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On the classes of hereditarily $\ell_p(c_0)$ Banach spaces

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Abstract

Hagler and Azimi introduced a class of hereditarily l_1 Banach spaces which fail the Schur property. Then, Azimi extended these spaces to a class of hereditarily l_p Banach spaces for $1 \leq p < \infty$ and we used these spaces to introduce a new class of hereditarily $l_p(c_0)$ Banach spaces analogous of the space of Popov. In particular, for $p = 1$ the spaces are further examples of hereditarily l_1 Banach spaces failing the Schur property. In this paper we show for $1 \leq p < \infty$, these spaces are dual spaces with nonseparable duals and fail the Dunford-Pettis property. Also for $p = 1$, spaces contain asymptotically isometric copies of ℓ_1.

1 Introduction

A class of hereditarily l_1 Banach spaces has been introduced by Hagler and Azimi, which among the other interesting properties fails the Schur property [3]. Then Azimi extended these spaces to a new class of hereditarily l_p Banach spaces, the $X_{\alpha,p}$ [1]. In 2005, Popov constructed a new class of hereditarily l_1 subspace of L_1 without the Schur property [9] and generalized his result to a class of hereditarily l_p Banach spaces [10]. In [4] we used the $X_{\alpha,p}$ spaces to introduce and study a new class of hereditarily l_p spaces, analogous of the space of Popov. Indeed, if $p_1 > p_2 > ... > 1$, the subspace Z_p for $p \in [1, \infty) \cup \{0\}$ of $X_p = (\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n})_p$ is hereditarily $\ell_p(c_0)$. In particular, we showed that for $p = 1$ the spaces are further examples of hereditarily l_1 Banach spaces which fail the Schur property. This would be the fourth example of this type. The first was constructed by J. Bourgain [6], the second by Hagler and Azimi, and

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the third by Popov. In [5] we showed the Banach spaces $X_{\alpha,p}$ for $1 \leq p < \infty$ contains asymptotically isometric copies of ℓ_p. In this paper we show that Z_1 contains asymptotically isometric copies of ℓ_1. For $p \geq 1$, Z_p is a dual space and fails the Dunford-Pettis property.

Before introducing these new spaces, let us recall the definition of the $X_{\alpha,p}$. Let $\alpha = (\alpha_i)$ be a sequence of reals in $[0,1]$ (whose terms are used as weighting factor in the definition of the norm) which satisfies the following properties:

(1) $1 = \alpha_1 \geq \alpha_2 \geq ... > 0$,

(2) $\lim_i \alpha_i = 0$,

(3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.

By a block F we mean an interval (finite or infinite) of integers. For a block F and $x = (t_1, t_2, ...)$ a sequence of scalars such that $\sum_j t_j$ converges, define $< x, F >= \sum_{j \in F} t_j$. A sequence $F_1, F_2, ..., F_n, ...$where each F_i is a finite block is admissible if

$$\max F_i < \min F_{i+1} \text{ for } i = 1, 2, 3, ...$$

For $x = (t_1, t_2, ...)$ a finitely nonzero sequence of scalars, define

$$\| x \|= \max \left(\sum_{i=1}^{n} \alpha_i \, |< x, F_i >|^p\right)^{\frac{1}{p}},$$

where the max is taken over all n, admissible sequences F_1, $F_2, ..., F_n$ and $1 \leq p < \infty$. Then $X_{\alpha,p}$ is the completion of the finitely nonzero sequences of scalars $x = (t_1, t_2, ...)$ in this norm. For a good information concerning these spaces, referred to [1] and [3].

Now we go through the construction of the spaces X_p analogous of the space of Popov. Let α be a fixed sequence, and $(X_{\alpha,p_n})_{n=1}^{\infty}$ a sequence of Banach spaces as above with $\infty > p_1 > p_2 > ... > 1$. The direct sum of these spaces in the sense of l_p is defined as the linear space

$$X_p = \left(\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n}\right)_p$$

with $p \in [1, \infty)$ which is the space of all sequences $x = \left(x^1, x^2, ...\right)$, $x^n \in X_{\alpha,p_n}, n = 1, 2, ...$ with

$$\| x \|_p= \left(\sum_{n=1}^{\infty} \| x^n \|_{\alpha,p_n}^p\right)^{\frac{1}{p}} < \infty.$$

The direct sum of the spaces (X_{α,p_n}) in the sense of c_0 is the linear space

$$X_0 = \left(\sum_{n=1}^{\infty} \oplus X_{\alpha,p_n}\right)_0$$

of all sequences $x = \left(x^1, x^2, ...\right)$, $x^n \in X_{\alpha,p_n}$, $n = 1, 2, ...$for which $\lim_n \| x^n \|_{\alpha,p_n}= 0$ with the norm

$$\| x \|_0 = \max_n \| x^n \|_{\alpha,p_n} .$$

We follow the same notations and terminology as in [8]. The construction and idea of the proof follow [10] but the nature of these spaces is different. In fact these spaces are a rich class of spaces which depend on the sequences (α_i) and (p_n) as above.

