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A survey of $\mathcal{RF}$-theory

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Abstract

Given a group G, $\mathcal{RF}$-theory constructs a new group $\mathcal{RF}(G)$ and an action $\mathcal{RF}(G) \to \mathrm{Isom}(\mathbf{X}_G)$ by isometries of $\mathcal{RF}(G)$ on a canonically associated $\mathbb{R}$-tree $\mathbf{X}_G$. We give an overview mostly focusing on the more basic aspects of this theory, up to and including the universal property of $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees.

Introduction

The present paper represents an extended version of a talk given during the International Conference & Humboldt Kolleg on Fundamental Structures of Algebra in honour of Professor Şerban Basarab's seventieth birthday, which took place 14-18 April 2010 in Ovidius University, Constanta (Romania).

No attempt has been made to cover the theory of Λ-trees itself, which is underlying the work reported here; the reader can find a convenient introduction into those aspects of particular importance for $\mathcal{RF}$-theory in [9, App. A], while the book [6] by Chiswell remains the standard reference for all aspects of Λ-tree theory. We also mention Shalen's earlier article [27], which provides a highly readable account of some basic aspects of $\mathbb{R}$-trees.

The original motivation behind $\mathcal{RF}$-theory was to introduce a class of groups with canonically associated $\mathbb{R}$-tree action by mimicking in a continuous setting the (old-fashioned) construction of free groups as sets of reduced words with reduced multiplication as the group law; the hope being that these new groups might turn out to have some important universal property vaguely corresponding to the universality of free groups for the class of all discrete groups with respect to epimorphisms. Somewhat more specifically, given a (discrete) group G, $\mathcal{RF}$-theory constructs a new group $\mathcal{RF}(G)$ and an action by isometries of $\mathcal{RF}(G)$ on a canonically associated $\mathbb{R}$-tree $\mathbf{X}_G$, and investigates properties of these groups and $\mathbb{R}$-trees.

Analysis of the groups $\mathcal{RF}(G)$ and their associated $\mathbb{R}$-trees $\mathbf{X}_G$ is difficult, and far from complete. At the time of writing, the present author has thought about this project off and on for more than six years, in part together with Ian Chiswell and Jan-Christoph Schlage-Puchta. During this time, starting from humble beginnings, a rich and deep theory has begun to unfold and, most recently, the long sought universal property of $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees has been identified: $\mathcal{RF}$-theory provides universal objects (with respect to inclusion) for free $\mathbb{R}$-tree actions; cf. [8]. Our survey draws a line at this point, focusing on those parts of the theory which by now appear stable and sufficiently well developed: cyclic reduction with some of its consequences, the classification of bounded subgroups, conjugacy and centralizers of hyperbolic elements, universality, some functorial aspects, and the theory of test functions with certain of its applications.

Proofs are mostly avoided, the reader being referred instead to some convenient place in the literature to read up on such details if desired (mostly Chiswell's and my recent book [9], or the paper on test function theory [23]). Those few arguments given usually serve the purpose of familiarizing the reader with an important definition, like that of a reduced function, a test function, local incompatibility, etc, or to illustrate the power of one of the more important results.

It was my pleasure to speak on this work in Constanta, and I would like to thank Professor Basarab and the organizers for the opportunity, and for inviting this report. Last, but certainly not least, the author would like to wish Professor Basarab many more happy and fruitful years, resulting in many more good theorems.

1 $\mathbb{R}$-free groups

The simplest case of Bass-Serre theory, which describes the structure of a group acting on a (simplicial) tree in terms of standard constructions of combinatorial group theory, occurs when the action is free. One has to assume here, of course, that no group element is an inversion; i.e., interchanges the endpoints of an edge, so that one can form the quotient graph for the action. It is not clear at present, whether there exists a useful analogue of Bass-Serre theory for actions on Λ-trees, where Λ is an ordered abelian group different from $\mathbb{Z}$; not even for $\Lambda = \mathbb{R}$, a case of particular interest to us here.* However, the analogy with Bass-Serre theory does at least suggest that one should begin by studying actions on Λ-trees, which are free and without inversion.

The principal purpose of this section is to draw together what is known at present concerning groups having a free action on some $\mathbb{R}$-tree (called $\mathbb{R}$-

*Cf. Basarab's article [3] in this context.

free groups for short); our main motivation for this being the universality of $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees, which is discussed in Section 6. We begin with the more general concept of a Λ-free group, where Λ is some ordered abelian group, and present results known to hold for arbitrary Λ, before concentrating on the class of $\mathbb{R}$-free groups. Note that an action by isometries on an $\mathbb{R}$-tree (or, more generally, on a Λ-tree with $\Lambda = 2\Lambda$) is automatically without inversion; cf. Lemma 1.2 in [6, Chap. 3].

1.1 Λ-free groups

As usual, let Λ denote a (totally) ordered abelian group. By a Λ-*free* group, we mean a group G having a free action without inversions on some Λ-tree (X, d). This is equivalent to requiring that, for any base point $x_0 \in X$, the displacement function $L_{x_0} : G \to \Lambda$ given by

$$L_{x_0}(g) = d(x_0, gx_0), \quad g \in G,$$

satisfies $L_{x_0}(g^2) > L_{x_0}(g)$ for every non-trivial group element g (length functions with this property are usually called *free*). A group is call *tree-free*, if it is Λ-free for some ordered abelian group Λ; tree-free groups are automatically torsion-free. Clearly, every subgroup of a Λ-free group is itself Λ-free. Also, if a group G acts freely and without inversions on a Λ-tree (X, d), and Λ embeds into an ordered abelian group Λ', then G acts freely and without inversions on the Λ'-tree $\Lambda' \oplus_\Lambda X$; cf., for instance, Lemma 2.1 in Chapter 3 of [6]. The abelian group Λ itself, and hence each of its subgroups is Λ-free, since Λ acts freely and without inversions on itself (viewed as a Λ-tree with metric induced by the absolute value) by translation; in particular, every torsion-free abelian group of rank at most $2^{\aleph_0}$ is $\mathbb{R}$-free, since these are precisely the groups which embed into the additive reals. Also, free groups are $\mathbb{Z}$-free, and thus $\mathbb{R}$-free by the corresponding remark above. A somewhat less trivial result is the following.

Proposition 1.1. *If $\{G_\sigma\}_{\sigma \in S}$ is a family of Λ-free groups, then the free product $G = \ast_{\sigma \in S} G_\sigma$ is again Λ-free.*

See [6, Chap. 5, Prop. 1.1] for the (rather straightforward) proof of this result. Proposition 1.1 again implies that free groups are $\mathbb{R}$-free, making use of the fact that the infinite-cyclic group $\mathbb{Z}$ is $\mathbb{R}$-free.

Denote by $\mathfrak{TF}(\Lambda)$ the class of groups consisting of all Λ-free groups. The only such class which is completely understood is $\mathfrak{TF}(\mathbb{Z})$; it consists precisely of all free groups. This fact follows immediately from Bass-Serre Theory, but goes back at least to Reidemeister and the early 1930s. By our remarks above, $\mathfrak{TF}(\Lambda)$ consists of torsion-free groups, is closed under taking subgroups and forming free products, and contains Λ itself. Call a group G *commutative-transitive*, if

$$[g,h] = [h,k] = 1 \text{ and } h \neq 1 \implies [g,k] = 1, \quad g,h,k \in G.$$

All tree-free groups are commutative-transitive; see [6, Lemma 5.1.2]. This result, as well as Proposition 1.1, essentially go back to Harrison's 1972 paper [15]. The main result of Harrison's paper however lies considerably deeper and characterizes $\mathbb{R}$-free groups with at most two generators. Her result was later generalized to the case of arbitrary Λ by Chiswell [7] and by Urbański and Zamboni [29] to give the following.

Theorem 1.2. *Let G be a group acting freely and without inversions on a Λ-tree, and let $g,h \in G - \{1\}$ be two non-trivial elements. Then $\langle g,h\rangle$ is either free of rank 2 or abelian (and one can refine the result by distinguishing between these cases in terms of the elements g,h).*

1.2 The class $\mathfrak{T}\mathfrak{F}(\mathbb{R})$

We now take a closer look at the case where $\Lambda = (\mathbb{R},+)$, which is of particular interest to us here. Like the other classes $\mathfrak{T}\mathfrak{F}(\Lambda)$ for $\Lambda \neq \mathbb{Z}$, $\mathfrak{T}\mathfrak{F}(\mathbb{R})$ is far from being understood; however, due to deep lying results of Morgan and Shalen in the one direction and Rips in the other, we can at least characterise the finitely generated members of this class. In the positive direction, we have the following.

Theorem 1.3. *The fundamental group of a closed surface (i.e., a compact connected 2-manifold without boundary) is $\mathbb{R}$-free, except for the non-orientable surfaces of genus 1, 2, and 3 (the connected sum of 1, 2, or 3 real projective planes).*

This result is due to Morgan and Shalen [20]. Its difficult proof uses deep ideas and results from the theory of measured geodesic laminations on hyperbolic surfaces; see Sections 4.4 and 4.5 in [6, Chap. 4] for a readable account of these ideas and techniques. The proof of Theorem 1.3 itself is sketched in Section 5.4, pp. 234–236 of the last reference. In the orientable case, a considerably simpler argument is known thanks to Stallings [28], using measured foliations rather than measured geodesic laminations.

The fact that the fundamental group of a non-orientable surface of genus n is not $\mathbb{R}$-free for $1 \leq n \leq 3$, is more elementary. For $n = 1$, the group in question is cyclic of order 2, thus not torsion-free. For $n = 2$, the group (usually denoted Γ_2^-) has the presentation

$$\Gamma_2^- \cong \big\langle x,y \,\big|\, x^2y^2 = 1\big\rangle \cong \big\langle a,b \,\big|\, a^2 = b^2\big\rangle,$$

so is the free product of two infinite-cyclic groups amalgamated along their subgroups of index 2 (the corresponding surface is often referred to as the "Klein bottle"). Γ_2^- is non-abelian and, its abelianisation being isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, is also not free. The fact that Γ_2^- is not $\mathbb{R}$-free follows now from Theorem 1.2. For $n = 3$, the corresponding group Γ_3^- is easily seen to admit the presentation

$$\Gamma_3^- \cong \big\langle a, b, c \,\big|\, [a, b] = c^2 \big\rangle,$$

thus is a free product with amalgamation of the free group $F = \langle a, b\rangle$ and the infinite-cyclic group generated by c, amalgamating the cyclic subgroups generated by $[a, b]$ and c^2. In this case, the argument is slightly less trivial, involving the ideas of characteristic set and hyperbolic length, which are standard in $\mathbb{R}$-tree theory; cf. [6, Theorem 5.4.5] for details. In what follows, we shall refer to the groups $\Gamma_1^-, \Gamma_2^-, \Gamma_3^-$ as *exceptional* surface groups, all other surface groups are called *non-exceptional*.

Coupling Proposition 1.1 with Theorem 1.3 plus the fact that free-abelian groups of finite rank embed into the additive reals (and hence are $\mathbb{R}$-free), we find in particular that each free product of finitely many groups, each of which is either free-abelian of finite rank, or a non-exceptional surface group, is a finitely generated $\mathbb{R}$-free group. The perhaps best result to date concerning $\mathbb{R}$-free groups, essentially due to Rips, provides a converse to the last statement, whence the characterisation of finitely generated $\mathbb{R}$-free groups mentioned above.

Theorem 1.4. *A finitely generated $\mathbb{R}$-free group G can be decomposed as a free product $G = G_1 * \cdots * G_s$, where each G_σ is either free-abelian of finite rank, or a non-exceptional surface group.*

In 1991, at a conference on the Isle of Thorns (a conference center of the University of Sussex, England), Rips outlined a proof of Theorem 1.4 which, characteristically, was not published; however, versions of it appeard a few years later in [5] and [14]. Rips' basic idea, which dominates all published proofs of this result, is to associate with an $\mathbb{R}$-tree action what is known as a *system of isometries*: a disjoint union D of finitely many finite trees (i.e., $\mathbb{R}$-trees spanned by a finite set of points), together with finitely many isometric isomorphisms $\varphi_j : A_j \to B_j$, where A_j, B_j are finite subtrees of D. To such a system one can associate a group G and, by taking a direct limit, it is enough to show that G has the form claimed in Theorem 1.4. The reader is referred to Chapter 6 of [6], which presents, with some modifications, the argument given by Gaboriau, Levitt, and Paulin in [14], making use of ideas from [13] and [26].

As concerns arbitrary (not necessarily finitely generated) $\mathbb{R}$-free groups, by what has been said so far, the following might appear as a reasonable conjecture:

(∗) *Every* $\mathbb{R}$*-free group decomposes as a free product of (non-exceptional) surface groups and subgroups of the additive reals.*

Counterexamples to (∗) have been given by Dunwoody [12] and by Zastrow [31]; see pp. 231–232 in [6, Chap. 5] for a sketch of Zastrow's argument, which involves the fundamental group of the Hawaiian earring. To date, the problem of characterising arbitrary $\mathbb{R}$-free groups is still wide open; however, a recently established embedding result (Theorem 6.1) of Chiswell and the present author might well turn out to be a key to its solution (see the corresponding remark in Section 6.1).

2 Group-valued functions and their concatenation

The construction of free groups begins with the concept of a *word* over an alphabet X, and the idea of forming a product of words by means of *concatenation.* Including the empty word, one arrives at a monoid (semigroup with identity element) $\mathcal{W}(X)$, the free (word) monoid over X. This section deals with a continous analogue of these simple ideas. We begin by taking a closer look at free word monoids, and then develop the concept of a group-valued function defined on a closed real interval and the star product of such functions as continuous analogues of words and concatenation of words.

