetilde{\nabla}_YE^T)g(\xi,\upsilon)+\eta(X)g(\widetilde{\nabla}_Y\xi,\upsilon),\end{aligned}$$

where ξ,υ are unit normal vector fields on M. Hence in view of (12),(17),(21) and (23) the last equation turns into

$$\begin{aligned}\overset{*}{\widetilde{R}}(X,Y,\xi,\upsilon) &= R^{\perp}(X,Y,\xi,\upsilon)+g\left(h(Y,A_\xi X),\upsilon\right)-g\left(h(X,A_\xi Y),\upsilon\right)\\&\quad +[g(X,\nabla_YE^T)-g(Y,\nabla_XE^T)]g(\xi,\upsilon),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\overset{*}{\widetilde{R}}(X,Y,\xi,\upsilon) &= R^{\perp}(X,Y,\xi,\upsilon)+g\left(\left(A_{\upsilon}A_{\xi}-A_{\xi}A_{\upsilon}\right)X,Y\right)\\ &\quad +[g(X,\nabla_Y E^T)-g(Y,\nabla_X E^T)]g(\xi,\upsilon)\\ &= R^{\perp}(X,Y,\xi,\upsilon)+g\left(\left[A_{\upsilon},A_{\xi}\right]X,Y\right)\\ &\quad +[g(X,\nabla_Y E^T)-g(Y,\nabla_X E^T)]g(\xi,\upsilon). \qquad (37)\end{aligned}$$

The equation (37) is the equation of Ricci with respect to the semi-symmetric non-metric connection.

Now assume that $\widetilde{M}$ is a space of constant curvature c with the semi-symmetric non-metric connection. Then

$$\begin{aligned}\overset{*}{\widetilde{R}}(X,Y)Z &= c\left(g(Y,Z)X-g(X,Z)Y\right)-\alpha(Y,Z)X+\alpha(X,Z)Y\\ &\quad -g(Y,Z)QX+g(X,Z)QY+\beta(Y,X)Z\\ &\quad -\beta(X,Y)Z+\beta(Y,Z)X-\beta(X,Z)Y. \qquad (38)\end{aligned}$$

Hence

$$\left(\overset{*}{\widetilde{R}}(X,Y)Z\right)^{\perp}=-g(Y,Z)(QX)^{\perp}+g(X,Z)(QY)^{\perp},$$

which gives us

$$\begin{aligned}\left(\overset{*}{\widetilde{R}}(X,Y)Z\right)^{\perp} &= -g(Y,Z)\left\{h(X,U^T)+\nabla_X^{\perp}U^{\perp}-\omega(X)U^{\perp}\right\}\\ &\quad +g(X,Z)\left\{h(Y,U^T)+\nabla_Y^{\perp}U^{\perp}-\omega(Y)U^{\perp}\right\}. \qquad (39)\end{aligned}$$

So from (32) and (39) the Ricci equation becomes

$$\begin{aligned}&\left(\overset{*}{\overline{\nabla}}_X\overset{*}{h}\right)(Y,Z)-\left(\overset{*}{\overline{\nabla}}_Y\overset{*}{h}\right)(X,Z)+\omega(Y)\overset{*}{h}(X,Z)-\omega(X)\overset{*}{h}(Y,Z)\\ &+\eta(Y)\overset{*}{h}(X,Z)-\eta(X)\overset{*}{h}(Y,Z)=-g(Y,Z)\{h(X,U^T)+\nabla_X^{\perp}U^{\perp}-\omega(X)U^{\perp}\}\\ &+g(X,Z)\left\{h(Y,U^T)+\nabla_Y^{\perp}U^{\perp}-\omega(Y)U^{\perp}\right\}.\end{aligned}$$

Since $\widetilde{M}$ is a space of constant curvature c, from (38)

$$\overset{*}{\widetilde{R}}(X,Y,\xi,\upsilon)=g(\xi,\upsilon)\{g(X,\nabla_Y E^T)-g(Y,\nabla_X E^T)\}.$$

Therefore using (37), we obtain

$$R^{\perp}(X,Y,\xi,\upsilon)=-g\left(\left[A_{\xi},A_{\upsilon}\right]X,Y\right).$$

Hence using (25), we can state the following theorem:

Theorem 5. *Let M be a submanifold of a space of constant curvature with the semi-symmetric metric connection in the sense of [11]. Then the normal connection $\nabla^{\perp}$ is flat if and only if all second fundamental tensors with respect to the semi-symmetric non-metric connection and the Levi-Civita connection are simultaneously diagonalizable.*

