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Depth and minimal number of generators of square free monomial ideals

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Abstract

Let I be an ideal of a polynomial algebra over a field generated by square free monomials of degree $\geq d$. If I contains more monomials of degree d than $(n-d)/(n-d+1)$ multiplied with the number of square free monomials of S of degree d then $\operatorname{depth}_S I \leq d$, in particular the Stanley's Conjecture holds in this case.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra in n-variables over a field K and $I \subset S$ a square free monomial ideal. Let d be a positive integer and $\rho_d(I)$ be the number of all square free monomials of degree d of I.

The proposition below was repaired using an idea of Y. Shen to whom we owe thanks.

Proposition 1. *If I is generated by square free monomials of degree $\geq d$ and $\rho_d(I) > ((n-d)/(n-d+1))\binom{n}{d}$ then* $\operatorname{depth}_S I \leq d$.

Proof. Apply induction on n. If $n = d$ then there exists nothing to show. Suppose that $n > d$. Let ν_i be the number of the square free monomials of degree d from $I \cap (x_i)$. We may consider two cases renumbering the variables if necessary.

Case 1 $\nu_1 > ((n-d)/(n-d+1))\binom{n-1}{d-1}$.

Let $S' := K[x_2, \ldots, x_n]$ and $x_1c_1, \ldots, x_1c_{\nu_1}$, $c_i \in S'$ be the square free monomials of degree d from $I \cap (x_1)$. Then $J = (I : x_1) \cap S'$ contains $(c_1, \ldots, c_{\nu_1})$ and so $\rho_{d-1}(J) \geq \nu_1 > ((n-d)/(n-d+1))\binom{n-1}{d-1}$. By induction hypothesis, we get $\operatorname{depth}_{S'} J \leq d-1$. It follows $\operatorname{depth}_S JS \leq d$ and so $\operatorname{depth}_S I \leq d$ by [7, Proposition 1.2].

Case 2 $\nu_i \leq ((n-d)/(n-d+1))\binom{n-1}{d-1}$ for all $i \in [n]$.

We get $\sum_{i=1}^n \nu_i \leq n((n-d)/(n-d+1))\binom{n-1}{d-1}$. Let A_i be the set of the square free monomials of degree d from $I \cap (x_i)$. A square free monomial from I of degree d will be present in d-sets A_i and it follows

$$\rho_d(I) = |\cup_{i=1}^n A_i| \leq (n/d)((n-d)/(n-d+1))\binom{n-1}{d-1} = ((n-d)/(n-d+1))\binom{n}{d}$$

if $n \geq d+1$. Contradiction! □

Remark 2. If I is generated by square free monomials of degree $\geq d$, then $\operatorname{depth}_S I \geq d$. Indeed, since I has a square free resolution the last shift in the resolution of I is at most n. Thus if I is generated in degree $\geq d$, then the resolution can have length at most $n-d$, which means that the depth of I is greater than or equal to d (this argument belongs to J. Herzog). Hence in the setting of the above proposition we get $\operatorname{depth}_S I = d$.

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Corollary 3. *Let I be an ideal generated by $\mu(I)$ square free monomials of degree d. If $\mu(I) > ((n-d)/(n-d+1))\binom{n}{d}$ then* $\operatorname{depth}_S I = d$.

Example 4. Let $I = (x_1x_2, x_2x_3) \subset S := K[x_1, x_2, x_3]$. Then $d = 2$ and $\mu(I) = 2 > (1/2)\binom{3}{2}$. It follows that $\operatorname{depth}_S I = 2$ by the above corollary.

Example 5. Let $I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_5, x_3x_4, x_3x_5, x_4x_5) \subset S := K[x_1, \ldots, x_5]$. Then $d = 2$ and $\mu(I) = 8 > (3/4)\binom{5}{2}$ and so $\operatorname{depth}_S I = 2$.

Next lemma presents a nice class of square free monomial ideals I with $\mu(I) = \binom{n}{d+1} \leq ((n-d)/(n-d+1))\binom{n}{d}$ but $\operatorname{depth}_S I = d$. We suppose that $n \geq 3$. Let w be the only square free monomial of degree n of S, that is $w = \Pi_{j=1}^n x_i$. Set $f_i = w/(x_ix_{i+1})$ for $1 \leq i < n$, $f_n = w/(x_1x_n)$ and let $L_n := (f_1, \ldots, f_{n-1})$, $I_n := (L, f_n)$ be ideals of S generated in degree $d = n-2$. We will see that $\operatorname{depth}_S I_n = n-2$ even $\mu(I_n) = n = \binom{n}{d+1}$.

Lemma 6. *Then* $\operatorname{depth}_S L_n = n-1$ *and* $\operatorname{depth}_S I_n = n-2$.