Fix a sequence (α_i) of reals which satisfies the above conditions, and a sequence (p_n) of reals with $\infty > p_1 > p_2 > ... > 1$. Consider the sequence space X_p as above. For each $n \geq 1$, denote by $(\overline{e}_{i,n})_{i=1}^{\infty}$ the unit vector basis of X_{α,p_n} similar to usual unit vector basis of ℓ_1 and by $(e_{i,n})_{i=1}^{\infty}$ its natural copy in X_p :

$$e_{i,n} = (\underbrace{0, ...0}_{n-1}, \overline{e}_{i,n}, 0, ...) \in X_p.$$

Let $\delta_n > 0$ and $\Delta = (\delta_n)$ such that $\sum_{i=1}^{\infty} \delta_n^p = 1$ if $p \geq 1$, and $lim_n \delta_n = 0$ and $\max_n \delta_n = 1$ if $p = 0$. For each $i \geq 1$ put $z_i = \sum_{n=1}^{\infty} \delta_n e_{i,n}$. Then

$$\| z_i \|_p = \left(\sum_{n=1}^{\infty} \| \delta_n e_{i,n} \|_{\alpha,p_n}^p\right)^{\frac{1}{p}} = (\sum_{n=1}^{\infty} \delta_n^p)^{\frac{1}{p}} = 1.$$

Since $\| e_{i,n} \|_{\alpha,p} = 1$ and

$$\| z_i \|_0 = \max_n \| \delta_n e_{i,n} \|_{\alpha,p_n} = 1.$$

It is clear that for any sequence $(t_i)_{i=1}^{m}$ of scalars,

$$\| \sum_{i=1}^{m} t_i z_i \|_p^p = \sum_{n=1}^{\infty} \delta_n^p \| \sum_{i=1}^{m} t_i e_{i,n} \|_{\alpha,p_n}^p \text{ if } 1 \leq p < \infty$$

and

$$\| \sum_{i=1}^{m} t_i z_i \|_0 = \max \delta_n \| \sum_{i=1}^{m} t_i e_{i,n} \|_{\alpha,p_n} \text{ if } p = 0.$$

Let Z_p be the closed linear span of $(z_i)_{i=1}^{\infty}$. For each $I \subseteq \mathbb{N}$ the projection P_I denotes the natural projection of X_p onto $[e_{i,n} : i \in \mathbb{N}, n \in I]$. Denote also $Q_n = P_{\{n,n+1,...\}}$.

Definition 1.1. A Banach space X is hereditarily l_p if every infinite dimensional subspace of X contains a subspace isomorphic to l_p.

A Banach space X has the Schur property if norm convergence and weak convergence coincide. It is well known that l_1 has the Schur property.

Here is the main result of [4].

Theorem 1.2. *(i) the Banach space Z_p is hereditarily l_p for $p > 1$.*

(ii) for $p = 1$ the space Z_1 is hereditarily l_1 and fails the Schur property.

(iii) The space Z_0 is hereditarily c_0.

2 The results

Definition 2.1. We say that a Banach space X contains asymptotically isometric copies of ℓ_1 if for some sequence $\varepsilon_n \downarrow 0$ $(0 < \epsilon_n \leq 1)$, there is a norm-one sequence (x_n) in X such that for all m and scalars $(t_n : 0 \leq n \leq m)$

$$\sum_{n=0}^{m}(1-\varepsilon_n)|t_n| \leq \|\sum_{n=0}^{m} t_n x_n\| \leq \sum_{n=0}^{m}|t_n|, \quad (t_n) \in \ell_1.$$

In [5], we showed the Banach space $X_{\alpha,p}$ contains asymptotically isometric copies of ℓ_p. Now, we show Z_1 contains asymptotically isometric copies of ℓ_1. First, we recall the following lemma that obtained of proof of theorem 2.7 of [4](which is similar to proof of theorem 2.5 of [10]).