2.1 The monoid $\mathcal{W}(X)$

A finite word is usually defined as an expression of the form

$$w = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_n}^{e_n},$$

where X is some given set, $e_1, \ldots, e_n \in \{1, -1\}$, and

$$X^{-1} = \big\{x^{-1} : x \in X\big\}$$

is a set in one-to-one correspondence with X via the map $x \mapsto x^{-1}$, such that $X \cap X^{-1} = \emptyset$. One extends this map to an involution on $X \cup X^{-1}$ by setting $(x^{-1})^{-1} = x$.

A word w can be though of as a function

$$w : \{1, 2, \ldots, n\} \to X \cup X^{-1},$$

for some integer $n \geq 0$, the unique word of length 0 being the empty word ε. The set $\{1, \ldots, n\}$ may be viewed as the interval $[1, n]$ in the discretely ordered abelian group $\Lambda = (\mathbb{Z}, +)$. Given words $u : [1, m] \to X \cup X^{-1}$ and

$v : [1, n] \to X \cup X^{-1}$, the idea of forming $u * v$, the concatenation of u and v, can then be formally introduced via

$$(u * v)(\sigma) = \begin{cases} u(\sigma), & 1 \leq \sigma \leq m \\ v(\sigma - m), & m + 1 \leq \sigma \leq m + n \end{cases} \quad (1 \leq \sigma \leq m + n), \qquad (1)$$

so that $u * v$ is a word on the interval $[1, m + n]$. It is easy to see that the multiplication $*$ thus introduced is associative, and that the empty word ε acts as a two-sided identity element; that is, we have

$$w * \varepsilon = w = \varepsilon * w.$$

If $w : [1, n] \to X \cup X^{-1}$ is a word, then its *length* $L(w)$ is defined as the number of its letters, $L(w) = n$. The *formal inverse* of w as above is the word w^{-1} of length n given by

$$w^{-1}(\sigma) = w(n - \sigma + 1)^{-1}, \quad 1 \leq \sigma \leq n.$$

We note that the formal inverse w^{-1} is clearly not an inverse to w with respect to star multiplication, since $*$ is length additive.

2.2 The monoid $\mathcal{F}(G)$

When trying to replace $(\mathbb{Z}, +)$ by the densely ordered group $\Lambda = (\mathbb{R}, +)$, the first problem which arises is that there is no longer a least positive element, so we replace a domain $[1, n]$ with an interval $[0, \alpha]$, where α is a non-negative real number. A more serious problem however is that concatenation can no longer be defined as in (1). Our solution is to replace the set $X \cup X^{-1}$ by a (discrete) group G. Let

$$\mathcal{F}(G) = \bigcup_{\substack{\alpha \in \mathbb{R} \\ \alpha \geq 0}} G^{[0,\alpha]}$$

be the set of all functions with values in G defined on an interval of the additive reals of the form $[0, \alpha]$ for some $\alpha \geq 0$. Concatenation is then defined as follows: given functions $f, g \in \mathcal{F}(G)$ on domains $[0, \alpha]$ and $[0, \beta]$, respectively, we let $f * g$ be the function given on the interval $[0, \alpha + \beta]$ of $(\mathbb{R}, +)$ via

$$(f * g)(\xi) = \begin{cases} f(\xi), & 0 \leq \xi < \alpha \\ f(\alpha)g(0), & \xi = \alpha \\ g(\xi - \alpha), & \alpha < \xi \leq \alpha + \beta \end{cases} \quad (\xi \in [0, \alpha + \beta]). \qquad (2)$$

The function $\mathbf{1}_G$ defined on the one-point interval $[0,0]=\{0\}$ by $\mathbf{1}_G(0)=1_G$ (where 1_G is the identity element of G) is a two-sided identity element with respect to the star operation; that is, we have

$$f * \mathbf{1}_G = f = \mathbf{1}_G * f, \quad f \in \mathcal{F}(G). \tag{3}$$

There is also a notion of *formal inverse* f^{-1} of a function $f \in \mathcal{F}(G)$: if f is defined on the domain $[0,\alpha]$, then f^{-1} is the function given on the same interval by

$$f^{-1}(\xi) = \big(f(\alpha-\xi)\big)^{-1}, \quad 0 \le \xi \le \alpha.$$

Straight from this definition, we have that

$$(f^{-1})^{-1} = f, \quad f \in \mathcal{F}(G).$$

The length $L(f)$ of $f \in \mathcal{F}(G)$ is defined as the length of its domain $[0,\alpha]$; that is, $L(f)=\alpha$. With these definitions, we now have the following result.

Proposition 2.1. *The set $\mathcal{F}(G)$, equipped with the star product* (2), *is a cancellative monoid.*

This Proposition 2.1 in [9, Chap. 2].

3 The group $\mathcal{RF}(G)$

Here, we introduce our main object of study, the group $\mathcal{RF}(G)$ formed with respect to a given (discrete) group G. We start, by way of motivation, with a description of the (old-fashioned) construction of free groups as sets of reduced words with reduced multiplication as the group operation. In Section 3.2, we then explain our continuous analogue of reduced words, i.e., reduced G-valued functions with domain a closed real interval of the form $[0,\alpha]$, and we define the (reduced) product of such reduced functions. The collection of all reduced functions $f:[0,\alpha]\to G$ for arbitrary real $\alpha \ge 0$ forms the group $\mathcal{RF}(G)$ under reduced multiplication as the group operation.

3.1 Reduced words and free groups

As for finite words, the formal inverse f^{-1} of a function $f \in \mathcal{F}(G)$ is not inverse to f, since the star operation is length additive. In order to remedy this defect, we need to introduce *reduced words* (respectively functions), and we have to define a new multiplication called the *reduced product*. A finite word $w = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_n}^{e_n}$ is reduced if $x_{i_j}^{e_j} \ne x_{i_{j+1}}^{-e_{j+1}}$ for all $1 \le j < n$; that is, if w does not contain a subword of the form $x^e x^{-e}$. Clearly, the empty word ε itself is reduced and, if a word w is reduced, then so is its formal inverse w^{-1}. In constructing free groups, one defines the (reduced) product of two reduced words u and v, by forming the concatenation $w = u * v$, and then deleting inverse pairs $x^e x^{-e}$ if necessary, starting from the boundary between u and v, and working outwards. Given an arbitrary set X, the free group $F(X)$ on X can then be defined as the set of reduced words over X with reduced multiplication as the group operation. The empty word ε is the identity, the formal inverse w^{-1} of a reduced word w becomes the inverse of w, and the only fact which is not completely obvious is the associativity of reduced multiplication; see pp. 126–127 in [16, § 18] for this argument.

3.2 Reduced functions and reduced multiplication

The notions corresponding to reduced words and reduced multiplication in the continuous setting are necessarily somewhat more elaborate.

A function $f \in \mathcal{F}(G)$ defined on the real interval $[0,\alpha]$ is called *reduced* if, for each point $\xi_0 \in (0,\alpha)$ with $f(\xi_0) = 1_G$ and every real number ε with $0 < \varepsilon \le \min\{\alpha - \xi_0, \xi_0\}$, there exists some real number δ such that $0 < \delta \le \varepsilon$, and such that $f(\xi_0 + \delta) \ne \big(f(\xi_0 - \delta)\big)^{-1}$. The set of all reduced functions in $\mathcal{F}(G)$ is denoted by $\mathcal{RF}(G)$. Given a function $f : [0,\alpha] \to G$ in $\mathcal{F}(G)$, let us call an ε-neighbourhood

$$[\xi_0 - \varepsilon, \xi_0 + \varepsilon] \subseteq [0,\alpha]$$

around a point $\xi_0 \in (0,\alpha)$ with $f(\xi_0) = 1_G$ a *cancelling neighbourhood around* ξ_0, if $f(\xi_0 - \delta) = \big(f(\xi_0 + \delta)\big)^{-1}$ for all $0 < \delta \le \varepsilon$. Then we can say that a function $f \in \mathcal{F}(G)$ as above is reduced if, and only if, there does not exist a cancelling neighbourhood around any interior point of the domain $[0,\alpha]$ of f satisfying $f(\xi_0) = 1_G$.

Functions of length 0 are automatically reduced (as their domain has no interior points), and an element $f \in \mathcal{F}(G)$ is reduced if, and only if, its formal inverse f^{-1} is reduced. If $f \in \mathcal{RF}(G)$, then f is not identically equal to 1_G on any non-degenerate subinterval of its domain; in particular, we have the reassuring fact that

$$\mathcal{RF}(\{1_G\}) = \{\mathbf{1}_G\}. \tag{4}$$

Also, if $f \in \mathcal{F}(G)$ has positive length, then $f * f^{-1}$ is not reduced; in particular, the star product of reduced functions need not be reduced.

We now proceed to define another multiplication on $\mathcal{F}(G)$ with the property that the product of two reduced functions is again reduced.

Given $f, g \in \mathcal{F}(G)$ of lengths α and β, respectively, we let

$$\varepsilon_0 = \varepsilon_0(f,g) := \begin{cases} \sup \mathcal{E}(f,g), & f(\alpha) = g(0)^{-1} \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\mathcal{E}(f,g) := \Big\{ \varepsilon \in [0, \min\{\alpha, \beta\}] : f(\alpha - \delta) = g(\delta)^{-1} \text{ for all } \delta \in [0, \varepsilon] \Big\},$$

and define fg on the interval $[0, \alpha + \beta - 2\varepsilon_0]$ via

$$(fg)(\xi) := \begin{cases} f(\xi), & 0 \le \xi < \alpha - \varepsilon_0 \\ f(\alpha - \varepsilon_0) g(\varepsilon_0), & \xi = \alpha - \varepsilon_0 \\ g(\xi - \alpha + 2\varepsilon_0), & \alpha - \varepsilon_0 < \xi \le \alpha + \beta - 2\varepsilon_0. \end{cases} \tag{5}$$

The function fg defined in (5) is called the *(reduced) product* of the functions f and g in $\mathcal{F}(G)$. In order to familiarize oneself with reduced multiplication, the reader might want to prove the following.

Lemma 3.1. *The reduced product fg of two reduced functions $f, g \in \mathcal{RF}(G)$ is again reduced.*

As a consequence of the product definition plus the fact that the product of two reduced functions is again reduced, we have the following result, which relates the star product to reduced multiplication.

Lemma 3.2. *For $f, g \in \mathcal{RF}(G)$, the following assertions are equivalent:*

(i) $\varepsilon_0(f,g) = 0$,

(ii) $fg = f * g$,

(iii) $f * g$ *is reduced.*

Remark 3.3. We note that the implication (i) $\Rightarrow$ (ii) of Lemma 3.2 holds in fact for $f, g \in \mathcal{F}(G)$.

Definition 3.4. For $f, g \in \mathcal{F}(G)$, we write $f \circ g$ to mean $f * g$ together with the information that $\varepsilon_0(f,g) = 0$, so that we have

$$f \circ g = f * g = fg$$

by the remark after Lemma 3.2.

Combining Remark 3.3 with (3), we see that $\mathbf{1}_G$ is a (two-sided) neutral element for $\mathcal{F}(G)$ with respect to (reduced) multiplication. One also finds that

$$ff^{-1} = \mathbf{1}_G = f^{-1}f, \quad f \in \mathcal{F}(G).$$

Furthermore, we note the following relation between the star and circle operation, respectively, and inversion.

Lemma 3.5. (Inversion of star products).

(a) Let $f_1, f_2 \in \mathcal{F}(G)$; then $\varepsilon_0(f_1, f_2) = \varepsilon_0(f_2^{-1}, f_1^{-1})$.

(b) Let $f_1, f_2 \in \mathcal{F}(G)$, and let $f = f_1 * f_2$. Then $f^{-1} = f_2^{-1} * f_1^{-1}$; in particular, $f = f_1 \circ f_2$ implies $f^{-1} = f_2^{-1} \circ f_1^{-1}$.

It is not hard to see that reduced multiplication is not associative on the whole of $\mathcal{F}(G)$; for instance, let f, g be the functions of lengths 1 and $\frac{1}{2}$, respectively, given by

$$f(\xi) := \begin{Bmatrix} x, & 0 \leq \xi < \frac{1}{2} \\ 1_G, & \xi = \frac{1}{2} \\ x^{-1}, & \frac{1}{2} < \xi \leq 1 \end{Bmatrix} \quad \text{and} \quad g(\xi) := \begin{Bmatrix} x, & 0 \leq \xi < \frac{1}{2} \\ 1_G, & \xi = \frac{1}{2} \end{Bmatrix},$$

where x is some fixed element of the group G. Then $f \neq \mathbf{1}_G$, $fg = g$, and thus

$$(fg)g^{-1} = gg^{-1} = \mathbf{1}_G \neq f = f(gg^{-1}).$$

Nevertheless, one can show that reduced multiplication is associative on the subset of $\mathcal{F}(G)$ consisting of all reduced functions, so that we have the following.

Theorem 3.6. *For every group G, the set $\mathcal{RF}(G)$ forms a group under reduced multiplication.*

The proof of Theorem 3.6 is long and fairly technical, using a non-trivial amount of cancellation theory, which has to be developed first; cf. Sections 2.3 and 2.4 in [9, Chapter 2].

4 The $\mathbb{R}$-tree associated with $\mathcal{RF}(G)$

The basic observation here is that the function $L : \mathcal{RF}(G) \to \mathbb{R}$ introduced in Section 2.2, which assigns to each reduced function f its length $L(f)$, satisfies the axioms of a (real) Lyndon length function; see Proposition 4.2 below.

Section 4.1 recalls the axioms of a (Λ-valued) Lyndon length function, and explains the intimate connection between group actions by isometries on Λ-trees and Λ-valued length functions; cf. Theorem 4.1. Combining this result with Proposition 4.2 then allows us (in Section 4.2) to associate with the group $\mathcal{RF}(G)$ an $\mathbb{R}$-tree action $\Phi : \mathcal{RF}(G) \to \mathrm{Isom}(\mathbf{X}_G)$ with canonical base point $x_0 \in \mathbf{X}_G$ such that $L = L_{x_0}$, where L_{x_0} is the displacement function associated with Φ and the point x_0.

