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ON THE NONLINEAR ELASTIC SIMPLY SUPPORTED BEAM EQUATION

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Abstract

Using a direct variational approach, we consider the existence of solutions and their dependence on a functional parameter for the elastic beam equation by means of investigating the critical points to the relevant Euler action functional.

1 Introduction

In this research we intend to investigate a fourth order Dirichlet problem connected with the elastic beam equation with simply supported ends via direct variational approach. In the recent literature, see for example [3], [14], [17], where also critical point theory is applied, mainly the simplified form of the beam equation

$$\frac{d^4}{dt^4}x = f(t,x) \tag{1}$$

pertaining to rigidly fastened boundary conditions

$$x(0) = x(1) = \dot{x}(0) = \dot{x}(1) = 0 \tag{2}$$

or simply supported conditions

$$x(0) = x(1) = \ddot{x}(0) = \ddot{x}(1) = 0 \tag{3}$$

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is considered. Since equation (1) does not fully reflect the real physical object, we investigate the following model

$$\frac{d^2}{dt^2}\left(E(t)\,I(t)\,\frac{d^2}{dt^2}x(t)\right)+w(t)\,x(t)=f(t,x(t)) \tag{4}$$

with suitable assumptions on f; here $E:[0,1]\to R$ is Young's modulus of elasticity for the beam, $I:[0,1]\to R$ is the moment of inertia of cross section of the beam and w is the load density (force per unit length of a beam); it is natural to assume that $w(t)>0$, $E(t)\geq E_0>0$, $I(t)\geq I_0>0$ for $t\in[0,1]$ and $E,I,w\in L^\infty(0,1)$. However, the simplified version (1) of the beam equation (4) seems to be easier tackled by mathematical methods and therefore a variety of methods could be applied in investigating the existence of solutions. The three critical point theorem due to Ricerri, the Sturm comparison theorem combined with the shooting method and also the Guo-Krasnosel'skij fixed point theorem of cone-expansion compression type were used in [3], [14], [17]. Apart from these methods, there were used the method of upper and lower solutions together with a type of a Landesman-Lazer condition, Leray-Schauder fixed point theorem, degree-theoretic methods, semiorder method on cones of Banach space, minimax method, a priori estimates together with the Krasnosel'skij theorem on cones, see [1], [2], [7], [8], [12], [16].

The case is not as easy with (4) due to the form of the left hand side of the beam equation. Although we may put functions E, I to be fixed constants, we may not put $w=0$ on $[0,1]$ without altering the original model. Let $H=H_0^1(0,1)\cap H^2(0,1)$ considered with the norm

$$\sqrt{\left\|\frac{d}{dt}x\right\|^2_{L^2(0,1)}+\left\|\frac{d^2}{dt^2}x\right\|^2_{L^2(0,1)}}.$$

Via a direct approach in the space we will look for solutions to the following problem

$$\begin{aligned}&\tfrac{d^2}{dt^2}\left(E(t)\,I(t)\,\tfrac{d^2}{dt^2}x(t)\right)+w(t)\,x(t)+F^1_x(t,x(t))=F^2_x(t,x(t))\,u(t)\,,\\&x(0)=x(1)=\ddot{x}(0)=\ddot{x}(1)=0.\end{aligned} \tag{5}$$

A functional parameter $u:[0,1]\to R$ belongs to the set

$$L_M=\{u:[0,1]\to R:\ u\text{ is measurable, }|u(t)|\leq m\text{ for }a.e.\ t\in[0,1]\}\,,$$

$m>0$ is a fixed real number; functions F^1, F^2 are subject to the following conditions:

A1 $F^1, F_x^1 : [0,1] \times R \to R$ *are Caratheodory functions;* F^1 *is continuously differentiable and convex with respect to the second variable in* R *for a. e.* $t \in [0,1]$; $t \to F^1(t,0)$ *is integrable on* $[0,1]$; *function* $t \to F_x^1(t,0)$ *belongs to* $L^2(0,1)$; *function* $t \to \max_{x\in[-d,d]} \left|F^1(t,x)\right|$ *is integrable for any* $d > 0$.