Lemma 2.2. *Let $\{\varepsilon_s\}$ be a real decreasing sequence such that $0 < \varepsilon_s \leq 1$ for all s. There exist a sequence $\{u_s\}$ of $S(Z_1)$ and a sequence of integers $1 \leq n_1 < n_2 < ...$ such that*
(i) $||u_s - Q_{n_s}u_s|| \leq \frac{\varepsilon_s}{4}$;
(ii) $||Q_{n_{s+1}}u_s|| \leq \frac{\varepsilon_s}{4}$.

Theorem 2.3. *Z_1 contains asymptotically isometric copies of ℓ_1.*

Proof. Let $\{\varepsilon_s\}$ be a real decreasing sequence such that for all s, $0 < \varepsilon_s \leq 1$. Using the previous lemma, we have a $\{u_s\} \subset S(Z_1)$ and a sequence of integers $1 \leq n_1 < n_2 < ...$ such that
(i) $||u_s - Q_{n_s}u_s|| \leq \frac{\varepsilon_s}{4}$;
(ii) $||Q_{n_{s+1}}u_s|| \leq \frac{\varepsilon_s}{4}$.

Put $v_s = Q_{n_s}u_s - Q_{n_{s+1}}u_s$ for $s \in \mathbb{N}$. Since $v_s = u_s - (u_s - Q_{n_s}u_s + Q_{n_{s+1}}u_s)$, then $||v_s|| \geq 1 - \frac{\varepsilon_s}{2}$. Then for each scalars $\{a_s\}_{s=1}^m$ one has

$$\sum_{s=1}^{m}(1-2\varepsilon_s)|a_s| \leq \sum_{s=1}^{m}|a_s|||v_s|| = ||\sum_{s=1}^{m} a_s v_s|| \leq \sum_{s=1}^{m}|a_s|.$$

But

$$||\sum_{s=1}^{m} a_s(u_s - v_s)|| \leq ||\sum_{s=1}^{m} a_s(u_s - Q_{n_s}u_s|| + ||\sum_{s=1}^{m} a_s Q_{n_{s+1}}u_s|| \leq$$

$$|| \leq \sum_{s=1}^{m}|a_s|||(u_s - Q_{n_s}u_s|| + \sum_{s=1}^{m}|a_s|||Q_{n_{s+1}}u_s|| \leq \sum_{s=1}^{m}|a_s|\frac{\varepsilon_s}{2}.$$

Then

$$||\sum_{s=1}^{m} a_s u_s|| \geq ||\sum_{s=1}^{m} a_s v_s|| - ||\sum_{s=1}^{m} a_s(u_s - v_s)||$$

$$\geq \sum_{s=1}^{m}(1-\frac{\varepsilon_s}{2}|a_s|-\sum_{s=1}^{m}\frac{\varepsilon_s}{2}|a_s|\geq \sum_{s=1}^{m}(1-\varepsilon_s)|a_s|.$$

□

Remark 2.4. *Recall by [7, p. 80] that for any family of Banach spaces $\{X_n : n \in \mathbb{N}\}$, If $p \geq 1$, $(\sum_n \oplus X_n)_p^* = (\sum_n \oplus X_n^*)_q$ where $\frac{1}{p}+\frac{1}{q}=1$, and If $p=0$, $(\sum_n \oplus X_n)_0^* = (\sum_n \oplus X_n^*)_1$.*
We know the Banach spaces X_{α,p_n} are dual spaces([1]). Let Y_{p_n} be the predual of X_{α,p_n}, that is, $Y_{p_n}^* = X_{\alpha,p_n}$. Then $(\sum_n \oplus Y_{p_n})_q^* = (\sum_n \oplus X_{\alpha,p_n})_p$. That is, $(\sum_n \oplus X_{\alpha,p_n})_p$, for $1 \leq p < \infty$, is a dual space with predual $(\sum_n \oplus Y_{p_n})_q$ where $\frac{1}{p}+\frac{1}{q}=1$.

Now we show that the subspace Z_p of $(\sum_n \oplus X_{\alpha,p_n})_p$ is a dual space.