4.1 Length functions

Let G be a group, Λ an ordered abelian group. A mapping $L : G \to \Lambda$ is called a (Λ-valued) *Lyndon length function* if the following holds.

(i) $L(1_G) = 0$.

(ii) For all $g \in G$, we have $L(g) = L(g^{-1})$.

(iii) For all g, h, $k \in G$, we have $c(g,h) \geq \min\{c(h,k), c(k,g)\}$, where $c(g,h) := \frac{1}{2}[L(g) + L(h) - L(g^{-1}h)]$.

Axiom (iii) is equivalent to: for all g, h, $k \in G$, at least two of $c(g,h)$, $c(h,k)$, $c(k,g)$ are equal, and not greater than the third. Note that c is symmetric by Condition (ii). It is not hard to verify that Axioms (i)–(iii) imply the following:

(iv) For all $g \in G$, $L(g) \geq 0$.

(v) For all g, $h \in G$, $L(gh) \leq L(g) + L(h)$.

(vi) For all g, $h \in G$, $0 \leq c(g,h) \leq \min\{L(g), L(h)\}$.

Property (v) is called the *triangle inequality*; (vi) is a consequence of (ii) and (v).

An important (and in a sense the most interesting) example of a Lyndon length function is the displacement function $L_{x_0} : G \to \Lambda$ arising from an action by isometries of the group G on a Λ-tree $\mathbf{X} = (X, d)$ and the choice of a base point $x_0 \in X$. The function L_{x_0} is defined by $L_{x_0}(g) = d(x_0, gx_0)$, and is easily seen to be a Lyndon length function on G satisfying

(vii) $c(g,h) \in \Lambda$ for all g, $h \in G$.

The next result is basic in the theory of Λ-trees; it says in particular that all Λ-valued length functions satisfying Property (vii) arise as displacement functions from isometric actions on Λ-trees; cf. Theorem A.29 in [9, App. A] for the proof. Note also that, if L is any length function, then $2L$ is a length function satisfying (vii).

Theorem 4.1. *Let G be a group and let $L : G \to \Lambda$ be a Lyndon length function satisfying Condition* (vii)*. Then there exist a Λ-tree $\mathbf{X} = (X, d)$, an action of G on $\mathbf{X}$, and a point $x_0 \in X$ such that $L = L_{x_0}$, and $\mathbf{X}$ is spanned by the orbit Gx_0.*

4.2 The $\mathbb{R}$-tree $\mathbf{X}_G$

We now have the following.

Proposition 4.2. *The map $L : \mathcal{RF}(G) \to \mathbb{R}$ assigning to each function $f \in \mathcal{RF}(G)$ the length $L(f)$ of its domain is a (real) Lyndon length function.*

Axioms (i) and (ii) for a length function obviously hold by definition of $\mathcal{RF}(G)$, hence the proof of Proposition 4.2 may focus on Axiom (iii), which in turn follows easily when we observe that

$$c(f, g) = \varepsilon_0(f^{-1}, g), \quad f, g \in \mathcal{RF}(G);$$

see [9, Prop. 3.1] for details.

Combining Theorem 4.1 with Proposition 4.2 yields existence of an $\mathbb{R}$-tree $\mathbf{X}_G = (X_G, d_G)$ on which $\mathcal{RF}(G)$ acts, with a canonical base point x_0, and such that $L = L_{x_0}$, where

$$L_{x_0}(f) = d_G(x_0, fx_0), \quad f \in \mathcal{RF}(G)$$

is the displacement function associated with the action of $\mathcal{RF}(G)$ on $(\mathbf{X}_G, x_0)$. One can show that $\mathbf{X}_G$ is always metrically complete (see Propositions 3.3 in Chapter 2 and A.43 in Appendix A of [9]), and that the action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$ is transitive; cf. Section 5.2. Furthermore, $\mathbf{X}_G$ is spanned by the orbit of x_0, that is,

$$X_G = \bigcup_{f \in \mathcal{RF}(G)} [x_0, fx_0],$$

where $[x_0, fx_0]$ denotes the segment spanned in $\mathbf{X}_G$ by the points x_0, fx_0. It follows in particular from these remarks that the stabilizer $\mathrm{stab}_{\mathcal{RF}(G)}(x_0)$ of the point x_0 under the action of $\mathcal{RF}(G)$ is given by

$$\mathrm{stab}_{\mathcal{RF}(G)}(x_0) = G_0 := \big\{f \in \mathcal{RF}(G) : L(f) = 0\big\}. \tag{6}$$

We note the following properties of centralizers and normalizers taken in $\mathcal{RF}(G)$ of elements $f \in G_0$ respectively subgroups $U \leq G_0$; cf. [9, Prop. 2.20].

Proposition 4.3. *(i) If $f \in G_0$ is a non-trivial element, then we have*

$$C_{\mathcal{RF}(G)}(f) = C_{G_0}(f).$$

(ii) For every non-trivial subgroup U of G_0, we have

$$N_{\mathcal{RF}(G)}(U) = N_{G_0}(U). \tag{7}$$

In particular, the subgroup G_0 is self-normalizing in $\mathcal{RF}(G)$.

5 The action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$

5.1 Classification of group elements and hyperbolic length

The action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$ leads to a classification of the elements of $\mathcal{RF}(G)$ according to whether they are *elliptic* (i.e., have a fixed point) or *hyperbolic* (that is, act as a fixed-point free isometry); a third class of elements, the inversions (i.e., group elements which are fixed-point free but whose square has a fixed point) do not arise for $\Lambda = \mathbb{R}$. Hyperbolic elements have some local geometry associated to them, leading to another type of length function on $\mathcal{RF}(G)$: if $f \in \mathcal{RF}(G)$ is hyperbolic, then there exists an isometric copy A_f of the real line in $\mathbf{X}_G$ (called the *axis* of f), such that f acts on A_f as a non-trivial translation; in particular, hyperbolic elements have infinite order. The translation length of a hyperbolic element f on its axis A_f is called the *hyperbolic length* of f, denoted $\ell(f)$; and ℓ is extended to the whole of $\mathcal{RF}(G)$ by setting $\ell(f) = 0$ if f is elliptic. Clearly, ℓ is invariant under conjugation.

5.2 Strongly regular length functions and transitivity

Definition 5.1. A Lyndon length function $L : G \to \Lambda$ on a group G is termed *strongly regular* if, for each $g \in G$ and every $\gamma \in \Lambda$ such that $0 \leq \gamma \leq L(g)$, there exist elements $g_1, g_2 \in G$ such that $g = g_1 g_2$, $L(g) = L(g_1) + L(g_2)$, and $L(g_1) = \gamma$.

We use this terminology, since there already exists a notion of *regular* length function in the literature; cf., for instance, [22]. One can show that a strongly regular length function satisfying Condition (vii) of Section 4.1 is regular; see [9, Prop. A.33].

By [9, Lemma 2.14] (Dissection of reduced functions), the real length function L on $\mathcal{RF}(G)$ is strongly regular. Strong regularity of a length function satisfying $c(g, h) \in \Lambda$ for all $g, h \in G$ and transitivity of the corresponding group action on a Λ-tree are related as follows.

Proposition 5.2. *Suppose that a group G acts by isometries on a Λ-tree $\mathbf{X} = (X, d)$, and let $x_0 \in X$ be any point. Then the following assertions are equivalent:*

(i) The group G is transitive on the subtree of $\mathbf{X}$ spanned by the orbit of x_0.

(ii) The displacement function L_{x_0} is strongly regular.

See Proposition A.37 in [9, Appendix A] for the proof this result.

As a consequence of Proposition 5.2 plus the fact that the length function L on $\mathcal{RF}(G)$ is strongly regular, we have the following.

Proposition 5.3. *The group $\mathcal{RF}(G)$ acts transitively on the points of its associated $\mathbb{R}$-tree $\mathbf{X}_G$. In particular, the set of elliptic elements of $\mathcal{RF}(G)$ equals $\bigcup_{t\in\mathcal{RF}(G)} tG_0t^{-1}$.*

Corollary 5.4. *$\mathcal{RF}(G)$ is torsion-free if, and only if, G is torsion-free.*

5.3 Cyclic reduction

Recalling again the theory of free groups, the notion of a cyclically reduced word (i.e., a reduced word not simultaneously beginning in a letter x^ε and ending in $x^{-\varepsilon}$) is crucial; for instance, observing that every non-trivial element of $F(X)$ is conjugate to a non-trivial cyclically reduced element, one sees that free groups are torsion-free. The cyclically reduced core of a non-trivial element $w \in F(X)$ also plays an important role in the solution of the conjugacy problem for finitely generated free groups; cf., for instance, [19, Theorem 1.3]. The aim of this section is to explain a continuous analogue of the core of a freely reduced word, and to use this concept (and technique) to further investigate the action of $\mathcal{RF}(G)$ on its tree $\mathbf{X}_G$.

Definition 5.5. A function $f \in \mathcal{RF}(G)$ is called *cyclically reduced* if $\varepsilon_0(f,f) = 0$ or, equivalently, if $L(f^2) = 2L(f)$.

Clearly, every function of length 0 is cyclically reduced. Also, if $f \in \mathcal{RF}(G)$ is cyclically reduced, then so is f^k for every integer k, and we have

$$L(f^k) = kL(f), \quad k \in \mathbb{Z}. \tag{8}$$

Our next result is crucial: it establishes, in analogy with the case of free groups, existence and uniqueness of cyclically reduced cores for the elements of the group $\mathcal{RF}(G)$. Applications include an algebraic characterization of hyperbolic elements in $\mathcal{RF}(G)$, as well as the computation of hyperbolic length ℓ in terms of the canonical length function L on $\mathcal{RF}(G)$. As for free groups, cyclically reduced cores also figure prominently in the conjugacy theorem for hyperbolic elements; see Section 7.

Lemma 5.6. (a) *Let $f \in \mathcal{RF}(G)$. Then there exist $t, f_1 \in \mathcal{RF}(G)$ with f_1 cyclically reduced, such that*

$$f = t \circ f_1 \circ t^{-1}.$$

(b) *If*

$$f = t \circ f_1 \circ t^{-1} = s \circ f_2 \circ s^{-1}, \tag{9}$$

where $t, s, f_1, f_2 \in \mathcal{RF}(G)$ *and* f_1, f_2 *are cyclically reduced, then* $s = tg$ *and* $f_2 = g^{-1} f_1 g$ *for some* $g \in G_0$; *in particular,* $L(f_1) = L(f_2)$.

See [9, Lemma 3.7] for the proof of Lemma 5.6, which uses a fair amount of the cancellation theory developed in [9, Sec. 2.3].

Definition 5.7. The cyclically reduced function f_1 found in Lemma 5.6 for a given reduced function $f \in \mathcal{RF}(G)$, which is unique up to conjugation by a G_0-element, is called the (cyclically reduced) *core* of f, denoted $c(f)$. The passage from f to f_1 itself is called *cyclic reduction* of the element f.

As a first minor application, we note that every non-trivial elliptic element of $\mathcal{RF}(G)$ lies in exactly one conjugate of G_0; in other words, the union $\bigcup_{t \in \mathcal{RF}(G)} tG_0t^{-1}$ forms an amalgam with trivial intersection. Indeed, if $x \in tG_0t^{-1} \cap sG_0s^{-1}$ and $x \neq \mathbf{1}_G$, then

$$t \circ g \circ t^{-1} = x = s \circ h \circ s^{-1}$$

for some $g, h \in G_0$, since x is non-trivial. By Part (b) of Lemma 5.6, this implies $s = tk$ for some $k \in G_0$, so $sG_0s^{-1} = tG_0t^{-1}$, as claimed.

Our main application of cyclic reduction in this section however is as follows.

Proposition 5.8. *For* $f \in \mathcal{RF}(G)$, *we have* $\ell(f) = L(c(f))$. *In particular, an element of* $\mathcal{RF}(G)$ *is hyperbolic if, and only if, its core has positive length.*

For the proof, see Proposition 3.13 and Corollary 3.21 in Chapter 3 of [9].

5.4 Bounded subgroups

Definition 5.9. A subgroup $\mathcal{H} \leq \mathcal{RF}(G)$ is called *bounded*, if there exists a real number $c \geq 0$ such that $L(f) \leq c$ for all $f \in \mathcal{H}$.

Obvious candidates are the conjugates of G_0, and their subgroups. The main result of this section (Proposition 5.12 below) establishes a converse to this observation, thereby characterising the bounded subgroups of $\mathcal{RF}(G)$. As applications we obtain an improved torsion result, as well as the fact that $\mathcal{RF}(G)$ does not contain non-trivial bounded subnormal subgroups. The crucial tool for establishing these facts is the following observation, which is also of independent interest.

Lemma 5.10. *Let* a_1 *and* a_2 *be two elliptic elements of* $\mathcal{RF}(G)$, *which do not lie in the same conjugate of* G_0. *Then their product* a_1a_2 *is hyperbolic.*

For different proofs of this result see Lemma 3.22, Part (ii) of Proposition 3.29, and Exercise 3.3 in [9, Chap. 3].

Remark 5.11. According to Lemma 5.10, the subgroup $E(G)$ of $\mathcal{RF}(G)$ generated by the elliptic elements abounds in hyperbolic elements. This observation motivates the question whether there are (non-trivial) groups G such that $\mathcal{RF}(G)$ coincides with $E(G)$; i.e., is generated by its elliptic elements. In fact, this is never the case. For G not an elementary abelian 2-groups this can be shown by suitably generalising the concept of an exponent sum map $e_x : F(X) \to \mathbb{Z}$, where $F(X)$ is a free group with basis X and $x \in X$; cf. [9, Chap. 5]. In the general case, this follows from the theory of test functions. In fact, one can show that $E(G)$ is as far from generating $\mathcal{RF}(G)$ as possible in the sense that

$$|\mathcal{RF}(G) : E(G)| = |\mathcal{RF}(G)| = |G|^{2^{\aleph_0}};$$

see Section 10.6 and Part (i) of [23, Theorem 37].