A2 $F^1, F_x^1 : [0,1] \times R \to R$ *are Caratheodory functions, functions* $t \to F^1(t,0)$ *and* $t \to \left(F^1\right)^*(t,0)$ *are integrable on* $[0,1]$; *function*

$$t \to \max_{x\in[-d,d]} \left|F^1(t,x)\right|$$

is integrable for any $d > 0$.

A3 $F^2, F_x^2 : [0,1] \times R \to R$ *are Caratheodory functions, there exists a function* $a \in L^2(0,1)$ *such that*

$$\left|F^2(t,x)\right| \le a(t) \quad \text{for a.e. } t \in [0,1] \text{ and all } x \in R. \tag{6}$$

$\left(F^1\right)^*$ denotes the Fenchel-Young transform of a function F^1 with respect to the second variable, [9], namely

$$\left(F^1\right)^*(t,v) = \sup_{x\in R}\left\{xv - F^1(t,x)\right\} \text{ for a.e. } t \in [0,1].$$

Remark 1.1. *We observe that for any* $x \in H$ *the following estimation holds*

$$|\dot{x}(t) - \dot{x}(s)| = \left|\int_s^t \ddot{x}(\tau)\,d\tau\right| \le \sqrt{t-s}\int_s^t \ddot{x}^2(\tau)\,d\tau$$

$$\le \sqrt{|t-s|}\,\|\ddot{x}\|_{L^2(0,1)} \le \|\ddot{x}\|_{L^2(0,1)}.$$

For any bounded sequence $\{x_k\}_{k=1}^\infty \subset H$, *the sequence of derivatives* $\{\dot{x}_k\}_{k=1}^\infty$ *is uniformly convergent (up to the subsequence) by the Ascoli-Arzela Theorem and thus strongly convergent in* $H_0^1(0,1)$. *Moreover, we have the following Poincaré type inequalities for any* $v \in H$, *see [10]*

$$\|v\|_{L^2(0,1)} \le \frac{1}{\pi}\|\dot{v}\|_{L^2(0,1)} \text{ and } \|\dot{v}\|_{L^2(0,1)} \le \frac{1}{\pi}\|\ddot{v}\|_{L^2(0,1)}. \tag{7}$$

The paper is organized as follows. Firstly we investigate the dependence on a functional parameter for the action functionals. Next we investigate the existence of a solution for problem (5) and its dependence on a parameter.

2 Dependence of the argument of a minimum on a functional parameter

The Euler action functional $J_u : H \to R$ associated with (5) is given by

$$J_u(x) = \frac{1}{2}\int_0^1 E(t) I(t) \left(\frac{d^2}{dt^2}x(t)\right)^2 dt + \frac{1}{2}\int_0^1 w(t) x^2(t)\, dt +$$

$$-\int_0^1 F^2(t, x(t))\, u(t)\, dt + \int_0^1 F^1(t, x(t))\, dt.$$

J_u is well defined with either **A1-A3** or **A2-A3**. We mention here that assumptions **A1-A3** or **A2-A3** do not provide the Gâteaux differentiability of J_u. It is interesting to note that the dependence on functional parameter u can be investigated for the arguments of a minimum for J_u without invoking its differentiability contrary to what is done in [6].

Lemma 2.1. *Suppose that either **A1-A3** or **A2-A3** hold. For any fixed $u \in L_M$ functional is coercive and weakly l.s.c. on H. For any fixed $u \in L_M$ there exists $x_u \in H$ such that* $\inf_{x\in H} J_u(x) = J_u(x_u)$.

Let us fix $u \in L_M$ and let $\{x_n\}_{n=1}^{\infty} \subset H$ be such a sequence that x_n converges to x weakly in H. By Remark 1.1 sequence $\{x_n\}_{n=1}^{\infty}$ contains a subsequence, denoted by $\{x_n\}_{n=1}^{\infty}$, convergent strongly in $H_0^1(0,1)$ and also convergent uniformly. The Lebesgue Dominated Convergence Theorem and (6) show that

$$\int_0^1 F^2(t, x_n(t))\, u(t)\, dt \to \int_0^1 F^2(t, x(t))\, u(t)\, dt \text{ as } n \to \infty.$$

Since $\{x_n\}_{n=1}^{\infty}$ is uniformly convergent, there exists a number $d > 0$ such that $|x_n(t)| \le d$ for all $t \in [0,1]$. Hence, by the Lebesgue dominated convergence

$$\int_0^1 F^1(t, x_n(t))\, dt \to \int_0^1 F^1(t, x(t))\, dt \text{ as } n \to \infty.$$

Since the remaining terms of J_u are convex and l.s.c., these are also weakly l.s.c. on H and so J_u is weakly l.s.c.

