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Bounds of Stanley depth

Dorin Popescu

Abstract

We answer positively a question of Asia Rauf for the case of intersections of three prime ideals generated by disjoint sets of variables and we present several inequalities on Stanley depth.

This is a detailed presentation of our talk at the conference on "Fundamental structures of algebra" in honor of Prof. Serban Basarab at his 70-th anniversary. Let $S = K[x_1, \ldots, x_n]$ be a polynomial algebra over a field K, $I \subset J \subset S$ two monomial ideals and $M = J/I$. The depth of M is a homological invariant and depends on the characteristic of the field K. For example if I is the Stanley-Reisner ideal associated to the triangulation of the projective real plane $\mathbf{P}^2_{\mathbf{R}}$ then $\operatorname{depth} S/I = 3$ if and only if the characteristic of K is not 2, otherwise $\operatorname{depth} S/I = 2$ (see [16]). This is because the singular homology $\tilde{H}_1(\mathbf{P}^2_{\mathbf{R}}; K) = 0$ if and only if the characteristic of K is not 2, otherwise $\tilde{H}_1(\mathbf{P}^2_{\mathbf{R}}; K) = K$. In 1982 Stanley [18] introduces a new invariant the so-called *the Stanley depth*, which is combinatorially defined and so does not depend on the characteristic of the field K. Given a monomial $u \in (J \setminus I)$ and $Z \subset \{x_1, \ldots, x_n\}$, we say that $\hat{u}K[Z]$, $\hat{u} = u + J$, is a *Stanley space* of dimension $|Z|$ if it is free over $K[Z]$. A *Stanley decomposition* of J/I is a finite direct sum of Stanley spaces, $\mathcal{D}: \quad J/I = \oplus_{i=1}^{s} u_i K[Z_i]$, and we call $\operatorname{sdepth} \mathcal{D} = \min\{|Z_i|\}$ the Stanley depth of $\mathcal{D}$. For example the Stanley decomposition $\mathcal{D} : K[x, y]/(x^2, xy) = K[y] \oplus xK$ has $\operatorname{sdepth} \mathcal{D} = 0$. We define

$$\operatorname{sdepth}_S J/I = \max\{\operatorname{sdepth} \mathcal{D} : \mathcal{D} \text{ Stanley decomposition of } J/I\}.$$

There exists an infinite set of Stanley decompositions and apparently it is impossible to find sdepth in general. Herzog-Vladoiu-Zheng [5] reduced the problem to find a partition of a finite ordered set. Stanley conjectured that

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sdepth $J/I \geq$ depth J/I. In [11] and [12] we showed that if $n \leq 5$ and either $J = S$ or $I = 0$ then the Stanley's Conjecture holds. Stanley depth shares some common properties, as Apel noticed [1], with the usual depth as, for example,

$$\text{sdepth}\, S/I \leq \min_{P \in \text{Ass}\, S/I} \dim S/P,$$

where Ass J/I denotes the associated prime ideals of J/I.

A. Rauf stated in [15] the following result:

Proposition 1. $\text{depth}_S\ S/(I:v) \geq \text{depth}_S\ S/I$, *for each monomial* $v \notin I$.

It is worth to mention that these results hold only in monomial frame. One could think about similar questions on Stanley depth. The following proposition can be seen as a possible analog of the above proposition and it is given in the arXiv version of [12] but not in the printed version, where the paper had to be shorter.

Proposition 2. $\text{sdepth}_S\ (I:v) \geq \text{sdepth}_S\ I$ *for each monomial* $v \notin I$.

Proof. By recurrence it is enough to consider the case when v is a variable, let us say $v = x_n$. Let $\mathcal{D}:\ I = \oplus_{i=1}^r u_i K[Z_i]$ be a Stanley decomposition of I such that sdepth $\mathcal{D} = \text{sdepth}_S\ I$. We will show that

$$\mathcal{D}':\ (I:x_n) = (\oplus_{x_n|u_i}\ (u_i/x_n)K[Z_i]) \oplus (\oplus_{u_j \notin (x_n), x_n \in Z_j}\ u_j K[Z_j])$$

is a Stanley decomposition of $(I:x_n)$. Indeed, if a is a monomial such that $x_n a \in I$ then we have $x_n a = u_i w_i$ for some i and a monomial w_i of $K[Z_i]$. If $x_n \nmid u_i$ then $x_n | w_i$ and so $x_n \in Z_i$. If $x_n | u_i$ then $a = (u_i/x_n) w_i$, which shows that

$$(I:x_n) = (\Sigma_{x_n|u_i}\ (u_i/x_n)K[Z_i]) + (\Sigma_{u_j \notin (x_n), x_n \in Z_j}\ u_j K[Z_j]).$$

It remains to show that the above sum is direct. If $x_n | u_i$, $u_j \notin (x_n)$, $x_n \in Z_j$ and $u_j w_j = (u_i/x_n) w_i$ for some monomials $w_j \in K[Z_j]$, $w_i \in K[Z_i]$ then $u_j(x_n w_j) = u_i w_i$ belongs to $u_i K[Z_i] \cap u_j K[Z_j]$, which is not possible.