Armed with Lemma 5.10, we can now prove the following.

Proposition 5.12. *Let $\mathcal{H}$ be a subgroup of $\mathcal{RF}(G)$. Then the following assertions are equivalent.*

(i) *$\mathcal{H}$ is bounded.*

(ii) *$\mathcal{H}$ consist entirely of elliptic elements.*

(iii) *$\mathcal{H}$ is conjugate to a subgroup of G_0.*

Proof. Clearly, (iii) implies (i). Next we show that (i) implies (ii). Indeed, if $f \in \mathcal{H}$ is hyperbolic then, by Lemma 5.6 and Proposition 5.8, we have $f = t \circ f_1 \circ t^{-1}$ with $t, f_1 \in \mathcal{RF}(G)$, f_1 cyclically reduced, and $L(f_1) > 0$; thus, by Equation (8),

$$L(f^n) = L(t \circ f_1^n \circ t^{-1}) = 2L(t) + nL(f_1) \to \infty \ \text{ as } \ n \to \infty,$$

contradicting Assumption (i). Finally, suppose that $\mathcal{H}$ does not contain a hyperbolic element. In showing that (ii) implies (iii), we may suppose that $\mathcal{H} \neq \{\mathbf{1}_G\}$. Fix a non-trivial element $a = t \circ g \circ t^{-1}$ in $\mathcal{H}$, $L(g) = 0$, and let $b \in \mathcal{H}$ be an arbitrary element. Since the union $\bigcup_{t \in \mathcal{RF}(G)} tG_0t^{-1}$ forms an amalgam with trivial intersection (see the observation preceding Proposition 5.8), a lies in tG_0t^{-1}, and in no other conjugate of G_0. Hence, our assumption that $\mathcal{H}$ consists entirely of elliptic elements together with Lemma 5.10 implies that $b \in tG_0t^{-1}$; so $\mathcal{H} \subseteq tG_0t^{-1}$, since b was arbitrary. $\square$

Corollary 5.13. *Every finite subgroup of $\mathcal{RF}(G)$ is conjugate to a subgroup of G_0; in particular, $\mathcal{RF}(G)$ is torsion-free if, and only if, G is torsion-free.*

Corollary 5.14. *The only bounded subnormal subgroup of $\mathcal{RF}(G)$ is the trivial group $\{\mathbf{1}_G\}$.*

Proof. Let $\mathcal{N} \leq \mathcal{RF}(G)$ be a non-trivial, bounded, and subnormal subgroup of $\mathcal{RF}(G)$. Then $G \neq \{1_G\}$, so $G_0 < \mathcal{RF}(G)$ (for instance, $\mathcal{RF}(G)$ contains functions of positive length with constant value equal to some element $x \in G - \{1_G\}$). Further, by Proposition 5.12, we have $\mathcal{N} \leq tG_0t^{-1}$ for some $t \in \mathcal{RF}(G)$. Let

$$\mathcal{N} = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r \triangleleft \mathcal{RF}(G)$$

be a strict subnormal series connecting $\mathcal{N}$ with $\mathcal{RF}(G)$. Then $r \geq 0$, since $\mathcal{RF}(G)$ itself is not bounded. Conjugating by t, we find that $\mathcal{N}^t \leq G_0$, and that

$$\{\mathbf{1}_G\} \neq \mathcal{N}^t = N_0^t \triangleleft N_1^t \triangleleft \cdots \triangleleft N_r^t = N_r \triangleleft \mathcal{RF}(G).$$

Suppose that $N_i^t \leq G_0$ for some $0 \leq i < r$ and some $r \geq 1$. Then, making use of Part (ii) of Proposition 4.3 plus the fact that $N_i^t \neq \{\mathbf{1}_G\}$, we get that

$$N_{i+1}^t \leq N_{\mathcal{RF}(G)}(N_i^t) = N_{G_0}(N_i^t) \leq G_0.$$

Since $N_0^t = \mathcal{N}^t \leq G_0$, this shows that $N_r \leq G_0$. Applying Part (ii) of Proposition 4.3 again, it follows that

$$\mathcal{RF}(G) = N_{\mathcal{RF}(G)}(N_r) = N_{G_0}(N_r) \leq G_0 < \mathcal{RF}(G),$$

a contradiction. Hence, such $\mathcal{N}$ does not exist. □

Remark 5.15. An alternative proof for the implication (i)⇒(iii) in Proposition 5.12 can be given as follows. Let $\mathcal{H} \leq \mathcal{RF}(G)$ be a bounded subgroup. Since the $\mathbb{R}$-tree $\mathbf{X}_G$ is complete, we can apply a result of Wilkens [30] (see, for instance, Lemma 2.5 in [6, Chapter 4]) to obtain that $\mathcal{H}$ has a global fixed point $x \in \mathbf{X}_G$. Since the action of G on $\mathbf{X}_G$ is transitive by Proposition 5.3, we can find an element $t \in \mathcal{RF}(G)$ such that $\mathcal{H}^t$ stabilizes the base point x_0; that is, $\mathcal{H}^t \subseteq G_0$, as required.

6 Universality

As is well known, free groups enjoy (and may even be characterized) by a certain universality property: if $F(X)$ is free with basis X then, given any group G and any map $\Theta : X \to G$, there exists a unique homomorphism $\Theta' : F \to G$ extending Θ; that is, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{inclusion}} & F(X) \\ {\scriptstyle\Theta}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{F(X)}} \\ G & \xleftarrow[\Theta']{} & F(X) \end{array}$$

commutes; in particular, mapping X (taken sufficiently large) onto a generating system of G, we see that free groups are universal (among the class of all groups) with respect to forming quotients. Since, roughly speaking, the idea behind the definition of the groups $\mathcal{RF}(G)$ can be described as an attempt to adapt the (old-fashioned) construction of free groups to a continuous setting, one would expect $\mathcal{RF}$-groups to exhibit some sort of universal property. The search for this universality property was the motivation for the research described in this section.

We present two embedding results from [8]. The first (Theorem 6.1) applies to Λ-trees for arbitrary Λ, and says, roughly speaking, that if a group G acts freely and without inversions on a Λ-tree $\mathbf{X} = (X, d)$, then this action can be extended to a transitive one. The second result (Theorem 6.2) says that, if G is a group acting freely and transitively on an $\mathbb{R}$-tree (X, d), then the whole action can be embedded into one involving an appropriate $\mathcal{RF}$-group and its associated $\mathbb{R}$-tree. Combining these two results, one finds that $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees are *universal*, with respect to inclusion, for free $\mathbb{R}$-tree actions.

Theorems 6.1 and 6.2 below arose out of an attempt to understand some of the key results in [2], one of the seminal papers on $\mathbb{R}$-tree theory. More specifically, Theorem 6.1 is inspired by [2, Theorem 3.4], while Theorem 6.2 is based to some extent on the argument for [2, Theorem 4.2]. In both cases, important details are omitted in [2], or appear highly problematic, and seem difficult (if not right-out impossible) to fill in. Consequently, [8] adopts a different approach involving, among other things, string rewriting and length functions.

6.1 Two embedding results

Our first result is as follows.

Theorem 6.1. *Let G be a group acting freely and without inversions on a Λ-tree $\mathbf{X} = (X, d)$. Then there exists a group $\hat{G}$ acting freely, without inversions, and transitively on a Λ-tree $\widehat{\mathbf{X}} = (\widehat{X}, \hat{d})$, together with a group embedding $\varphi : G \to \hat{G}$ and a G-equivariant isometry $\mu : \mathbf{X} \to \widehat{\mathbf{X}}$.*

See [8, Theorem 5.4] for the proof. Our second embedding result concerns $\mathbb{R}$-tree actions, which are free and transitive; cf. [8, Theorem 8.6].

Theorem 6.2. *Let G be a group acting freely and transitively on an $\mathbb{R}$-tree $\mathbf{X} = (X, d)$. Then there exists a group H, an injective group homomorphism $\psi : G \to \mathcal{RF}(H)$, and a G-equivariant isometry $\nu : \mathbf{X} \to \mathbf{X}_H$.*

6.2 Universality of $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees

Combining Theorems 6.1 and 6.2, we obtain the following important result.

Theorem 6.3. *Let G be a group acting freely on an $\mathbb{R}$-tree $\mathbf{X} = (X, d)$. Then there exists a group H, a group embedding $\chi : G \to \mathcal{RF}(H)$, and a G-equivariant isometry $\lambda : \mathbf{X} \to \mathbf{X}_H$ containing the canonical base point x_0 in its image.*

Somewhat informally, Theorem 6.3 says that $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees are universal, with respect to inclusion, for free $\mathbb{R}$-tree actions. Given a group G as in Theorem 6.3, one may ask, which groups H will satisfy the conclusion of the theorem. A partial answer to this question is given by the following refinement of Theorem 6.3.

Theorem 6.4. *Let G be a group acting freely on an $\mathbb{R}$-tree $\mathbf{X}$ by means of a homomorphism $\varphi : G \to \mathrm{Isom}(\mathbf{X})$, and let $\kappa_{\mathbf{X},\varphi} := |\hat{G}/\approx|$ be the cardinal number of directions of the tree $\widehat{\mathbf{X}}$. Moreover, let $\{b_i\}_{i \in I}$ be a set of representatives for the G-orbits on $\mathbf{X}$. Then we have the following.*

(i) $$\kappa_{\mathbf{X},\varphi} \leq \begin{cases} \aleph_0, & G = 1 \text{ and } 1 < |I| < \infty \\ \max\big\{|G|,\, |I|\big\}, & \text{otherwise.} \end{cases}$$

(ii) For every group H containing a subgroup of cardinality $\kappa_{\mathbf{X},\varphi}$, there exists an injective group homomorphism $\chi : G \to \mathcal{RF}(H)$ and a G-equivariant isometry $\lambda : \mathbf{X} \to \mathbf{X}_H$, whose image contains the base point of $\mathbf{X}_H$.

This is Theorem 6.4 in [9, Chap. 6].

6.3 A realisation theorem

By a *real group*, we mean any subgroup of the additive reals $(\mathbb{R}, +)$. It follows in particular from the discussion in Section 1, that every free product of real groups is $\mathbb{R}$-free; that is, has a free action on some $\mathbb{R}$-tree. Consequently, by our universality result, Theorem 6.3, every free product $\Lambda = \ast_{i \in I} \Lambda_i$ of real groups embeds as a hyperbolic subgroup in $\mathcal{RF}(G)$ for some suitable group G.

One can also consider a modified embedding problem, where some non-trivial group G is given in advance, and the question is asked which free products $\Lambda = \ast_{i\in I}\Lambda_i$ of (non-trivial) real groups embed as a hyperbolic subgroup in $\mathcal{RF}(G)$ for this given group G. Our corresponding result is as follows.

Theorem 6.5. *Let G be a given non-trivial group, and let $\{\Lambda_i\}_{i\in I}$ be a family of non-trivial real groups. Then the free product $\Lambda = \ast_{i\in I}\Lambda_i$ embeds as a hyperbolic subgroup in $\mathcal{RF}(G)$ if, and only if, $|I| \leq |G|^{2^{\aleph_0}}$.*

We note that the answer afforded by Theorem 6.5 to the relative embedding problem formulated above is best possible in that, according to this theorem, Λ is realized as a hyperbolic subgroup in $\mathcal{RF}(G)$ whenever Λ can be embedded in $\mathcal{RF}(G)$ as a mere subset. The recent paper [21], among other things, provides two rather different proofs of Theorem 6.5; see Corollary 5.4 and Section 6 of that paper.

7 Conjugacy of hyperbolic elements

As is easy to see, two elliptic elements $a = sgs^{-1}$ and $b = tht^{-1}$, with $g, h \in G_0 - \{\mathbf{1}_G\}$, are conjugate in $\mathcal{RF}(G)$ if, and only if, g and h are conjugate in G_0. Hence, nothing further can be said in general about conjugacy of elliptic elements in $\mathcal{RF}(G)$, without restricting or specifying the group G. For hyperbolic elements however, the situation is different, and much more interesting. This is the topic of the present section. We begin by recalling the solution of the corresponding (conjugacy) problem for finitely generated free groups, following the account in [19, Section 1.4].