Now assume that X and Y are orthonormal unit tangent vector fields on M. Then in view of (31) we can write

$$\begin{aligned}\overset{*}{\widetilde{R}}(X,Y,Y,X) &= \overset{*}{R}(X,Y,Y,X) - g\left(h(Y,Y),h(X,X)\right) + g\left(h(X,Y),h(Y,X)\right) \\ &\quad + g\left(Y,Y\right)\omega\left(h(X,X)\right) + g\left(X,X\right)\left[\omega\left(h(Y,Y)\right) - \eta(h(Y,Y))\right] \\ &\quad + \eta\left(U^{\perp}\right) - \omega\left(U^{\perp}\right).\end{aligned}$$

So we get

$$\begin{aligned}\overset{*}{\widetilde{K}}(\pi) &= \overset{*}{K}(\pi) \\ &\quad - g\left(h(Y,Y),h(X,X)\right) + g\left(h(X,Y),h(Y,X)\right) \\ &\quad + \omega\left(h(X,X)\right) + \omega\left(h(Y,Y)\right) \\ &\quad - \eta(h(Y,Y)) + \eta\left(U^{\perp}\right) - \omega\left(U^{\perp}\right).\end{aligned} \tag{40}$$

Now let M be a submanifold of a Riemannian manifold of $\widetilde{M}$ with the semi-symmetric non-metric connection in the sense of [11] and π be a subspace of the tangent space T_pM spanned by the orthonormal base $\{X,Y\}$. Denote by $\overset{*}{\widetilde{K}}(\pi)$ and $\overset{*}{K}(\pi)$ the sectional curvatures of $\widetilde{M}$ and M at a point $p \in \widetilde{M}$, respectively with respect to the semi-symmetric non-metric connection. Let γ be a geodesic in $\widetilde{M}$ which lies in M and T be a unit tangent vector field of γ in M. Then from (17) we have

$$\begin{aligned} h(T,T) &= 0, \\ \overset{*}{h}(T,T) &= -U^{\perp}. \end{aligned} \tag{41}$$

Let π be the subspace of the tangent space T_pM spanned by X, T and $\widetilde{U}$ which are the vector field tangent to M. Then from (41) we have $\overset{*}{h}(T,T) = 0$. Hence using (40) we obtain

$$\overset{*}{\widetilde{K}}(\pi) = \overset{*}{K}(\pi) + g\left(h(X,T),h(X,T)\right).$$

Let X be a unit tangent vector field on M which is parallel along γ in M and orthogonal to T. So $\nabla_T X = 0$ and $g(X,T) = 0$. We have the following theorem:

Theorem 6. *Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with the semi-symmetric non-metric connection in the sense of [11] and γ be a geodesic of $\widetilde{M}$ which lies in M and T be the unit tangent vector field of γ in M. π be the subspace of the tangent space T_pM spanned by X and T. If the vector field $\widetilde{U}$ is tangent to M, then*

i) $\overset{*}{\widetilde{K}}(\pi) \geq \overset{*}{K}(\pi)$ *along* γ.

ii) If X is a unit tangent vector field on M which is parallel along γ in M and orthogonal to T then the equality case of (i) holds if and only if X is parallel along γ in $\widetilde{M}$.

Theorem 7. *Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with the semi-symmetric non-metric connection in the sense of [11]. If the second fundamental form of M with respect to the van der Waerden-Bortolotti connection and with respect to the van der Waerden-Bortolotti connection with the semi-symmetric non-metric connection is parallel and $\widetilde{U}$ normal then $U^\perp$ is parallel in the normal bundle.*

Proof. Applying (13), (16) and (17) in (33), we obtain

$$\begin{aligned}\left(\overset{*}{\overline{\nabla}}_X \overset{*}{h}\right)(Y,Z) &= (\overline{\nabla}_X h)(Y,Z) - g(Y,Z)\nabla^{\perp}_X U^{\perp} - \omega(Y)h(X,Z) \\ &\quad + g(X,Y)h(U^T,Z) + 2\eta(X)\,h(Y,Z) + \eta(Y)\,h(X,Z) \\ &\quad - 2\eta(X)\,g(Y,Z)U^{\perp} - \eta(Y)\,g(X,Z)U^{\perp} - \omega(Z)\,h(Y,X) \\ &\quad + g(X,Z)h(Y,U^T) + \eta(Z)\,h(Y,X) - \eta(Z)\,g(Y,X)U^{\perp}\end{aligned} \tag{42}$$

Since the conditions $\left(\overset{*}{\overline{\nabla}}_X \overset{*}{h}\right)(Y,Z) = 0$ and $(\overline{\nabla}_X h)(Y,Z) = 0$ holds on M and contraction with g^{YZ}, we get

$$\nabla^{\perp}_X U^{\perp} = 0.$$

So $U^\perp$ is parallel in the normal bundle. Thus the proof of the theorem is completed. □