By the convexity of F^1 with respect to the second variable and by **A1** we see that

$$\begin{aligned} &\int_0^1 F^1(t, x(t))\, dt \ge \int_0^1 F^1(t,0)\, dt + \int_0^1 F_x^1(t,0)\, x(t)\, dt \ge \\ &\int_0^1 F^1(t,0)\, dt - \left\|F_x^1(\cdot,0)\right\|_{L^2(0,1)} \|x\|_{L^2(0,1)} \end{aligned} \tag{8}$$

for any $x \in H$. By (8) and by relation

$$-\int_0^1 \left|F^2(t,x(t))\,u(t)\right| dt \geq -m\int_0^1 |a(t)|\,dt. \tag{9}$$

J_u is coercive on H with assumptions **A1-A3.**

Let as assume **A2-A3.** By inequality

$$\int_0^1 F^1(t,x(t))\,dt \geq -\int_0^1 \left(F^1\right)^*(t,0)\,dt \tag{10}$$

and by (9) we see that J_u is coercive.

Finally, since J_u is coercive and weakly l.s.c. in both cases, there exists $x_u \in H$ such that $J_u(x_u) = \inf_{x\in H} J_u(x)$.

Theorem 2.1. *We suppose that either **A1, A3** or **A2, A3** hold. Let $\{u_k\}_{k=1}^{\infty}$, $u_k \in L_M$, be such a sequence that $\lim_{k\to\infty} u_k = \overline{u}$ weakly in $L^2(0,1)$. For each $k = 1,2,...$ the set*

$$V_{u_k} = \left\{x \in H : J_u(x) = \inf_{v\in H} J_u(v)\right\}$$

is nonempty and for any sequence $\{x_k\}_{k=1}^{\infty}$, $x_k \in V_{u_k}$, of arguments of a minimum of J_{u_k} corresponding to u_k, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty} \subset H$ and an element $\overline{x} \in V_{\overline{u}}$ such that $\lim_{n\to\infty} x_{k_n} = \overline{x}$ (strongly in $C(0,1)$, strongly in $H_0^1(0,1)$, weakly in $H^2(0,1)$) and $J_{\overline{u}}(\overline{x}) = \inf_{x\in H} J_{\overline{u}}(x)$.

Proof. Firstly, we investigate the convergence of the sequence of the arguments of a minimum. Secondly, we show the last assertion.

By Lemma 2.1 for each $k = 1,2,...$ there exists

$$x_k \in V_{u_k} \subset S_k = \{x : J_{u_k}(x) \leq J_{u_k}(0)\}.$$

With **A1, A3** for any $x \in S_k$ we have

$$-\int_0^1 F^2(t,0)\,u_k(t)\,dt + \int_0^1 F^2(t,x(t))\,u_k(t)\,dt \leq 2m\int_0^1 |a(t)|\,dt. \tag{11}$$

By (8) we obtain

$$\int_0^1 F^1(t,0)\,dt - \int_0^1 F^1(t,x(t))\,dt \leq -\left\|F_x^1(\cdot,0)\right\|_{L^2(0,1)} \|x\|_{L^2(0,1)}.$$

By writing $0 \leq J_{u_k}(0) - J_{u_k}(x_k)$ explicitly we see that

$$\begin{aligned} &0 \leq -\tfrac{1}{2}\int_0^1 E(t)\, I(t) \left(\tfrac{d^2}{dt^2} x_k(t)\right)^2 dt - \tfrac{1}{2}\int_0^1 w(t)\, x_k^2(t)\, dt \\ &\leq 2m\int_0^1 |a(t)|\, dt - \left\|F_x^1(\cdot,0)\right\|_{L^2(0,1)} \|x_k\|_{L^2(0,1)}. \end{aligned}$$