Thus $\mathcal{D}'$ is a Stanley decomposition of $(I:x_n)$ with sdepth $\mathcal{D}' \geq$ sdepth $\mathcal{D} = \text{sdepth}_S\ I$, which ends the proof. □

Corollary 3. *(*Ishaq, [8]*) Let* $I \in S$ *be a monomial ideal with* $\text{Ass}(S/I) = \{P_1, \ldots, P_s\}$. *Then* $\text{sdepth}(I) \leq min\{\text{sdepth}(P_i) : 1 \leq i \leq s\}$.

Proof. (After [8]) Let $P_i \in \text{Ass}(S/I)$. Then P_i is a monomial ideal and there exists a monomial $w_i \in I$ such that $I : w_i = P_i$. By the above proposition, we have $\text{sdepth}(I) \leq \text{sdepth}(I : w_i) = \text{sdepth}(P_i)$. □

Another interesting result of Ishaq is the following

Theorem 4. *(*Ishaq, [7]*)* $\operatorname{sdepth}(J/I) \leq \operatorname{sdepth}(\sqrt{J}/\sqrt{I})$.

When $J = S$ the result is given in [2], or in the case of depth in [3].

Next we present some bounds of $\operatorname{sdepth}(I), \operatorname{sdepth}(S/I)$ given when I has a small number of primary components.

Theorem 5. *(*Popescu-Qureshi, [14]*) Let Q, Q' be two primary monomial ideals of S. If $Q + Q'$ is the maximal ideal of S then* $\operatorname{sdepth} S/(Q \cap Q') \leq$

$$\max\{\min\{\dim S/Q', \lceil \frac{\dim(S/Q)}{2} \rceil\}, \min\{\dim(S/Q), \lceil \frac{\dim(S/Q')}{2} \rceil\}\},$$

and the equality holds when Q, Q' are irreducible (for example prime).

Always we can reduce the problem to the case when $Q+Q'$ is the maximal ideal of S, since a free variable increases depth and sdepth by 1 as it is showed in [5].

Corollary 6. *If Q, Q' are irreducible monomial ideals then the Stanley's Conjecture holds for $S/(Q \cap Q')$.*

Theorem 7. *(*Popescu-Qureshi, [14]*) If Q, Q' are irreducible monomial ideals and $Q + Q'$ is the maximal ideal of S then*

$$\operatorname{sdepth} Q \cap Q' \geq \lceil \frac{\dim(S/Q)}{2} \rceil + \lceil \frac{\dim(S/Q')}{2} \rceil.$$

Corollary 8. *Let Q, Q', Q'' be irreducible monomial ideals then the Stanley's Conjecture holds for $Q \cap Q'$ and $S/(Q \cap Q' \cap Q'')$.*

The above corollary is completed by Adrian Popescu as follows:

Theorem 9. *(*A. Popescu, [10]*) The Stanley's Conjecture holds for intersections of three prime ideals.*

The proof of the above theorem relies on a special Stanley decomposition which we extend in [13]. Let $r < n$ be a positive integer and $S' = K[x_{r+1}, \ldots, x_n]$, $S'' = K[x_1, \ldots, x_r]$. We suppose that one prime ideal P_i is generated in some of the first r variables. If $P_i = (x_1, \ldots, x_r)$ we say that P_i is a *main prime*. For a subset $\tau \subset [s]$ we set

$$S_\tau = K[\{x_i : 1 \leq i \leq r, x_i \notin \Sigma_{i\in\tau} P_i\}]$$

and let $\mathcal{F}$ be the set of all nonempty subsets $\tau \subset [s]$ such that

$$L_\tau = (\cap_{i\in\tau} P_i) \cap S' \neq (0), J_\tau = (\cap_{i\in[s]\setminus\tau} P_i) \cap S_\tau \neq (0).$$

For $\tau \in \mathcal{F}$ we consider the ideals $I_0 = (I \cap K[x_1, \dots, x_r])S$, and

$$I_\tau = J_\tau L_\tau S_\tau[x_{r+1}, \dots, x_n].$$

Define the integers

$$A_\tau = \operatorname{sdepth}_{S_\tau[x_{r+1},\dots,x_n]} I_\tau \geq \operatorname{sdepth}_{S_\tau} J_\tau + \operatorname{sdepth}_{S'} L_\tau$$

and $A_0 = \operatorname{sdepth}_S I_0$ if $I_0 \neq (0)$. Then

Theorem 10. *(*D. Popescu, [13]*)* $\operatorname{sdepth}_S I \geq \min\{A_0, \{A_\tau\}_{\tau\in\mathcal{F}}\}$.