7.1 The conjugacy problem for free groups

Let F be a free group with basis $X = \{x_1, x_2, \dots, x_n\}$. The first step is to introduce a specific process σ for cyclically reducing an arbitrary word w in the free generators x_i. Roughly speaking, σ cyclically reduces w by first freely reducing it, and then cancelling first and last symbols, if necessary. For instance,

$$\begin{aligned}\sigma(x_1x_2x_3x_3^{-1}x_2x_1x_2^{-1}x_1^{-1}) &\equiv \sigma(x_1x_2^2x_1x_2^{-1}x_1^{-1})\\ &\equiv \sigma(x_2^2x_1x_2^{-1})\\ &\equiv \sigma(x_2x_1)\\ &\equiv x_2x_1,\end{aligned}$$

where $\equiv$ means identity as words. More precisely, one first introduces a process ρ, which freely reduces a given word w in the generators x_i by going through the word w from left to right, deleting every inverse pair of the form $x_i^{\varepsilon}x_i^{-\varepsilon}$, when we first hit upon it. For instance, in order to compute the word

$$\rho(x_1x_2^{-1}x_3x_3^{-1}x_2x_2^{-1}),$$

one successively computes

$$\begin{aligned}\rho(x_1) &\equiv x_1,\\ \rho(x_1x_2^{-1}) &\equiv x_1x_2^{-1},\\ \rho(x_1x_2^{-1}x_3) &\equiv x_1x_2^{-1}x_3,\\ \rho(x_1x_2^{-1}x_3x_3^{-1}) &\equiv x_1x_2^{-1},\\ \rho(x_1x_2^{-1}x_3x_3^{-1}x_2) &\equiv x_1,\\ \rho(x_1x_2^{-1}x_3x_3^{-1}x_2x_2^{-1}) &\equiv x_1x_2^{-1}.\end{aligned}$$

In general, ρ is defined by induction on word length; see the proof of Theorem 1.2 in [19, Sec. 1.4]. One then defines σ for an arbitrary word w in the generators x_i by $\sigma(w) \equiv \sigma(\rho(w))$, where σ is defined inductively for freely reduced words via

$$\sigma(w) \equiv \begin{cases} \varepsilon, & w \equiv \varepsilon\\ x_i^e, & w \equiv x_i^e\\ x_i^{e_1} * v * x_j^{e_2}, & w \equiv x_i^{e_1} * v * x_j^{e_2} \text{ with } i \neq j \text{ or } e_1 = e_2\\ \sigma(v), & w \equiv x_i^{e_1} * v * x_j^{e_2} \text{ with } i = j \text{ and } e_1 = -e_2.\end{cases}$$

Here, ε again denotes the empty word, $i, j \in [n]$, and $e_1, e_2 \in \{1, -1\}$.†

The next step, usually not carried out in this formality, consists in defining an equivalence relation τ on F via

$$w_1 \tau\, w_2 \;:\Longleftrightarrow\; w_1 \equiv u * v \text{ and } w_2 \equiv v * u.$$

Here, elements of F are viewed as reduced words. If $w_1 \tau\, w_2$ for elements $w_1, w_2 \in F$, then we call w_2 a *cyclic permutation* of w_1. Reflexivity and

†As usual, for a non-negative integer n, $[n]$ denotes the standard set $\{1, 2, \ldots, n\}$ of cardinality n.

symmetry of τ are clear, so we may focus on transitivity of τ. Suppose that $w_1 \tau w_2$ and $w_2 \tau w_3$. Then we have word identities

$$w_1 \equiv u * v,$$

$$w_2 \equiv v * u \equiv u' * v',$$

$$w_3 \equiv v' * u'.$$

Assume that $L(v) \leq L(u')$, so that $u' \equiv v * u'_1$ and $u \equiv u'_1 * v'$. Then $w_1 \equiv u'_1 * (v' * v)$ and $w_3 \equiv (v' * v) * u'_1$, hence $w_1 \tau w_3$ as claimed. The case where $L(v) > L(u')$ is treated in a similar way.

We now have the following classical result, effectively resolving the conjugacy problem in a finitely generated free group with a specified basis.[‡]

Proposition 7.1. *If F is the free group on the free generators $x_1, x_2, \ldots, x_n$, then two words w_1 and w_2 in the generators x_i define conjugate elements of F if, and only if, $\sigma(w_1) \tau \sigma(w_2)$.*

As we shall see, conjugacy of hyperbolic elements in $\mathcal{RF}(G)$ is governed by a result which (apart from necessarily being not effective) is an almost complete analogue of Proposition 7.1; cf. Theorem 7.4 below.

7.2 The equivalence relation τ_G and the conjugacy theorem

There is in general no analogue, finite or transfinite, for the process of reduction; that is, for the map ρ introduced above. Instead, we shall have to work with the elements of the group $\mathcal{RF}(G)$ directly. There exists however a kind of analogue for the process σ of cyclic reduction, given by Lemma 5.6, which establishes existence and uniqueness of the (cyclically reduced) core $c(f)$ of a reduced function $f \in \mathcal{RF}(G)$. Allowing ourselves to be guided by the situation for free groups, we now introduce an equivalence relation on $\mathcal{RF}(G)$ analogous to the relation τ on a free group discussed above.

Definition 7.2. Given a group G, we define a binary relation τ_G on $\mathcal{RF}(G)$ by

$$f_1 \, \tau_G \, f_2 \ :\Longleftrightarrow \ f_1 = p \circ q \ \text{ and } \ f_2 = q \circ p \text{ for some } p, q \in \mathcal{RF}(G).$$

If $f_1 \tau_G f_2$, we say that f_2 is a *cyclic permutation* of f_1.

Next, we show that τ_G is indeed an equivalence relation on $\mathcal{RF}(G)$. Characteristically, this task, while certainly being rather straightforward, is not as easy as establishing the corresponding observation for free groups.

[‡]Cf. Theorem 1.3 in Section 1.4 of [19].

Lemma 7.3. *Relation τ_G is an equivalence relation on $\mathcal{RF}(G)$.*

Proof. Symmetry is clear by definition of τ_G, and reflexivity holds since we may write a given element $f \in \mathcal{RF}(G)$ as $f = f \circ \mathbf{1}_G = \mathbf{1}_G \circ f$. Hence, it only remains to establish transitivity of τ_G. If $f_1 \tau_G f_2$ and $f_2 \tau_G f_3$, then there exist elements $p_1, p_2, q_1, q_2 \in \mathcal{RF}(G)$ such that we have

$$f_1 = p_1 \circ q_1,$$

$$f_2 = q_1 \circ p_1 = p_2 \circ q_2,$$

$$f_3 = q_2 \circ p_2.$$

Suppose first that $L(q_1) \leq L(p_2)$. Then we can apply [9, Lemma 2.14] (dissection of reduced functions), to write $p_2 = q_1 \circ u$ with some $u \in \mathcal{RF}(G)$. Hence, applying [9, Cor. 2.18] (associativity of the circle product), we find that

$$f_2 = q_1 \circ p_1 = (q_1 \circ u) \circ q_2 = q_1 \circ (u \circ q_2);$$

and [9, Prop. 2.1] gives that $p_1 = u \circ q_2$. Applying again [9, Cor 2.18], we now obtain that

$$f_1 = (u \circ q_2) \circ q_1 = u \circ (q_2 \circ q_1)$$

and

$$f_3 = q_2 \circ (q_1 \circ u) = (q_2 \circ q_1) \circ u,$$

so $f_1 \tau_G f_3$ as desired. An analogous argument serves in the case where $L(p_2) < L(q_1)$. □

We can now state the main result of this section, whose (fairly difficult) proof occupies Sections 7.4 and 7.5 in [9, Chap. 7].

Theorem 7.4 (The conjugacy theorem for hyperbolic elements)**.** *Let $f_1, f_2 \in \mathcal{RF}(G)$ be hyperbolic elements. Then f_1 is conjugate to f_2 in $\mathcal{RF}(G)$ if, and only if, $c(f_1) \tau_G\, c(f_2)$.*

We note that, if $f_1 = g f_2 g^{-1}$ with $g \in G_0$, then $f_1 = p \circ q$ and $f_2 = q \circ p$, where $p := g$ and $q := f_2 g^{-1}$; in other words, conjugating by a G_0-element does not change the τ_G-class of a reduced function. In particular, in view of Lemma 5.6, the core $c(f)$ of a reduced function $f \in \mathcal{RF}(G)$ is well determined modulo the equivalence relation τ_G, so the criterion stated in Theorem 7.4 for conjugacy of two hyperbolic elements actually makes sense.

7.3 Normalizers of cyclic hyperbolic subgroups

Suppose that $w \in F$ is a non-trivial element of the free group F (again considered as a reduced word in a basis of F), and that $u \in F$ normalizes the subgroup $\langle w \rangle$ generated by w.[§] Since free groups are torsion-free, w has infinite order, thus $\langle w \rangle \cong C_\infty$; and, consequently, $w^{\pm 1}$ are the only two generators of $\langle w \rangle$. Since u normalizes $\langle w \rangle$, it follows that

$$u^{-1}wu = w^{\pm 1}.$$

However, w is not conjugate to its inverse w^{-1}. Indeed, if it were, then, by Proposition 7.1, we would have word identities

$$\sigma(w) \equiv u * v$$

and

$$\sigma(w^{-1}) \equiv v * u.$$

Moreover, since $\sigma(w^{-1}) \equiv (\sigma(w))^{-1}$, these identities in turn imply that

$$v^{-1} * u^{-1} \equiv v * u,$$

from which we conclude that $u \equiv u^{-1}$ and $v \equiv v^{-1}$; in particular, $u^2 = v^2 = 1$. Since F is torsion-free, we must have $u = v = 1$, thus $\sigma(w) \equiv \varepsilon$, and so $w = 1$, a contradiction. It follows that

$$N_F(\langle w \rangle) \subseteq C_F(w),$$

and since the reverse inclusion is trivial,

$$N_F(\langle w \rangle) = C_F(w) \tag{10}$$

holds for all $w \in F$, which in turn allows us to conclude that normalizers of infinite-cyclic subgroups in a free group are themselves cyclic.

Since $\mathcal{RF}$-groups are in general not torsion-free, and Theorem 7.4 necessarily just deals with hyperbolic elements, we can only expect a limited analogue of (10) to hold for $\mathcal{RF}(G)$. This is the following.

Proposition 7.5. *Let $f \in \mathcal{RF}(G)$ be a hyperbolic element.*

(a) If f is not a product of two involutions, then we have

$$N_{\mathcal{RF}(G)}(\langle f \rangle) = C_{\mathcal{RF}(G)}(f). \tag{11}$$

[§]With suitable modification, we follow the argument of Proposition 2.19 in Chapter I of [17].

(b) *If f is a product of two involutions, $f = pq$ with $p^2 = q^2 = \mathbf{1}_G$, then the centralizer $C_{\mathcal{RF}(G)}(f)$ has index 2 in the normalizer $N_{\mathcal{RF}(G)}(\langle f\rangle)$, with the non-trivial coset being generated by p; that is, we have*

$$N_{\mathcal{RF}(G)}(\langle f\rangle) = C_{\mathcal{RF}(G)}(f) \cup p\, C_{\mathcal{RF}(G)}(f). \tag{12}$$

Corollary 7.6. *Suppose that G does not contain proper involutions. Then we have*

$$N_{\mathcal{RF}(G)}(\langle f\rangle) = C_{\mathcal{RF}(G)}(f)$$

for every hyperbolic element $f \in \mathcal{RF}(G)$.

The main point in proving Proposition 7.5 is to understand when a hyperbolic element is conjugate to its inverse. This is cleared up in our next result.

Lemma 7.7. *Let $f \in \mathcal{RF}(G)$ be a hyperbolic element. Then the following are equivalent.*

(i) *f is conjugate in $\mathcal{RF}(G)$ to its inverse f^{-1}.*

(ii) *f is the product of two involutions lying in different conjugates of G_0.*

Proof. (i) ⇒ (ii). Suppose that f is conjugate to f^{-1}, and write $f = t \circ f_1 \circ t^{-1}$ with f_1 cyclically reduced according to Lemma 5.6. Then f^{-1} is hyperbolic as well, and we have $f^{-1} = t \circ f_1^{-1} \circ t^{-1}$, where f_1^{-1} is again cyclically reduced. Hence, $c(f) = f_1$ and $c(f^{-1}) = f_1^{-1}$, and Theorem 7.4 tells us that $f_1 = p \circ q$ and $f_1^{-1} = q \circ p$ for some $p, q \in \mathcal{RF}(G)$. Applying Lemma 3.5 (inversion of star products) yields that

$$p \circ q = f_1 = p^{-1} \circ q^{-1}. \tag{13}$$

Comparing values of the left-hand and right-hand sides of (13), we find that

$$p(\xi) = (p \circ q)(\xi) = (p^{-1} \circ q^{-1})(\xi) = p^{-1}(\xi), \quad 0 \le \xi < L(p);$$

whereas, for $0 < \xi \le L(q)$,

$$q(\xi) = (p \circ q)(L(p) + \xi) = (p^{-1} \circ q^{-1})(L(p) + \xi) = q^{-1}(\xi).$$

Since $L(f_1) > 0$ by Proposition 5.8, we must have $L(p) > 0$ or $L(q) > 0$. If $L(q) > 0$, then we have

$$q(0) = (q \circ p)(0) = (q^{-1} \circ p^{-1})(0) = q^{-1}(0);$$

thus $q = q^{-1}$, implying $p = p^{-1}$. If, on the other hand, $L(q) = 0$, then we must have $L(p) > 0$, and a calculation similar to the one above yields that $p = p^{-1}$, again implying $q = q^{-1}$. In both cases, we have found that

$$p^2 = \mathbf{1}_G = q^2,$$

so that f_1, and hence also f itself, is a product of two involutions. Moreover, these two involutions cannot lie in the same conjugate of G_0, since their product, f, would then be contained in the same G_0-conjugate, hence would be elliptic, contradicting our hypothesis that f is hyperbolic.

(ii) $\Rightarrow$ (i). If f is the product of two involutions, say $f = pq$ with $p^2 = \mathbf{1}_G = q^2$, then

$$p^{-1}fp = qp = q^{-1}p^{-1} = f^{-1},$$

so that f is indeed conjugate to f^{-1}. □

8 Centralizers of hyperbolic elements

By Part (i) of Proposition 4.3, the centralizers of elliptic elements of $\mathcal{RF}(G)$ are determined, up to isomorphism, by the isomorphism types of centralizers in the group G itself; hence, without restricting the structure of G, nothing more can be said here.

The situation is very different, and much more interesting, for the centralizers of hyperbolic elements; their theory is the topic of the present section.

Our main result (Theorem 8.2 below) provides considerable insight into the structure of the centralizer $\mathfrak{C}_f := C_{\mathcal{RF}(G)}(f)$, where $f \in \mathcal{RF}(G)$ is some hyperbolic element. We also obtain a criterion deciding whether or not $\mathfrak{C}_f$ is cyclic. As it turns out, the centralizers of hyperbolic functions are always real groups, thus abelian and relatively "small". We also obtain a presentation for $\mathfrak{C}_f$ in terms of a generating system exhibiting considerable internal structure. This is remarkable, since test function theory allows us to show that *every* non-trivial subgroup of the additive reals is realized as the centralizer of some hyperbolic element; see Corollary 10.14. We wonder, whether this new information might be useful in obtaining a classification of real groups, which so far have resisted any such attempt. Theorem 8.2 also allows us to deduce that $\mathcal{RF}$-groups do not contain soluble normal subgroups (see [9, Sec. 8.9]); however, test function theory offers another and more revealing approach to this problem; cf. Part (iii) of Theorem 10.16.