Proof. Apply induction on $n \geq 3$. If $n = 3$ then $L_3 = (x_3, x_1)$, $I_3 = (x_1, x_2, x_3)$ and the result is trivial. Assume that $n > 3$. Note that $(L_n : x_n) = L_{n-1}S = (I_n : x_n)$ because $f_n, f_{n-1} \in (L_n : x_n)$. We have

$$L_n = (L_n : x_n) \cap (x_n, L_n) = (L_{n-1}S) \cap (x_n, f_{n-1}),$$

$$I_n = (I_n : x_n) \cap (x_n, I_n) = (L_{n-1}S) \cap (x_n, f_{n-1}, f_n) = (L_{n-1}S) \cap (x_n, u) \cap (x_1, x_{n-1}, x_n),$$

where $u = w/(x_1x_{n-1}x_n)$. But (x_1, x_{n-1}) is a minimal prime ideal of $L_{n-1}S$ and so we may remove (x_1, x_{n-1}, x_n) above, that is $I_n = (L_{n-1}S) \cap (x_n, u)$. On the other hand, $(L_{n-1}S) + (x_n, u) = (x_n, I_{n-1})$ and $(L_{n-1}S) + (x_n, f_{n-1}) = (x_n, L_{n-1})S$ because $f_{n-1} \in L_{n-1}S$. We have the following exact sequences

$$0 \to S/L_n \to S/L_{n-1}S \oplus S/(x_n, f_{n-1}) \to S/(x_n, L_{n-1}S) \to 0,$$

$$0 \to S/I_n \to S/L_{n-1}S \oplus S/(x_n, u) \to S/(x_n, I_{n-1}S) \to 0.$$

By induction hypothesis $\operatorname{depth} L_{n-1} = n-2$ and $\operatorname{depth} I_{n-1} = n-3$ and so $\operatorname{depth}_S S/(x_n, L_{n-1}S) = n-3$, $\operatorname{depth}_S S/(x_n, I_{n-1}S) = n-4$. As $\operatorname{depth}_S S/(x_n, f_{n-1}) = \operatorname{depth}_S S/(x_n, u) = n-2$, it follows $\operatorname{depth}_S S/L_n = n-2$, $\operatorname{depth}_S S/I_n = n-3$ by the Depth Lemma applied to the above exact sequences. □

Now, let I be an arbitrary square free monomial ideal and P_I the poset given by all square free monomials of I (a finite set) with the order given by the divisibility. Let $\mathcal{P}$ be a partition of P_I in intervals $[u, v] = \{w \in P_I : u|w, w|v\}$, let us say $P_I = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} \mathcal{P} = \min_i \deg v_i$ and $\operatorname{sdepth}_S I = \max_{\mathcal{P}} \operatorname{sdepth} \mathcal{P}$, where $\mathcal{P}$ runs in the set of all partitions of P_I. This is the so called the Stanley depth of I, in fact this is an equivalent definition given in a general form by [1].

For instance, in Example 4, we have $P_I = \{x_1x_2, x_2x_3, x_1x_2x_3\}$ and we may take $\mathcal{P} : \quad P_I = [x_1x_2, x_1x_2x_3] \cup [x_2x_3, x_2x_3]$ with $\operatorname{sdepth}_S \mathcal{P} = 2$. Moreover, it is clear that $\operatorname{sdepth}_S I = 2$.

Remark 7. If I is generated by $\mu(I) > \binom{n}{d+1}$ square free monomials of degree d then $\operatorname{sdepth}_S I = d$. Since $((n-d)/(n-d+1))\binom{n}{d} \geq \binom{n}{d+1}$, the Proposition 1 says that in a weaker case case $\operatorname{depth}_S I \leq \operatorname{sdepth}_S I$, which was in general conjectured by Stanley [8]. Stanley's Conjecture holds for intersections of four monomial prime ideals of S by [2] and [4] and for square free monomial ideals of $K[x_1, \ldots, x_5]$ by [3] (a short exposition on this subject is given in [5]). It is worth to mention that Proposition 1 holds in the stronger case when $\mu(I) > \binom{n}{d+1}$ (see [6]), but the proof is much more complicated and the easy proof given in the present case has its importance.

In the Example 5 we have $P_I = [x_1x_2, x_1x_2x_4] \cup [x_1x_3, x_1x_3x_5] \cup [x_1x_4, x_1x_4x_5] \cup [x_2x_3, x_1x_2x_3] \cup [x_3x_4, x_1x_3x_4] \cup [x_3x_5, x_3x_4x_5] \cup [x_4x_5, x_2x_4x_5] \cup [x_2x_3x_4, x_2x_3x_4] \cap [x_2x_3x_5, x_2x_3x_5] \cup (\cup_\alpha [\alpha, \alpha])$, where α runs in the set of square free monomials of I of degree $4, 5$. It follows that $\operatorname{sdepth}_S I = 3$. But as we know $\operatorname{depth}_S I = 2$.

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