Corollary 11. *(*D.Popescu,[13]*) The Stanley's Conjecture holds for intersections of four prime ideals.*

Our Theorem 10 has also some limits which can be seen in the next example.

Example 12. *([13]) Let* $n = 10$,

$$P_1 = (x_1, \dots, x_7), P_2 = (x_3, \dots, x_8),$$

$$P_3 = (x_1, \dots, x_4, x_8, \dots, x_{10}),$$

$$P_4 = (x_1, x_2, x_5, x_8, x_9, x_{10}),$$

$$P_5 = (x_5, \dots, x_{10}).$$

We have $P_1+P_3 = P_2+P_3 = P_1+P_4 = P_2+P_4 = P_3+P_5 = P_1+P_5 = m$, $P_2 + P_5 = m \setminus \{x_1, x_2\}$, $P_3 + P_4 = m \setminus \{x_6, x_7\}$, $P_4 + P_5 = m \setminus \{x_3, x_4\}$, $P_1 + P_2 = m \setminus \{x_9, x_{10}\}$. *We have* $t(I) = 2$, *where* $t(I)$ *is the big size of* I *(see Definition [13]), and* $\operatorname{depth}_S S/I = 4$. *Applying Proposition 10 for* P_1 *as main prime we see that* $A^{(1)}_{3,4} \geq 3$, *that is* A_τ *for* $\tau = \{3, 4\}$. *Indeed,*

$$A^{(1)}_{3,4} \geq \operatorname{sdepth}_{K[x_6,x_7]}(x_6, x_7)K[x_6, x_7]+$$

$$+\operatorname{sdepth}_{K[x_8,x_9,x_{10}]}(x_8, x_9, x_{10})K[x_8, x_9, x_{10}] = 3.$$

Similarly choosing P_2 *as a main prime we get* $A^{(2)}_{3,4} \geq 3$ *and taking* P_3,P_4 *as main primes we get* $A^{(3)}_{2,5} \geq 3$, *respectively* $A^{(4)}_{2,5} \geq 3$. *Thus from these we cannot conclude that* $\operatorname{sdepth}_S I \geq \operatorname{depth}_S I$. *Fortunately, choosing* P_5 *as a main prime one can see that all* $A_\tau \geq 4$, *which is enough.*

Let $I = \cap_{i=1}^s P_i$, $s \geq 2$ be a reduced intersection of monomial prime ideals of S. We assume that $\Sigma_{i=1}^s P_i = m = (x_1, \dots, x_n)$.

Definition 13. Let e be the minimal number such that there exists e-prime ideals among (P_i) whose sum is m. After Lyubeznik the *size* of I is $e-1$. We call the *big size* of I the minimal number $t = t(I) < s$ such that the sum of all possible $(t+1)$-prime ideals of $\{P_1, \ldots, P_s\}$ is m. In particular, there exist $1 \leq i_1 < \ldots < i_t \leq s$ such that $\Sigma_{k=1}^t P_{i_k} \neq m$ and for all $j \in [s] \setminus \{i_1, \ldots, i_t\}$ we have $P_j + \Sigma_{k=1}^t P_{i_k} = m$. Clearly the big size of I is bigger than the size of I.

Remark 14. By Lyubeznik, $\operatorname{depth}_S S/I$ is always greater than the size of I and so if the size of I is 1 then necessary $\operatorname{depth}_S I \geq 2$.

Example 15. Let $n = 5$, $s = 4$, $P_1 = (x_1, x_5)$, $P_2 = (x_2, x_5)$, $P_3 = (x_3, x_5)$, $P_4 = (x_1, x_2, x_3, x_4)$. Since $P_1 + P_2 + P_3 \neq m$ the big size of $I = \cap_{i=1}^4 P_i$ is 3 but $\operatorname{depth}_S S/I = 1$ because $P_i + P_4 = m$ for all $1 \leq i \leq 3$.