By (7) we obtain

$$\begin{aligned} &\tfrac{1}{2}E_0 I_0 \left\|\tfrac{d^2}{dt^2}x_k\right\|_{L^2(0,1)}^2 - \tfrac{1}{\pi^2}\left\|F_x^1(\cdot,0)\right\|_{L^2(0,1)} \left\|\tfrac{d^2}{dt^2}x_k\right\|_{L^2(0,1)} \leq \\ &2m\int_0^1 |a(t)|\, dt. \end{aligned} \tag{12}$$

With **A2, A3** we also have (11). By (10) we see that

$$\begin{aligned} &\tfrac{1}{2}\int_0^1 E_0 I_0 \left(\tfrac{d^2}{dt^2}x_k(t)\right)^2 dt \leq \\ &2m\int_0^1 |a(t)|\, dt + \int_0^1 F^1(t,0)\, dt + \int_0^1 \left(F^1\right)^*(t,0)\, dt. \end{aligned} \tag{13}$$

Therefore either by (12) or by (13) there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ weakly convergent in H, which up to a subsequence may be assumed to be strongly convergent in $H_0^1(0,1)$ and so convergent uniformly.

Next,by Lemma 2.1 applied with $\overline{u}$ there exists $x_0 \in H$ such that $J_{\overline{u}}(x_0) = \inf_{x\in H} J_{\overline{u}}(x)$. We suppose that $J_{\overline{u}}(x_0) < J_{\overline{u}}(\overline{x})$ and investigate the right hand side of the equivalent inequality

$$\begin{aligned} &\delta < \left(J_{u_{k_n}}(x_{k_n}) - J_{\overline{u}}(x_0)\right) - \left(J_{u_{k_n}}(x_{k_n}) - J_{u_{k_n}}(\overline{x})\right) \\ &- \left(J_{u_{k_n}}(\overline{x}) - J_{\overline{u}}(\overline{x})\right), \end{aligned} \tag{14}$$

where $\delta > 0$ is certain constant such that $\delta < J_{\overline{u}}(\overline{x}) - J_{\overline{u}}(x_0)$. By Lebesgue Dominated Convergence Theorem, we see that

$$\lim_{n\to\infty}\left(J_{u_{k_n}}(\overline{x}) - J_{\overline{u}}(\overline{x})\right) = 0.$$

By the generalized Krasnosel'skij Theorem, see [5], and by (15) we see that $\lim_{n\to\infty} F^2(\cdot, x_{k_n}(\cdot)) = F^2(\cdot, \overline{x}(\cdot))$ strongly in $L^2(0,1)$. Since $\lim_{n\to\infty} u_{k_n} = \overline{u}$ weakly in $L^2(0,1)$, we see that

$$\lim_{n\to\infty}\int_0^1 F^2(t, x_{k_n}(t))\, u_{k_n}(t)\, dt = \int_0^1 F^2(t, \overline{x}(t))\, \overline{u}(t)\, dt.$$

Thus we have $\lim_{n\to\infty}\left(J_{u_{k_n}}(x_{k_n})-J_{u_{k_n}}(\overline{x})\right)=0$. By similar arguments we show that $\lim_{k_n\to\infty}\left(J_{u_{k_n}}(x_0)-J_{\overline{u}}(x_0)\right)=0$. Now, since x_{k_n} minimizes $J_{u_{k_n}}$ over H we get

$$\lim_{n\to\infty}\left(J_{u_{k_n}}(x_{k_n})-J_{\overline{u}}(x_0)\right)\leq\lim_{k_n\to\infty}\left(J_{u_{k_n}}(x_0)-J_{\overline{u}}(x_0)\right)=0.$$

Therefore we obtain in (14) that $\delta<0$. Thus $J_{\overline{u}}(\overline{x})=\inf_{x\in H}J_{\overline{u}}(x)$ and so $\overline{u}\in V_{\overline{u}}$. □

3 Existence of solutions to beam equation and their dependence on a parameter

Now we proceed to investigate the existence of solutions to (5) and their dependence on a functional parameter u. We must make additional assumptions which would ensure that J_u is differentiable in the sense of Gâteaux.