Example. Let $\mathbb{T}^2 : S^1(1) \times S^1(1) \subset \mathbb{R}^4$ be a torus embedded in $\mathbb{R}^4$ defined by

$$\mathbb{T}^2 = \{(\cos u, \sin u, \cos v, \sin v) : u, v \in \mathbb{R}\}.$$

When $p = (\cos u, \sin u, \cos v, \sin v)$, $T_p(\mathbb{T}^2)$ is spanned by

$$e_1 = (-\sin u, \cos u, 0, 0),$$

$$e_2 = (0, 0, -\sin v, \cos v)$$

and $T_p^{\perp}(\mathbb{T}^2)$ is spanned by

$$e_3 = (\cos u, \sin u, 0, 0),$$

$$e_4 = (0, 0, \cos v, \sin v).$$

Differentiating these we get

$$\begin{aligned}
&\widetilde{\nabla}_{e_1} e_1 = -e_3 \quad , \quad \widetilde{\nabla}_{e_2} e_2 = -e_4, \\
&\widetilde{\nabla}_{e_1} e_2 = 0 \quad , \quad \widetilde{\nabla}_{e_2} e_1 = 0, \\
&\widetilde{\nabla}_{e_1} e_3 = e_1 \quad , \quad \widetilde{\nabla}_{e_2} e_4 = e_2, \\
&\widetilde{\nabla}_{e_1} e_4 = 0 \quad , \quad \widetilde{\nabla}_{e_2} e_3 = 0.
\end{aligned} \tag{43}$$

So by Gauss equation (12) we have

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0 \quad , \quad \nabla_{e_2} e_2 = 0, \\
&\nabla_{e_1} e_2 = 0 \quad , \quad \nabla_{e_2} e_1 = 0,
\end{aligned} \tag{44}$$

$$h(e_1, e_1) = -e_3, \quad h(e_2, e_2) = -e_4, \quad h(e_1, e_2) = 0 \tag{45}$$

[3]. Assume that the vector fields $\widetilde{U}$ and $\widetilde{E}$ defined in (10) and (11) are tangent to $\mathbb{T}^2$. Using (44) and (16) we get

$$\begin{aligned}
\overset{*}{\nabla}_{e_1} e_1 &= \omega(e_1)e_1 - U^T - 2\eta(e_1)e_1, \\
\overset{*}{\nabla}_{e_1} e_2 &= \omega(e_2)e_1 - \eta(e_1)e_2 - \eta(e_2)e_1, \\
\overset{*}{\nabla}_{e_2} e_1 &= \omega(e_1)e_2 - \eta(e_2)e_1 - \eta(e_1)e_2, \\
\overset{*}{\nabla}_{e_2} e_2 &= \omega(e_2)e_2 - U^T - 2\eta(e_2)e_2.
\end{aligned} \tag{46}$$

So using (46), (18) and (19) we have

$$\overset{*}{T}(e_1, e_2) = \omega(e_2)e_1 - \omega(e_1)e_2 \neq 0$$

and

$$\begin{aligned}
\left(\overset{*}{\nabla}_{e_1} g\right)(e_1, e_1) &= 4\eta(e_1) \neq 0 \quad , \quad \left(\overset{*}{\nabla}_{e_1} g\right)(e_1, e_2) = \eta(e_2) \neq 0 \\
\left(\overset{*}{\nabla}_{e_1} g\right)(e_2, e_2) &= 2\eta(e_1) \neq 0 \quad , \quad \left(\overset{*}{\nabla}_{e_2} g\right)(e_1, e_1) = 2\eta(e_2) \neq 0 \\
\left(\overset{*}{\nabla}_{e_2} g\right)(e_1, e_2) &= \eta(e_1) \neq 0 \quad , \quad \left(\overset{*}{\nabla}_{e_2} g\right)(e_2, e_2) = 4\eta(e_2) \neq 0
\end{aligned}$$

Hence $\overset{*}{\nabla}$ is a semi-symmetric non-metric connection. Furthermore using (17) we have

$$\overset{*}{h}(e_1, e_1) = -e_3, \quad \overset{*}{h}(e_1, e_2) = 0, \quad \overset{*}{h}(e_2, e_2) = -e_4. \tag{47}$$

Thus by the use of (45) and (47), the mean curvature normals of $\mathbb{T}^2$ with respect to the semi-symmetric non-metric connection and the Levi-Civita connection are

$$\overset{*}{H} = -\frac{1}{2}(e_3 + e_4) = H.$$

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