We begin by introducing the (strong) periods of a hyperbolic function f, as these turn out to play a crucial rôle in the analysis of the centralizer $\mathfrak{C}_f$.

8.1 Periods

Let $f \in \mathcal{RF}(G)$ be an element of length $L(f) = \alpha > 0$. Then the points $\omega \in [0, \alpha]$ satisfying

$$\forall\, \gamma, \delta \in (0, \alpha] :\, |\gamma - \delta| = \omega \to f(\gamma) = f(\delta)$$

are called *periods* of f. A period $\omega \in \Omega_f$ is called a *strong period*, if $\alpha - \omega \in \Omega_f$. Note that, according to this definition, the numbers 0 and α are always strong periods, called the *trivial periods*. We denote by Ω_f the set of all periods of f, and by Ω_f^0 the subset of strong periods.

Our next result collects together some useful properties of periods and strong periods.

Lemma 8.1. *Let $f \in \mathcal{RF}(G)$ be a function of length $\alpha > 0$.*

(i) If $\omega_1, \omega_2 \in \Omega_f$ and $\omega_1 + \omega_2 \in [0, \alpha]$, then $\omega_1 + \omega_2 \in \Omega_f$.

(ii) If $\omega_1, \ldots, \omega_r \in \Omega_f^0$ for some $r \geq 1$, and $\omega_1 + \cdots + \omega_r \in [0, \alpha]$, then we have $\omega_1 + \cdots + \omega_r \in \Omega_f^0$.

(iii) If $\omega_1, \omega_2 \in \Omega_f^0$ and $\omega_1 - \omega_2 \in [0, \alpha]$, then $\omega_1 - \omega_2 \in \Omega_f^0$.

(iv) Let $\omega_1, \ldots, \omega_r \in \Omega_f^0$, where $r \geq 1$. Then

$$\omega_1 + \cdots + \omega_r = k\alpha + \omega$$

for some non-negative integer k and $\omega \in \Omega_f^0 - \{\alpha\}$.

(v) Denote by $\langle \Omega_f^0 \rangle$ the subgroup of $(\mathbb{R}, +)$ generated by the set Ω_f^0. Then

$$\langle \Omega_f^0 \rangle \cap [0, \alpha] = \Omega_f^0. \tag{14}$$

The proofs are fairly straightforward; cf. Lemma 8.5 in [9, Chap. 8] for details.

It follows from Lemma 8.1 that every element ξ of the subgroup $\langle \Omega_f^0 \rangle$ of $(\mathbb{R}, +)$ generated by the set Ω_f^0 of strong periods can be written in the form

$$\xi = \sigma(k\alpha + \omega), \quad (k, \omega, \sigma) \in \mathbb{N}_0 \times (\Omega_f^0 - \{\alpha\}) \times \{1, -1\}. \tag{15}$$

Moreover, with the convention that 0 is written as $(+1)(0 \cdot \alpha + 0)$, and not as $(-1)(0 \cdot \alpha + 0)$, Representation (15) is unique. Indeed, let $\xi \in \langle \Omega_f^0 \rangle$, and set $k := \left[\frac{|\xi|}{\alpha}\right]$.¶ Then $k \in \mathbb{N}_0$ and $\omega := |\xi| - k\alpha$ satisfies $0 \leq \omega < \alpha$, thus,

¶As usual, $[x]$, the Gauß bracket of the real number x, denotes the largest integer less than or equal to x.

$\omega \in \Omega_f^0 - \{\alpha\}$ by Part (v) of Lemma 8.1. Hence, ξ can be written in the desired form (15). Next, if $\xi \neq 0$ and

$$\sigma_1(k_1\alpha + \omega_1) = \xi = \sigma_2(k_2\alpha + \omega_2),$$

then $k_1\alpha + \omega_1, k_2\alpha + \omega_2 > 0$, so $\sigma_1 = \sigma_2$. It follows that

$$|k_1 - k_2|\alpha = |\omega_2 - \omega_1| < \alpha,$$

and therefore $k_1 = k_2$ and $\omega_1 = \omega_2$. Finally, if $\xi = \sigma(k\alpha + \omega) = 0$, then $k\alpha + \omega = 0$, thus $k = \omega = 0$; and, by our convention, $\sigma = +1$.

8.2 The main result

From now on, we shall assume that $f \in \mathcal{RF}(G)$ is cyclically reduced, of length $L(f) = \alpha > 0$, and normalized in the sense that $f(0) = 1_G$ (every hyperbolic element is conjugate to a function satisfying these requirements).

Making use of Parts (ii) and (iv) of Lemma 8.1, we define a binary operation $\boxplus$ on the set $\Omega_f^0 - \{\alpha\}$ via

$$\omega_1 \boxplus \omega_2 := \begin{Bmatrix} \omega_1 + \omega_2, & \omega_1 + \omega_2 < \alpha \\ \omega_1 + \omega_2 - \alpha, & \omega_1 + \omega_2 \geq \alpha \end{Bmatrix} \quad (\omega_1, \omega_2 \in \Omega_f^0 - \{\alpha\}).$$

In this way, the set $\Omega_f^0 - \{\alpha\}$ becomes an abelian group; in fact, sending ω to $\omega + \langle\alpha\rangle$ gives a group isomorphism

$$\big(\Omega_f^0 - \{\alpha\}, \boxplus\big) \cong \big\langle \Omega_f^0 \big\rangle / \langle \alpha \rangle.$$

We now come to the main result of this section.

Theorem 8.2 (The Centralizer Theorem). *Let $f \in \mathcal{RF}(G)$ be cyclically reduced, of length $L(f) = \alpha > 0$, and normalized.*

(a) *The set*

$$C_f = \Big\{ f^k \circ f|_{[0,\omega]} : \ (k, \omega) \in \mathbb{N}_0 \times \big(\Omega_f^0 - \{\alpha\}\big),\ k + \omega > 0 \Big\}$$

forms a positive cone for the centralizer $\mathfrak{C}_f$ of f in $\mathcal{RF}(G)$, giving $\mathfrak{C}_f$ the structure of an ordered abelian group.

(b) *Every element of $\mathfrak{C}_f$ is cyclically reduced; in particular, $\mathfrak{C}_f$ is a hyperbolic subgroup of $\mathcal{RF}(G)$.*

(c) *The mapping* $\rho_f : \mathfrak{C}_f \to \langle \Omega_f^0 \rangle$ *given by* $(f^k \circ f|_{[0,\omega]})^\sigma \mapsto \sigma(k\alpha + \omega)$ *is an isomorphism of ordered abelian groups satisfying*

$$L(g) = |\rho_f(g)|, \quad g \in \mathfrak{C}_f. \tag{16}$$

(d) $\mathfrak{C}_f$ *has the presentation*

$$\Big\langle x_\omega \ (\omega \in \Omega_f^0) \ \Big| \ [x_\alpha, x_\omega] = 1 \, (\omega < \alpha), x_{\omega_1} x_{\omega_2} = x_\alpha^{[\frac{\omega_1+\omega_2}{\alpha}]} x_{\omega_1 \boxplus \omega_2} \, (\omega_1, \omega_2 < \alpha) \Big\rangle. \tag{17}$$

The proof of the centralizer theorem is long, technical, and fairly difficult; see Sections 8.4–8.7 in [9, Chap. 8] for full details. Theorem 8.2 allows us in particular to characterize those hyperbolic elements $f \in \mathcal{RF}(G)$, whose centralizer $\mathfrak{C}_f$ is cyclic.

Corollary 8.3. *Let* f *be as in Theorem* 8.2, *and set* $\omega_0 := \inf\big(\Omega_f^0 - \{0\}\big)$. *Then the following assertions are equivalent.*

(i) *The set* Ω_f^0 *is finite.*

(ii) *We have* $\omega_0 \in \Omega_f^0 - \{0\}$.

(iii) *The centralizer* $\mathfrak{C}_f$ *is cyclic.*

Moreover, if (i)–(iii) *hold, then* $\alpha = k_0\omega_0$ *for some positive integer* k_0, *we have* $f = f_0^{k_0}$ *with* $f_0 := f|_{[0,\omega_0]}$, *the positive cone* C_f *of* f *consists of the positive powers of* f_0, *and the centralizer* $\mathfrak{C}_f$ *is generated by* f_0.

Proof. Clearly, (i) implies (ii). Next, assume (ii), that is $\omega_0 \in \Omega_f^0 - \{0\}$, set $k_0 := \big[\frac{\alpha}{\omega_0}\big]$, and consider $\omega' := \alpha - k_0\omega_0$. Then $k_0 \in \mathbb{N}$, $0 \le \omega' < \omega_0$, and $\omega' \in \Omega_f^0$ by Parts (ii) and (iii) of Lemma 8.1. This forces $\omega' = 0$ by definition of ω_0; that is $\alpha = k_0\omega_0$. A similar argument shows that Ω_f^0 cannot contain any point which is not an integral multiple of ω_0. Indeed, let $\omega_1 \in \Omega_f^0$ be such that $\omega_1 > \omega_0$, and set $k_1 := \big[\frac{\omega_1}{\omega_0}\big]$. Then $k_1 \in \mathbb{N}$ and

$$0 \le \omega_1' := \omega_1 - k_1\omega_0 < \omega_0.$$

Again applying Parts (ii) and (iii) of Lemma 8.1, we see that ω_1' is a strong period of f, forcing $\omega_1' = 0$ by definition of ω_0. Since, on the other hand, again using Part (ii) of Lemma 8.1, the number $k\omega_0$ is a strong period of f for $k \in \{0, 1, 2, \ldots, k_0\}$, we conclude that

$$\Omega_f^0 = \big\{0,\, \omega_0,\, 2\omega_0,\, \ldots,\, (k_0 - 1)\omega_0,\, \alpha\big\},$$

and, hence, that $\langle\Omega_f^0\rangle = \langle\omega_0\rangle$. In particular, we have seen that (ii)⇒(i).

Translating our observations back by means of the isomorphism ρ_f of Part (c) of Theorem 8.2, we find that $f = f_0^{k_0}$, where $f_0 = f|_{[0,\omega_0]}$, that $C_f = \{f_0^k : k \in \mathbb{N}\}$, and that $\mathfrak{C}_f = \langle f_0\rangle$. This shows that (ii)⇒(iii), and establishes the claims concerning α, f, C_f, and $\mathfrak{C}_f$ under the assumption that Assertion (ii) holds.

(iii)⇒(ii). Suppose that $\omega_0 \notin \Omega_f^0 - \{0\}$. Then Ω_f^0 contains a strictly decreasing sequence $\{\omega_\kappa\}_{\kappa\geq 1}$ of points converging to ω_0; in particular, $\{\omega_\kappa\}$ is a Cauchy sequence. It follows that $\langle\Omega_f^0\rangle$, and hence $\mathfrak{C}_f$, is not cyclic. □

8.3 The centralizer partition property

Let F be a free group. Then, as is well known, F is commutative-transitive; cf. Proposition 2.17 in [17, Chap. 1]. Equivalently, the binary relation on the set $F - \{1\}$ given by

$$a \leftrightarrow b \;:\Longleftrightarrow\; a \text{ and } b \text{ commute}$$

is an equivalence relation. This is also equivalent to: the family of sets

$$C_F(a) - \{1\}, \quad a \in F - \{1\}$$

partitions the set $F - \{1\}$ of non-trivial elements of F. We call this the *centralizer partition property* of F. As a further application of Theorem 8.2, we establish an analogue of this fact for the hyperbolic elements of $\mathcal{RF}(G)$.

Proposition 8.4. *Let G be a non-trivial group, and let $f, g \in \mathcal{RF}(G)$ be hyperbolic elements. Then the following assertions are equivalent.*

(i) *The centralizers of f and g in $\mathcal{RF}(G)$ coincide.*

(ii) *f and g commute.*

(iii) $C_{\mathcal{RF}(G)}(f) \cap C_{\mathcal{RF}(G)}(g) \neq \{\mathbf{1}_G\}$.

As an immediate consequence of Proposition 8.4, we have the following analogue of the centralizer partition property of free groups, alluded to above.

Corollary 8.5 (The Centralizer Partition Property for $\mathcal{RF}$-Groups)**.** *Suppose that G is non-trivial. Then the sets*

$$C_{\mathcal{RF}(G)}(f) - \{\mathbf{1}_G\}$$

for hyperbolic functions $f \in \mathcal{RF}(G)$ form a partition of the set $\mathcal{RF}(G) - \bigcup_{t\in\mathcal{RF}(G)} tG_0t^{-1}$ of hyperbolic elements; equivalently, the binary relation

$$f \leftrightarrow g \;:\Longleftrightarrow\; f \text{ and } g \text{ commute}, \quad f, g \in \mathcal{RF}(G) - \bigcup_{t\in\mathcal{RF}(G)} tG_0t^{-1}$$

is an equivalence relation on the set $\mathcal{RF}(G) - \bigcup_{t\in\mathcal{RF}(G)} tG_0t^{-1}$.