A4 *For any $d\in R$ there exists a function $f\in L^2(0,1)$ (depending on d), $f_d(t)>0$ for a.e. $t\in[0,1]$, such that*

$$\max\left\{\left|F_x^1(t,-b)\right|,\left|F_x^1(t,b)\right|\right\}\leq f_d(t)\quad \text{for a.e. } t\in[0,1]. \tag{15}$$

there exists a function $b\in L^2(0,1)$ such that

$$\left|F_x^2(t,x)\right|\leq b(t)\quad \text{for all } x\in R \text{ and for a.e.} t\in[0,1];$$

A5 *For any $d>0$ there exists a function $f_d\in L^2(0,1)$ (depending on d), $f_d(t)>0$ for a.e. $t\in[0,1]$, such that*

$$\left|F_x^1(t,x)\right|\leq f_d(t)\,,\text{for all } x\in[-d,d], \text{ for a.e. } t\in[0,1]; \tag{16}$$

there exists a function $b\in L^2(0,1)$ such that

$$\left|F_x^2(t,x)\right|\leq b(t)\,\text{for all } x\in R \text{ and for a.e.} t\in[0,1].$$

Lemma 3.1. *Suppose that **A1-A3-A4** or **A2-A3-A5** hold. For any fixed $u\in L_M$ the functional J_u has an argument of a minimum over H which satisfies (5) in the weak sense, i.e. for any $g\in H$ we have*

$$\begin{aligned}&\textstyle\int_0^1 E(t)\,I(t)\,\frac{d^2}{dt^2}x(t)\,\frac{d^2}{dt^2}g(t)\,dt+\int_0^1 w(t)\,x(t)\,g(t)\,dt\\&+\textstyle\int_0^1(-F_x^2(t,x(t))\,u(t)\,g(t)+F_x^1(t,x(t))\,g(t))dt=0.\end{aligned} \tag{17}$$

Proof. It is easy to see that with either assumptions **A1-A3-A4** or **A2-A3-A5** functional J_u has a Gâteaux derivative $\frac{d}{dx}J_u$ at any $x \in H$. Only the differentiability of the term $\int_0^1 F^1(t, x(t))\,dt$ requires some explanation due to the lack of a global growth conditions. We observe that for any $v \in H$ there exists a constant $d_v > 0$ such that $v(t) \in [-d_v, d_v]$ for a.e. $t \in [0,1]$. Now, by either **A4** or **A5** we see that for any $\varepsilon > 0$ and any fixed $g \in H$ a function $t \to F_x^1(t, x(t) + \varepsilon g(t))$ belongs to $L^2(0,1)$. It is obvious with **A5** while with **A4** it follows by the same argument since the derivative of a convex function is nondecreasing. □

Proof. Summarizing J_u is coercive, weakly l.s.c. and Gâteaux differentiable on H and so it has an argument of a minimum x_u for which $\frac{d}{dx}J_u(x_u) = 0$, i.e. for which (17) holds. □

Finally, we have the following theorem

Theorem 3.1. *Suppose that either **A1, A3, A4** or **A2, A3, A5** hold. Let $u \in L_M$ be fixed. There exists*

$$x_u \in V_u = \left\{ x \in H : J_u(x) = \inf_{v \in H} J_u(v) \ \text{ and } \frac{d}{dx}J_u(x) = 0 \right\}$$

and such that x_u satisfies (5) in the weak sense (17). Moreover, x_u satisfies (5) for a.e. $t \in [0,1]$ and is subject to boundary conditions (3) and $\frac{d^2}{dt^2}\left(E(\cdot)\, I(\cdot)\, \frac{d^2}{dt^2}x_u(\cdot)\right) \in L^2(0,1)$.

Proof. By Lemma 3.1 it remains to be shown that x_u satisfies (5) for a.e. $t \in [0,1]$ and that it is subject to boundary conditions (3). We mention that the last assertion does not follow by the definition of the weak solution. Since relation (17) holds for any $g \in H$, it holds also for any $g \in C_0^\infty(0,1)$. Now by the application of the higher order version of the Fundamental Lemma of the calculus of variations, see [13], we obtain that x_u satisfies (5) for a.e. $t \in [0,1]$. Obviously now $\frac{d^2}{dt^2}\left(E(\cdot)\, I(\cdot)\, \frac{d^2}{dt^2}x_u(\cdot)\right) \in L^2(0,1)$. □