Theorem 2.5. *The sequence (z_i) is a normalized boundedly complete basis for $Z_p(1 \leq p < \infty)$. Thus Z_p is a dual space.*

Proof. Suppose that (t_j) is a sequence of scalars such that, for each integer n, $\sup_n ||\sum_{j=1}^n t_j z_j|| = A$, for some $A \in \mathbb{R}$. we know that the basis of Z_p is (strictly) monotone. Then for any integers n and m with $n > m$, $||\sum_{i=1}^m t_i z_i|| < ||\sum_{i=1}^n t_i z_i||$. In the other word, $(||\sum_{i=1}^n t_i z_i||)_{n=1}^\infty$ is a strictly increasing and bounded sequence of real numbers. That is, $A = ||\sum_{j=1}^\infty t_j z_j||$. Then $\sum_{j=1}^\infty t_j z_j$ converge and by [8, 1.b.4] Z_p is a dual space. □

Note: Here, strictly is necessary. A simple example is the Banach space c_0. We know that for any integer n, $sup_n||\sum_{j=1}^n e_j|| = 1$ but $\sum_{j=1}^\infty e_j \notin c_0$.

Definition 2.6. We say that a Banach space X has Dunford- Pettis property if, for each couple weakly null sequences (x_n) and (x_n^*) in X and X^*, respectively, we have $lim_n x_n^*(x_n) = 0$.

Azimi in [3] showed that for $p \geq 1$, the Banach space $X_{\alpha,p}$ fails the Dunford Pettis property. Now, we show the Banach space Z_p $(1 \leq p < \infty)$ fails the Dunford Pettis property.

Theorem 2.7. *The Banach space Z_p $(1 \leq p < \infty)$ fails the Dunford Pettis property.*

Proof. Let $u_i = z_{2i} - z_{2i-1}$ and $f_i : Z_p \to \mathbb{R}$ such that for any $x = (x_1, x_2, ...) \in Z_p$ with $x_i = (x_{i,1}, x_{i,2}, ...) \in X_{\alpha,p_i}$, we have $f_i(x) = x_{1,i}$ for integers i. Then for $g_n = f_{2n} - f_{2n-1}$, we have $g_n(u_n) = 2\delta_1$. To complete the proof we need to show that $u_n \to 0$ weakly, and $g_n \to 0$ weakly. The first one follows from the fact that, for every increasing sequence (n_k) of integers, we have

$$\begin{aligned}
\lim_{k\to\infty}\frac{||u_{n_1}+u_{n_2}+...+u_{n_k}||}{k} &= \lim_{k\to\infty}\frac{(\sum_{n=1}^{\infty}\delta_n^p(\sum_{i=1}^{2k}\alpha_i)^{\frac{p}{p_n}})^{\frac{1}{p}}}{k}\\
&\le \lim_{k\to\infty}\frac{(\sum_{n=1}^{\infty}\delta_n^p(\sum_{i=1}^{2k}\alpha_i)^{p})^{\frac{1}{p}}}{k}\\
&= \lim_{k\to\infty}\frac{(\sum_{i=1}^{2k}\alpha_i)(\sum_{n=1}^{\infty}\delta_n^p)^{\frac{1}{p}}}{k}\\
&= \lim_{k\to\infty}\frac{\sum_{i=1}^{2k}\alpha_i}{k}=0.
\end{aligned}$$

It remains to show that $g_n \to 0$ weakly. If not there are $F \in Z_p^{**}$ with $||F|| = 1$, $\delta > 0$ and a subsequence (g_{n_k}) such that $F(g_{n_k}) > \delta$ for all integers k. So for integer N we have $\sum_{k=1}^{N} F(g_{n_k}) > N\delta$ and hence

$$\frac{||\sum_{k=1}^{N} g_{n_k}||}{N} > \delta.$$

This implies that for any integer N, there exist $x^N = (x_1^N, x_2^N, ...) \in Z_p$ with $x_i^N = (x_{i,1}^N, x_{i,2}^N, ...) \in X_{\alpha,p_i}$ such that

$$\frac{1}{N}\sum_{k=1}^{N} g_{n_k}(x^N) > \delta.$$

We have $\lim_{n\to\infty} x_{1,n}^N = 0$ for integer N, since $\sum_{i=1}^{\infty}\alpha_i = \infty$. Therefore

$$\begin{aligned}
|\frac{1}{N}\sum_{k=1}^{N} g_{n_k}(x^N)| &= \frac{1}{N}|\sum_{k=1}^{N}(x_{1,2n_k}^N - x_{1,2n_k-1})^N|\\
&\le \frac{1}{N}|\sum_{k=1}^{N}|x_{1,2n_k}^N| + \frac{1}{N}\sum_{k=1}^{N}|x_{1,2n_k-1}^N| \to 0
\end{aligned}$$

as $N \to \infty$ which is a contradiction. □

3 The dual and predual of $X_{\alpha,p}$.

Some properties of the dual and predual of $X_{\alpha,1}$ and $X_{\alpha,p}$ have been studied in [2] and [5]. We give now a direct proof to show $X_{\alpha,p}^*$ is nonseperable.