Proof of Proposition 8.4. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear; thus, it suffices to show the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (i). This follows from the fact that centralizers of hyperbolic elements in $\mathcal{RF}(G)$ are abelian. Indeed, suppose that $[f,g] = \mathbf{1}_G$, so that $g \in C_{\mathcal{RF}(G)}(f)$, and let $h \in C_{\mathcal{RF}(G)}(f)$ be an arbitrary element. Then h and g both lie in $C_{\mathcal{RF}(G)}(f)$; and, since f is hyperbolic, $C_{\mathcal{RF}(G)}(f)$ is abelian by Theorem 8.2. Hence, $[h,g] = \mathbf{1}_G$, thus $h \in C_{\mathcal{RF}(G)}(g)$. This shows that

$$C_{\mathcal{RF}(G)}(f) \leq C_{\mathcal{RF}(G)}(g),$$

and the reverse inclusion is established in a similar way.

(iii) $\Rightarrow$ (ii). This follows from the fact that, by Theorem 8.2, centralizers of hyperbolic elements are both abelian and hyperbolic. To be more explicit, let

$$h \in C_{\mathcal{RF}(G)}(f) \cap C_{\mathcal{RF}(G)}(g)$$

be a non-trivial element. As f is hyperbolic, the set $C_{\mathcal{RF}(G)}(f) - \{\mathbf{1}_G\}$ consists entirely of hyperbolic elements; in particular, h itself is hyperbolic, implying that $C_{\mathcal{RF}(G)}(h)$ is abelian. Since f and g are both contained in $C_{\mathcal{RF}(G)}(h)$, we conclude that $[f,g] = \mathbf{1}_G$, as claimed. □

9 Functoriality

The $\mathcal{RF}$-construction gives rise in a natural way to several important functors, two of which are briefly discussed in this section; cf. [9, Chap. 6] for more information on this topic.

9.1 The categories Groups, $\widetilde{\textbf{Groups}}$, and $\widehat{\textbf{Groups}}$

We are going to work with three categories of groups:

- **Groups** – the category of groups and group homomorphisms,

- $\widetilde{\textbf{Groups}}$ – the category of groups and injective homomorphisms,

- $\widehat{\textbf{Groups}}$ – the category whose objects are groups, and whose morphisms $\varphi \in \mathrm{Mor}(G,H)$ are injective maps $\varphi : G \to H$ satisfying $\varphi(1_G) = 1_H$ and
  $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$.

Note that $\widetilde{\textbf{Groups}}$ is a subcategory of both **Groups** and $\widehat{\textbf{Groups}}$. An important observation concerning the category $\widehat{\textbf{Groups}}$ is as follows.

Proposition 9.1. *(a) A morphism φ in the category $\widehat{\mathbf{Groups}}$ is an isomorphism if, and only if, φ is bijective as a map.*

(b) Assuming the axiom of choice, two objects $G, H \in |\widehat{\mathbf{Groups}}|$ are isomorphic if, and only if, $|\mathrm{Inv}(G)| = |\mathrm{Inv}(H)|$ and $|G - \mathrm{Inv}(G)| = |H - \mathrm{Inv}(H)|$.

Here, for a group Γ,

$$\mathrm{Inv}(\Gamma) = \{\gamma \in \Gamma : \gamma^2 = 1_\Gamma\}$$

is the set of *involutions* (trivial and non-trivial) of Γ. See [9, pp. 95–96] for the proof of Proposition 9.1.

9.2 The functor $\widetilde{\mathcal{RF}}(-)$

Let G and H be groups, and let $\varphi \in \mathrm{Mor}_{\widetilde{\mathbf{Groups}}}(G, H)$ be a morphism connecting G to H (i.e., an injective homomorphism from G to H). Defining

$$\hat{\varphi}(f) = \varphi \circ f, \quad f \in \mathcal{F}(G),$$

we obtain a map $\hat{\varphi} : \mathcal{F}(G) \to \mathcal{F}(H)$, which can be shown to restrict to an injective homomorphism $\hat{\varphi}|_{\mathcal{RF}(G)} : \mathcal{RF}(G) \to \mathcal{RF}(H)$. Setting

$$\widetilde{\mathcal{RF}}(G) := \mathcal{RF}(G), \quad G \in |\widetilde{\mathbf{Groups}}|$$
$$\widetilde{\mathcal{RF}}(\varphi) := \hat{\varphi}|_{\mathcal{RF}(G)}, \quad \varphi \in \mathrm{Mor}_{\widetilde{\mathbf{Groups}}}(G, H),$$

we obtain a covariant functor $\widetilde{\mathcal{RF}}(-)$ on the category $\widetilde{\mathbf{Groups}}$ of groups and embeddings; cf. [9, Prop. 6.3].

Existence of the functor $\widetilde{\mathcal{RF}}(-)$ implies the reassuring fact that isomorphic groups give rise to isomorphic $\mathcal{RF}$-groups. The question as to what extent the converse holds, i.e., whether $\mathcal{RF}(G) \cong \mathcal{RF}(H)$ implies $G \cong H$, appears difficult, and only partial results are known at present; cf. [24]. The main problem here seems to be that, in general, the subgroup G_0 of functions of length 0 cannot be detected by a mere group isomorphism, without any recourse to the length function or its associated geometry.

Another noteworthy consequence is the following.

Corollary 9.2. *Every automorphism α of G extends to an automorphism $\hat{\alpha}$ of $\mathcal{RF}(G)$ (identifying G with G_0) in such a way that mapping α to $\hat{\alpha}$ gives an embedding of $\mathrm{Aut}(G)$ into $\mathrm{Aut}(\mathcal{RF}(G))$.*

Proof. If α is an automorphism of G, then $\hat{\alpha} := \widetilde{\mathcal{RF}}(\alpha)$ is an automorphism of $\mathcal{RF}(G)$ such that $\hat{\alpha}|_{G_0} = \alpha$. The mapping given by $\alpha \mapsto \hat{\alpha}$ is a homomorphism from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(\mathcal{RF}(G))$ since $\widetilde{\mathcal{RF}}(-)$ is a covariant functor; and the fact that $\hat{\alpha}$ extends α shows that this homomorphism is injective. □

9.3 The functor $\widehat{\mathcal{RF}}_0(-)$ and rigidity

The main result concerning functoriality of the $\mathcal{RF}$-construction however lies considerably deeper, and concerns a functor $\widehat{\mathcal{RF}}_0(-) : \widehat{\mathbf{Groups}} \to \mathbf{Groups}$, which acts on objects via

$$\widehat{\mathcal{RF}}_0(G) = \mathcal{RF}(G)/E(G).$$

Combining existence of the functor $\widehat{\mathcal{RF}}_0(-)$ with Part (b) of Proposition 9.1, we deduce the following interesting if somewhat mysterious result.

Theorem 9.3 (The Rigidity Theorem.)**.** *Let G and H such that* $|\mathrm{Inv}(G)| = |\mathrm{Inv}(H)|$ *and* $|G - \mathrm{Inv}(G)| = |H - \mathrm{Inv}(H)|$. *Then we have*

$$\mathcal{RF}(G)/E(G) \cong \mathcal{RF}(H)/E(H).$$

As in the theory of semisimple Lie groups,[‖] the word 'rigidity' refers here to a certain extension property of morphisms, whence the terminology.

Remarks 9.4. 1. Roughly speaking, Theorem 9.3 asserts that the isomorphism type of the quotient group $\mathcal{RF}_0(G) = \mathcal{RF}(G)/E(G)$ depends only on the (cardinal) number of involutions of G, and that of its non-involutions. What happens if, in particular, $\mathrm{Inv}(G) = \{1_G\}$; that is, if G has no involutions apart from the identity? Then the isomorphism type of $\mathcal{RF}_0(G)$ depends only on one cardinal number, $|G|$, the order of the group G. Could $\mathcal{RF}(G)/E(G)$ turn out to be free in this case, so that $\mathcal{RF}(G)$ would be a split extension of $E(G)$ by a (huge) free group? The answer is 'no'; see Part (i) of Theorem 10.16.

2. Existence of a third functor $\widehat{\mathcal{RF}}(G)(-)$ on the category $\widehat{\mathbf{Groups}}$ shows that under the hypothesis of Theorem 9.3, we also have that $|\mathcal{RF}(G)| = |\mathcal{RF}(H)|$, without however specifying what this cardinal number really is. For the computation of the cardinality of $\mathcal{RF}(G)$, see Corollary 10.15.

10 Test functions

In the course of previous sections, we have more than once come upon important questions which could not satisfactorily be resolved at that stage. Here is a sample.

(1) Is $\mathcal{RF}(G)$, in particular for G an elementary abelian 2-group, generated by its elliptic elements (cf. Remark 5.11)?

(2) What is the cardinality of $\mathcal{RF}(G)$ (cf. the second of Remarks 9.4)?

[‖] Cf. [18, Chap. VII].

(3) Is $\mathcal{RF}_0(G) = \mathcal{RF}(G)/E(G)$ a free group (cf. the first of Remarks 9.4)?

(4) Are the abelianized groups

$$\mathcal{RF}(G)/[\mathcal{RF}(G), \mathcal{RF}(G)] \quad \text{and} \quad \mathcal{RF}_0(G)/[\mathcal{RF}_0(G), \mathcal{RF}_0(G)]$$

free abelian?

(5) Given a group G, which real groups are realized as centralizer of some hyperbolic element in $\mathcal{RF}(G)$ (see the introduction to Section 8)?

(6) Does $\mathcal{RF}(G)$ contain soluble normal subgroups (see the introduction to Section 8)?

(7) Which free products of real groups are embedded in $\mathcal{RF}(G)$, for a given group G, as a hyperbolic subgroup (see Section 6.3)?

In order to be able to answer these and related questions, we introduce two new concepts: that of a *test function*, and that of a family of pairwise *locally incompatible* functions. These concepts, and the powerful techniques developed around them, make it possible, in particular, to answer all the questions above. The purpose of the present section is to explain these key concepts, to introduce some of their machinery, and to explain the connection with the above problems (1)–(7). Full details may be found in the papers [21] and [23].

10.1 Definition of a test function

Roughly speaking, a test function is an element of $\mathcal{RF}(G)$, which does not look locally like its own inverse. More precisely, we have the following.

Definition 10.1. A function $f \in \mathcal{F}(G)$ is called a *test function*, if it has positive length, and there do not exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, L(f))$, such that

$$f(\xi_1 + \eta) = f^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon.$$

Since f and f^{-1} occur symmetrically here, the inverse of a test function is again a test function.

Test functions are automatically cyclically reduced (see [23, Lemma 12]); and if f is a test function, then so is f^k for any non-zero integer k.

In order to get some feeling for Definition 10.1, let us show that test functions are reduced. Suppose that $f \in \mathcal{RF}(G)$ is a test function, and that there exists an interior point ξ_0 in the domain of f with $f(\xi_0) = 1_G$, and such that, for $|\eta| < \varepsilon$,

$$f(\xi_0 + \eta) = \big(f(\xi_0 - \eta)\big)^{-1} = f^{-1}(L(f) - \xi_0 + \eta) = f^{-1}(\xi_0' + \eta)$$

where $\xi_0' := L(f) - \xi_0$ and $\varepsilon > 0$. Since ξ_0' is again an inner point of the domain $[0, L(f)]$ of f, the resulting equation

$$f(\xi_0 + \eta) = f^{-1}(\xi_0' + \eta), \quad |\eta| < \varepsilon$$

contradicts the definition of a test function; so f is reduced, as claimed.

Test functions do in fact exist: for instance, let G be any non-trivial group, and let $x \in G - \{1_G\}$ be a non-trivial element. Then the function f_0 of length 1, say, given by

$$f_0(\xi) = \left\{\begin{matrix} x, & \xi^2 \in \mathbb{Q} \\ 1_G, & \text{otherwise} \end{matrix}\right\} \quad (\xi \in [0,1])$$

is a test function. In fact, much more is true: given any proper real group Λ, there exists a family $\mathfrak{F}$ of test functions, such that $|\mathfrak{F}| = |G|^{(\mathbb{R}:\Lambda)}$, $C_{\mathcal{RF}(G)}(f) \cong \Lambda$ for all $f \in \mathfrak{F}$, and such that any two elements of $\mathfrak{F}$ are independent in an appropriate sense; cf. Theorem 10.11 in Section 10.5 below.

10.2 Theory of a single test function

Here, we are going to explain, how a given test function $f \in \mathcal{RF}(G)$ gives rise to an associated homomorphism $\lambda_f : \mathcal{RF}(G) \to \mathbb{R}$. Our construction is based on the idea of locally comparing a function $g \in \mathcal{RF}(G)$ with a fixed test function and its inverse. More precisely, given a test function $f \in \mathcal{RF}(G)$ of length α, and an arbitrary element $g \in \mathcal{F}(G)$ of length β, say, define sets $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$ by

$$\mathcal{M}_f^+(g) := \Big\{\xi \in (0,\beta) : \exists\, \varepsilon > 0,\ \exists\, \xi' \in (0,\alpha) \text{ such that } g(\xi + \eta) = f(\xi' + \eta) \text{ for all } |\eta| < \varepsilon\Big\}$$

and

$$\mathcal{M}_f^-(g) := \Big\{\xi \in (0,\beta) : \exists\, \varepsilon > 0,\ \exists\, \xi' \in (0,\alpha) \text{ such that } g(\xi + \eta) = f^{-1}(\xi' + \eta) \text{ for all } |\eta| < \varepsilon\Big\}$$

(of course, our notation is supposed to imply that all function values written down are actually defined). The following is more or less immediate from the definition of the sets $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$.

Lemma 10.2. *Let $f \in \mathcal{RF}(G)$ be any fixed test function, and let $L(f) = \alpha$. Then*

(i) $\mathcal{M}_f^+(g) \cap \mathcal{M}_f^-(g) = \emptyset, \quad g \in \mathcal{F}(G)$;

(ii) $\mathcal{M}_f^+(f|_{[0,\beta]}) = (0, \beta)$ *and* $M_f^-(f|_{[0,\beta]}) = \emptyset, \quad 0 \le \beta \le \alpha$;

(iii) $\mathcal{M}_f^+(g) = \emptyset = \mathcal{M}_f^-(g), \quad g \in G_0$.