Proof. Next, given any $g \in H$, we integrate (17) by parts to obtain

$$\int_0^1 \frac{d^2}{dt^2}\left(E(t)\, I(t)\, \frac{d^2}{dt^2}x_u(t)\right) g(t)\,dt + \left(\dot{g}(1)\,\ddot{x}_u(1) - \dot{g}(0)\,\ddot{x}_u(0)\right)$$

$$\int_0^1 w(t)\, x_u(t)\, g(t)\,dt +$$

$$\int_0^1 (-F_x^2(t, x_u(t))\, u(t)\, g(t) + F_x^1(t, x_u(t))\, g(t))dt = 0.$$

Since x_u satisfies (5) a.e. we see that $\dot{g}(1)\ddot{x}_u(1) - \dot{g}(0)\ddot{x}_u(0) = 0$. Since g is arbitrary we must have $\ddot{x}_u(1) = \ddot{x}_u(0) = 0$. □

Theorem 3.2. *We suppose that either **A1, A3, A4** or **A2, A3, A5** hold. Let $\{u_k\}_{k=1}^{\infty}$, $u_k \in L_M$, be such a sequence that $\lim_{k\to\infty} u_k = \overline{u}$ weakly in $L^2(0,1)$. For each $k = 1,2,...$ the set V_{u_k} is nonempty and for any sequence $\{x_k\}_{k=1}^{\infty}$ of solutions $x_k \in V_{u_k}$ to the problem (5) corresponding to u_k, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty} \subset H$ and an element $\overline{x} \in H$ such that $\lim_{n\to\infty} x_{k_n} = \overline{x}$ (strongly in $C(0,1)$, strongly in $H_0^1(0,1)$, weakly in $H^2(0,1)$) and $J_{\overline{u}}(\overline{x}) = \inf_{x\in H} J_{\overline{u}}(x)$. Moreover, $\overline{x} \in V_{\overline{u}}$ and satisfies for a.e. $t \in [0,1]$*

$$\frac{d^2}{dt^2}\left(E(t)\,I(t)\,\frac{d^2}{dt^2}\overline{x}(t)\right) + w(t)\,\overline{x}(t) = F_x^2(t,\overline{x}(t))\,\overline{u}(t) - F_x^1(t,\overline{x}(t)),$$
$$\overline{x}(0) = \overline{x}(1) = \frac{d^2}{dt^2}\overline{x}(0) = \frac{d^2}{dt^2}\overline{x}(1) = 0. \tag{18}$$

Proof. All the assertions of the Theorem follow by Theorem 2.1 apart from the last one. Since $J_{\overline{u}}$ is differentiable in the sense of Gâteaux we have $\overline{x} \in V_{\overline{u}}$ and since $J_{\overline{u}}(\overline{x}) = \inf_{x\in H} J_{\overline{u}}(x)$ it follows that $\overline{x}$ satisfies (18). □

4 Examples

Finally, we give examples of nonlinear terms satisfying our assumptions.

Example 4.1 (Conditions **A1, A3, A4**). *Let $F^2(t,x) = a(t)\,f_2(x)$, $F^1(t,x) = g(t)\,f_1(x)$, where $a,\ g \in L^2(0,1)$, $f_1, f_2 \in C^1(R)$, f_1 is convex (say, $f_1(x) = e^x$) and f_2 is bounded and has a bounded derivative (say, $f_2(x) = \arctan x$). Then $\left|F_x^2(t,x)\right| = \left|a(t)\,\frac{d}{dx}f_2(x)\right| \le |a(t)|\sup_{x\in R}\left|\frac{d}{dx}f_2(x)\right|$ and since $F_x^1(t,x) = g(t)\,\frac{d}{dx}f_2(x)$ we see that for any fixed $x \in R$ function $t \to F_x^1(t,x)$ belongs to $L^2(0,1)$.*

Example 4.2 (Conditions **A2, A3, A5**). *Let $F^2(t,x) = f(t)\,g(x)$, $g \in C^1$ has a bounded derivative and $F^1(t,x) = \frac{1}{4}g_1(t)\,x^4 - \frac{1}{2}g_2(t)\,x^2$, where $f \in L^2(0,1)$, $g_1,\ g_2 \in L^\infty(0,1)$, $g_1(t), g_2(t) > 0$ for a.e. $t \in [0,1]$. Then*

$$\left|F_x^2(t,x)\right| = \left|f(t)\,\frac{d}{dx}g(x)\right| \le |f(t)|\sup_{x\in R}\left|\frac{d}{dx}g(x)\right| = a(t) \ \text{ and } a \in L^2(0,1)$$