Theorem 3.1. *For* $1 \le p < \infty$, $X_p^* = (\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n})_p^*$ *is nonseparable.*

Proof. Let $\{F_i\}$ be a sequence of blocks of integer such that $maxF_i < minF_{i+1}$ and $F = (F_1, F_2, ...)$. Now, for $x = (x_1, x_2, ...) \in Z_p$, we define the linear functional

$$f_F(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} < x_n, F_i >$$

on Z_p.

Let F_ϕ be a finite block of integer and x_ϕ be a corresponding unit vector in Z_p such that $1 = \|x_\phi\|$ and x_ϕ is normed by F_ϕ. We know, $(x_\phi)_n$ is normed by F_ϕ. Now, select blocks F_0 and F_1 disjoint from each other and disjoint from F_ϕ such that $maxF_\phi < minF_0$ and $maxF_\phi < minF_1$. Now, we select x_0 and x_1 in Z_p such that

$$1 = \|x_0\| \qquad , \qquad 1 = \|x_1\|.$$

and x_0 is normed by F_0 and x_1 is normed by F_1

We select F_{00} and F_{01} disjoint from each other and disjoint from F_0 such that

$$maxF_0 < minF_{00} \qquad , \qquad maxF_0 < minF_{01}.$$

select x_{00} and x_{01} such that

$$1 = \|x_{00}\| \qquad , \qquad 1 = \|x_{01}\|.$$

and x_{00} is normed by F_{00} and x_{01} is normed by F_{01}. We select F_{10} and F_{11} disjoint from each other and disjoint from F_1 such that

$$maxF_1 < minF_{10} \qquad , \qquad maxF_1 < minF_{11}.$$

select x_{10} and x_{11} such that

$$1 = \|x_{10}\| \qquad , \qquad 1 = \|x_{11}\|.$$

and x_{10} is normed by F_{10} and x_{11} is normed by F_{11}. In an obvious way we correspond to the dyadic tree, $T = \bigcup_{n=0}^{\infty}\{0,1\}^n$ disjoint sets

$$F_{10}, F_{11}, F_{000}, F_{001}, F_{010}, F_{011}, ...$$

of integers and corresponding sequences $x_{10}, x_{11}, x_{000}, x_{001}, x_{010}, x_{011}, ...$ as above.

Since for any two branches $F^1 = (F_\phi, F_0, F_{00}, ...)$ and $F^2 = (F_\phi, F_0, F_{01}, ...)$ we have

$$f_{F^1}(x_{00}) = 1 \qquad , \qquad f_{F^2}(x_{00}) = 0$$

hence $\|f_{F^1} - f_{F^2}\| \geq 1$.

Assertion of theorem follows from the fact that the set of all branches is uncountable. so Z_p^* is not separable. □

Definition 3.2. Let X be a linear space and C be a convex subset of X. A point $x \in C$ is said to be an extreme point of C if and only if $C \backslash \{x\}$ is still convex, that is, if any time $x = \lambda x_1 + (1-\lambda)x_2$ where $x_1, x_2 \in C$ and $0 < \lambda < 1$, then it must be that $x = x_1 = x_2$. Given such a set C, $ext(C)$ will denote the set of all extreme points of C.

Definition 3.3. Let L be a linear space and $A \subseteq L$. By convex hull of A, which we will denote by co(A), we mean the smallest convex subset of L containing A.

We will use the following theorem of Krein-Milman :

Theorem 3.4. *Let X be a locally convex linear topological space and C be a compact, convex subset of X . Then C contains extreme points. Moreover, $C = \overline{co}(ext(C))$. That is, any closed convex set is the closed convex hull of its extreme points.*

By use of Banach-Alaoglu theorem, the unit ball of $(\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n})_p$ is weak*-compact set in $(\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n})_p$. Since this set is obviously convex as well, we have

Theorem 3.5. *The closed unit ball of the dual space of a normed linear space is the weak*-closed convex hull of its extreme points.*

Since $(\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n})_p$, $(p \geq 1)$ is a dual space, by using the previous theorem we have

Theorem 3.6. *The closed unit ball of $(\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n})_p$, $(p \geq 1)$ is the weak*-closed convex hull of its extreme points.*

Dedicated to: the memory of professor Parviz Azimi

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