Since the sets $\mathcal{M}_f^+(g), \mathcal{M}_f^-(g)$ are defined by open conditions (i.e., conditions invariant under slight perturbation of the point considered), $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$ are open sets, hence Lebesgue measurable. Given a fixed test function $f \in \mathcal{RF}(G)$, we define a function $\lambda_f : \mathcal{RF}(G) \to \mathbb{R}$ by

$$\lambda_f(g) := \mu\big(\mathcal{M}_f^+(g)\big) - \mu\big(\mathcal{M}_f^-(g)\big), \quad g \in \mathcal{RF}(G),$$

where μ denotes Lebesgue measure. We note that, by Part (iii) of Lemma 10.2, we have

$$\lambda_f(G_0) = 0. \tag{18}$$

Also, by Part (ii) of Lemma 10.2,

$$\lambda_f(f|_{[0,\beta]}) = \beta, \quad 0 \le \beta \le L(f). \tag{19}$$

Our main result concerning the maps λ_f is now the following.

Theorem 10.3. *For each fixed test function $f \in \mathcal{RF}(G)$, the map*

$$\lambda_f : \mathcal{RF}(G) \to \mathbb{R}$$

defined above is a surjective group homomorphism, whose kernel contains $E(G)$.

Once it is known that λ_f is a homomorphism, surjectivity of λ_f follows from the fact (see Equation (19)) that

$$[0, L(f)] \subseteq \lambda_f(\mathcal{RF}(G)),$$

while the assertion that $\lambda_f(E(G)) = 0$ is an obvious consequence of (18) and the fact that $E(G)$ is the normal closure of G_0. The remaining claim (that λ_f is a homomorphism) is established in three steps: first, one shows that λ_f respects inverses; then, one proves that the equation

$$\lambda_f(gh) = \lambda_f(g) + \lambda_f(h) \tag{20}$$

holds for $g, h \in \mathcal{RF}(G)$ with $\varepsilon_0(g, h) = 0$; finally, combining these two results with [23, Lemma 18], it is straightforward to show that Equation (20) holds for arbitrary elements $g, h \in \mathcal{RF}(G)$; that is, that λ_f is a group homomorphism; see Lemma 17 and Theorem 19 in [23, Sec. 4] for details.

Theorem 10.3 has a number of important consequences, which we turn to next.

First, coupling Theorem 10.3 with the existence of test functions, we find that non-trivial $\mathcal{RF}$-groups are never generated by their elliptic elements.

Corollary 10.4. *Let G be a non-trivial group. Then the quotient group $\mathcal{RF}(G)/E(G)$ maps homomorphically onto $\mathbb{R}$; in particular, $\mathcal{RF}(G)$ is not generated by its elliptic elements.*

Second, we get the following.

Corollary 10.5. *If $f \in \mathcal{RF}(G)$ is a test function, then f is not contained in the normal subgroup $E(G)[\mathcal{RF}(G), \mathcal{RF}(G)]$.*

Third, Theorem 8.2 allows us to deduce the following interesting result.

Corollary 10.6. *Let f be a test function. Then every non-trivial element of the centralizer $C_{\mathcal{RF}(G)}(f)$ of f in $\mathcal{RF}(G)$ is itself a test function; in particular, we have*

$$C_{\mathcal{RF}(G)}(f) \cap E(G)[\mathcal{RF}(G), \mathcal{RF}(G)] = \{\mathbf{1}_G\}. \tag{21}$$

10.3 Local (in)compatibility

We shall need an appropriate notion of independence for test functions. The most useful concept turns out to be that of local incompatibility, which is introduced in the following definition in somewhat greater generality.

Definition 10.7. Two functions $f_1, f_2 \in \mathcal{F}(G)$ of lengths α_1 respectively α_2 are called *locally compatible* (loc. comp. for short), if there exist $\varepsilon > 0$ and points $\xi_i \in (0, \alpha_i)$ such that we either have

$$f_1(\xi_1 + \eta) = f_2(\xi_2 + \eta), \quad |\eta| < \varepsilon$$

or

$$f_1(\xi_1 + \eta) = f_2^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon.$$

If f_1 and f_2 both have positive length, but are not locally compatible, they are called *locally incompatible* (loc. incomp. for short).

Local compatibility is a symmetric relation on $\mathcal{F}(G)$; that is, we have

$$f_1 \text{ loc. comp. } f_2 \Longrightarrow f_2 \text{ loc. comp. } f_1 \quad (f_1, f_2 \in \mathcal{F}(G)). \tag{22}$$

The following result summarizes some important properties of local incompatibility.

Lemma 10.8. *(i) If $f_1, f_2 \in \mathcal{F}(G)$ are locally incompatible, then $\varepsilon_0(f_1, f_2) = 0$.*

(ii) If $f_1, f_2 \in \mathcal{F}(G)$ are locally incompatible, then so are the functions f_1^{-1} and f_2, as are the functions f_1^{-1} and f_2^{-1}.

(iii) Let $f_1, \ldots, f_r, g_1, \ldots, g_s \in \mathcal{RF}(G)$ be reduced functions. Suppose that each f_ρ is locally incompatible to every g_σ, and that the products $f_1 \cdots f_r$ and $g_1 \cdots g_s$ both have positive length. Then $f_1 \cdots f_r$ and $g_1 \cdots g_s$ are again locally incompatible.

(iv) For $k \geq 1$, let $f_1, f_2, \ldots, f_k$ be pairwise locally incompatible test functions, and let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be non-zero integers. Then $f_1^{\gamma_1} f_2^{\gamma_2} \ldots f_k^{\gamma_k}$ is again a test function.

For Parts (i) and (ii), see [23, Lemma 24], Part (iv) is the contents of Proposition 29 in the same paper, while Part (iii) is [21, Theorem 4.1].

10.4 A hyperbolicity criterion

Call a subgroup $\mathcal{H} \leq \mathcal{RF}(G)$ *hyperbolic*, if the set $\mathcal{H} - \{\mathbf{1}_G\}$ consists entirely of hyperbolic elements; that is, of functions whose cyclically reduced core has positive length (see Proposition 5.8). It appears difficult to decide in general, when a family $\{\mathcal{H}_\sigma\}_{\sigma \in S}$ of hyperbolic subgroups $\mathcal{H}_\sigma \leq \mathcal{RF}(G)$ has the property that its join

$$\mathcal{H} = \big\langle \mathcal{H}_\sigma : \sigma \in S \big\rangle$$

is again hyperbolic. Our next result in particular illustrates the usefulness of the concept of local incompatibility as the appropriate notion of "independence".

Definition 10.9. Let $\{\mathcal{H}_\sigma\}_{\sigma \in S}$ be a family of subgroups $\mathcal{H}_\sigma \leq \mathcal{RF}(G)$, with bijective indexing. We say that $\{\mathcal{H}_\sigma\}_{\sigma \in S}$ *satisfies Condition* (LI), if

$$f_1 \in \mathcal{H}_{\sigma_1} - \{\mathbf{1}_G\},\ f_2 \in \mathcal{H}_{\sigma_2} - \{\mathbf{1}_G\},\ \text{and } \sigma_1 \neq \sigma_2 \implies f_1 \text{ loc. incomp. } f_2.$$

Theorem 10.10. *Let $\{\mathcal{H}_\sigma\}_{\sigma \in S}$ be a family of subgroups $\mathcal{H}_\sigma \leq \mathcal{RF}(G)$ with bijective indexing. Suppose that each $\mathcal{H}_\sigma$ is hyperbolic, and that the family $\{\mathcal{H}_\sigma\}_{\sigma \in S}$ meets Condition* (LI)*. Then the join $\mathcal{H} = \langle \mathcal{H}_\sigma : \sigma \in S \rangle$ is a hyperbolic subgroup of $\mathcal{RF}(G)$, and is isomorphic to the free product $\ast_{\sigma \in S} \mathcal{H}_\sigma$.*

This is Theorem 1.2 in [21].

10.5 An existence theorem

The following result (Theorem 30 in [23]) is of great importance in $\mathcal{RF}$-theory.

Theorem 10.11. *Let G be a non-trival group, and let $0 < \Lambda < \mathbb{R}$ be any proper subgroup of the additive reals. Then there exists a family $\mathfrak{F}$ of pairwise locally incompatible normalized test functions in $\mathcal{RF}(G)$, such that $|\mathfrak{F}| = |G|^{(\mathbb{R}:\Lambda)}$, and such that $\langle \Omega_f^0 \rangle = \Lambda$ for all $f \in \mathfrak{F}$.*

Remark 10.12. Since test functions are cyclically reduced and of positive length, we have $C_{\mathcal{RF}(G)}(f) \cong \Lambda$ for each $f \in \mathfrak{F}$ by the centralizer theorem.

As a first illustration of the power of Theorem 10.11, we list a few immediate consequences. By choosing Λ in Theorem 10.11 countable, we obtain the following.

Corollary 10.13. *Suppose that G is non-trivial. Then there exists a family $\{f_\sigma\}_{\sigma\in S}$ of pairwise locally incompatible test functions in $\mathcal{RF}(G)$ with bijective indexing, such that $|S| = |G|^{2^{\aleph_0}}$ and $L(f_\sigma) = \alpha_\sigma$ for each $\sigma \in S$, where $\{\alpha_\sigma\}_{\sigma\in S}$ is any given family of positive real numbers indexed (not necessarily injectively) by the elements of S.*

Second, we find that each non-trivial real group is realized up to isomorphism as the centralizer of some hyperbolic function in $\mathcal{RF}(G)$.

Corollary 10.14. *Let G be a non-trivial group, and let A be a non-trivial torsion-free abelian group of rank at most $2^{\aleph_0}$. Then there exists a test function $f \in \mathcal{RF}(G)$ such that $C_{\mathcal{RF}(G)}(f) \cong A$.*

Third, since test functions are automatically reduced, Corollary 10.13 allows us to compute the cardinality of $\mathcal{RF}(G)$.**

Corollary 10.15. *We have $|\mathcal{RF}(G)| = |G|^{2^{\aleph_0}}$.*

10.6 A structure theorem

One of the most important applications of Theorem 10.11 to date is the following *structure theorem* for $\mathcal{RF}(G)$ and its quotient $\mathcal{RF}_0(G) = \mathcal{RF}(G)/E(G)$.

Theorem 10.16. *Let G be a non-trivial group, set $\mathfrak{c}_G := |G|^{2^{\aleph_0}}$, and assume the axiom of choice. Then the following assertions hold true.*

(i) The groups $\mathcal{RF}(G)$ and $\mathcal{RF}_0(G)$ contain a free subgroup of rank $\mathfrak{c}_G$, but are not free; in particular, $|\mathcal{RF}_0(G)| = \mathfrak{c}_G$.

**For an alternative approach to Corollary 10.15, see [23, Sec. 7].

(ii) *The abelianizations $\overline{\mathcal{RF}(G)}$ and $\overline{\mathcal{RF}_0(G)}$ of $\mathcal{RF}(G)$ and $\mathcal{RF}_0(G)$, respectively, contain a $\mathbb{Q}$-vector space of dimension $\mathfrak{c}_G$ as a direct summand; in particular, these groups contain a free-abelian subgroup of rank $\mathfrak{c}_G$, but are not free-abelian, and*

$$|\overline{\mathcal{RF}(G)}| = \mathfrak{c}_G = |\overline{\mathcal{RF}_0(G)}|.$$

(iii) *Every non-trivial normal subgroup $\mathcal{N} \trianglelefteq \mathcal{RF}(G)$ contains a free subgroup of rank $\mathfrak{c}_G$; in particular, $|\mathcal{N}| = \mathfrak{c}_G$, and $\mathcal{N}$ is not soluble.*

(iv) *If $\mathcal{N} \trianglelefteq \mathcal{RF}(G)$ has a non-trivial elliptic element, then $\mathcal{N}$ contains a subgroup isomorphic to a free power $U_0^{*\mathfrak{c}_G}$, where $U_0 := \mathcal{N} \cap G_0$.*

As an illustration how test function theory comes into the picture in establishing Theorem 10.16, we indicate the proof of Part (iii); for full details see [23, Sec. 8].

Let $\{f_\sigma\}_{\sigma\in S}$ be a family of test functions as described in Corollary 10.13 with $L(f_\sigma) = 1$ for all $\sigma \in S$, say. Since $\mathcal{N}$ is non-trivial, it must, according to Proposition 5.12 and Corollary 5.14, contain a hyperbolic element h and, since $\mathcal{N}$ is normal in $\mathcal{RF}(G)$, the core h_1 of h also lies in $\mathcal{N}$. Since h is hyperbolic, we have $L(h_1) > 0$ by Proposition 5.8. Moreover, using local incompatibility of the functions f_σ, it is easy to see that for all but at most two indices $\sigma \in S$, we have

$$f_\sigma h_1 f_\sigma^{-1} = f_\sigma \circ h_1 \circ f_\sigma^{-1}. \tag{23}$$

Deleting these exceptional functions, we obtain a family $\{f_\sigma\}_{\sigma\in S'}$ of pairwise locally incompatible test functions with $|S'| = |S|$, such that (23) holds for all $\sigma \in S'$. Since $\mathcal{N}$ is normal, the subgroup

$$\mathcal{F} := \left\langle f_\sigma h_1 f_\sigma^{-1} : \sigma \in S' \right\rangle$$

is contained in $\mathcal{N}$ and, using the facts that h_1 is cyclically reduced, and that the test functions f_σ are locally incompatible, one finds that $\mathcal{F}$ is freely generated by the elements $f_\sigma h_1 f_\sigma^{-1}$, whence the result.

To conclude, we remark that a very short proof of Theorem 6.5 can be given by combining the hyperbolicity criterion (Theorem 10.10) with Theorem 10.11, Theorem 8.2, and Part (iii) of Lemma 10.8; cf. [21, Cor. 5.4].

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