and

$$F^1(t,x) = g_1(t)\,x^3 - g_2(t)\,x.$$

Again, for any fixed $x \in R$ the function $t \to |g_1(t)|\,x^3 + |g_2(t)|\,x$ belongs to $L^2(0,1)$. We remark that F^1 need not be convex on R and that $t \to \left(F^1\right)^*(t,0)$

is integrable. Indeed, for a.e. (fixed) $t \in [0,1]$ *function* $x \to -\frac{1}{4}g_1(t)x^4 + \frac{1}{2}g_2(t)x^2$ *has its maximum* x_M *satisfying* $g_1(t)x^3 - g_2(t)x = 0$ *so either*

$$x_M = 0 \text{ and } \left(F^1\right)^*(t,0) = \sup_{x\in R}\left\{-\frac{1}{4}g_1(t)x^4 + \frac{1}{2}g_2(t)x^2\right\} = 0$$

or

$$x_M^2 = \frac{g_2(t)}{g_1(t)} \text{ and } \left(F^1\right)^*(t,0) = -\frac{1}{2}\frac{(g_2(t))^2}{g_1(t)}.$$

References

[1] P. Amster, P.P. Cárdenas Alzate, Existence of solutions for some nonlinear beam equations. *Port. Math.* (N.S.), **63** (2006), no. 1, 113–125.

[2] Z. Bai, H. Wang, On positive solutions of some nonlinear fourth-order beam equations, *J. Math. Anal. Appl.*, **270** (2002), no. 2, 357–368.

[3] G. Bonanno, B. Di Bella, A boundary value problem for fourth-order elastic beam equation, *J. Math. Anal. Appl.*, **342** (2008), 1166-1176.

[4] M. Galewski, On the optimal control problem governed by the nonlinear elastic beam equation, *App. Math. Comp.*, **203** (2008), no. 2, 916–920.

[5] D. Idczak, A. Rogowski, On a generalization of Krasnoselskii's theorem, *J. Austral. Math. Soc.*, **72** (2002), no. 3, 389-394.

[6] U. Ledzewicz, H. Schättler, S. Walczak, Optimal control systems governed by second-order ODEs with Dirichlet boundary data and variable parameters, *Illinois J. Math.*, **47** (2003), no.4, 1189-1206.

[7] X.-L. Liu, W.-T. Li, Positive solutions of the nonlinear fourth-order beam equation with three parameters. *J. Math. Anal. Appl.*, **303** (2005), no. 1, 150–163.

[8] T. F. Ma, Positive solutions for a beam equation on a nonlinear elastic foundation. *Math. Comput. Modelling*, **39** (2004), no. 11-12, 1195–1201.

[9] J. Mawhin, *Problèmes de Dirichlet variationnels non linéaires*, Les Presses de l'Université de Montréal, 1987.

[10] L. A. Peletier, R. K. A. M Van der Vorst; V.K. Troy, Stationary solutions of a fourth-order nonlinear diffusion equation, *Differ. Equ.*, **31** (1995), no. 2, 301–314.

[11] R. T. Rockafellar, Convex Integral Functionals and Duality, in E. Zarantonello (Ed) *Contributions to Nonlinear Functional Analysis*, Academic Press, New York, 1971, 215-236.

[12] L. Sanchez, Boundary value problems for some fourth order ordinary differential equations, *Appl. Anal.*, **38** (1990), no. 3, 161–177.

[13] J.L. Troutman, *Variational Calculus with Elementary Convexity*, Springer-Verlag, New York Inc, 1983

[14] X. Yang, K. Lo, Existence of a positive solution to a fourth-order boundary value problem, *Nonlinear Anal.*, **69** (2008), 2267–2273.

[15] B., Yang, Positive solutions for the beam equation under certain boundary conditions, *Electron. J. Differential Equations*, **2005**, no. 78

[16] Q. Yao, Solvability of an elastic beam equation with Caratheodory function, *Math. Appl. (Wuhan)*, **17** (2004), no. 3, 389–392.

[17] Q. Yao, Positive solutions of a nonlinear elastic beam equation rigidly fastened on the left and simply supported on the right, *Nonlinear Anal.*, **69** (2008), 2267–2273.

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