Corollary 16. *If the big size of I is 1 then the Stanley's Conjecture holds for I.*

It is easy to see that the above corollary holds for $n \leq 2$. If $n \geq 3$ then $\operatorname{sdepth}_S I \geq 2 = \operatorname{depth} I$ by Fløysted and Herzog [4]. A different proof is done in [13] using Theorem 10. This theorem is extended for all monomial ideals and has the following consequence:

Theorem 17. *(*Herzog, Popescu, Vladoiu, [6]*) sdepth $I \geq 1 +$ size I.*

Next we present some results on intersections of prime ideals generated by disjoint sets of variables. A helpful result is the following:

Theorem 18. *(*D. Popescu, [13]*) Let $I = \cap_{i=1}^s P_i$ be a reduced intersection of monomial prime ideals of S. Assume that $P_i \not\subset \Sigma_{1=j\neq i}^s P_j$ for all $i \in [s]$. Then*

$$\operatorname{sdepth}_S I \geq s = \operatorname{depth}_S I,$$

that is the Stanley's Conjecture holds for I.

The above result is useful to show the following:

Theorem 19. *(*Ishaq, [8]*) Let I be a monomial ideal such that the prime ideals of* $\operatorname{Ass} S/I$ *are generated by disjoint sets of variables. Then the Stanley's Conjecture holds for I and S/I.*

When I is square free the above theorem is stated in [10]. A. Rauf [15] asked if $\operatorname{sdepth} I \geq 1 + \operatorname{sdepth} S/I$. When I is the intersection of two irreducible monomial ideals, this question has a positive answer (see [14]).

Theorem 20. *Let* $1 \leq r \leq e \leq q$ *be some integers such that* $n = r + e + q$ *and assume that* $P_1 = (x_1, \ldots, x_r)$, $P_2 = (x_{r+1}, \ldots, x_{r+e})$, $P_3 = (x_{r+e+1}, \ldots, x_{r+e+q})$ *and* $I = P_1 \cap P_2 \cap P_3$. *Then*

1. $\operatorname{sdepth}_S\ I \geq \operatorname{sdepth}_S S/I$,
2. *moreover* $\operatorname{sdepth}_S\ I \geq 1 + \operatorname{sdepth}_S S/I$ *except possible in the case when either* $r = e$ *is even and* q *is even, or* r *is odd and* $e = r + 1$.

Proof. Choose P_1 to be main prime and apply Theorem 10. Set A_2, S_2, J_2, L_2 for $\tau = \{2\}$ and similarly for $\tau = \{3\}$ or $\tau = \{2, 3\}$. Note that $S_2 = S_3 = S_{23} = S''$ and $J_2 = J_3 = 0$, $J_{2,3} = P_1 \cap S''$. Then

$$A_{23} \geq \operatorname{sdepth}_{S_2}(P_1 \cap S_{23}) + \operatorname{sdepth}_{S'}(P_2 \cap P_3 \cap S') \geq \lceil \frac{r}{2} \rceil + \lceil \frac{q+e}{2} \rceil,$$

the inequality being strict by [7, Corollaries 2.9, 2.10] (see also [17]) if q, e are not both even, and $\lceil \frac{r}{2} \rceil$ denotes the smallest upper integer greater than $r/2$. It follows that $A_{23} \geq 1 + r + \lceil \frac{q}{2} \rceil$ except possible when $r = e$ is even and q is even. Using the next proposition $\operatorname{sdepth}_S S/I \leq r + \lceil \frac{q}{2} \rceil$ except possible when $e = r + 1$ and r is odd. Hence $\operatorname{sdepth}_S\ I \geq 1 + \operatorname{sdepth}_S S/I$ except possible in the cases when either $r = e$ is even and q is even, or r is odd and $e = r + 1$. In these two cases we may have only $\operatorname{sdepth}_S\ I \geq \operatorname{sdepth}_S S/I$. Finally, $A_0 = \operatorname{sdepth}_{S''}(I \cap S'') + n - r \geq 1 + \dim S/P_1 \geq 1 + \operatorname{sdepth}_S S/I$ if $I \cap S'' \neq 0$. The proof ends by applying Theorem 10. □

Proposition 21. *(*Ishaq,[8]*) In the hypothesis of the above theorem it holds*

$$\operatorname{sdepth}_S S/I < 1 + r + \min\{e, \lceil \frac{q}{2} \rceil\},$$

except in the case r *is odd and* $e = r+1$ *when the upper bound could be possible reached.*

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Institute of Mathematics "Simion Stoilow", University of Bucharest,
P.O.Box 1-764, Bucharest 014700, Romania
e-mail: dorin.popescu@imar.ro