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MULTIPLIERS ON SOME LORENTZ SPACES

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Abstract

This paper is concerned with the characterization of the spaces of bounded linear operators commuting with translation operators on some Lorentz spaces which are defined on a locally compact abelian group with Haar measure. These characterizations are motivated by those of Figà-Talamanca [8, 9], Avcı and Gürkanlı [1] wherein the concept of tensor product is used as a basic tool for obtaining them.

1 Introduction

Let G be a locally compact abelian group with Haar measure dx. For $1 < p_1, p_2 < \infty$, $1 \leq q_1, q_2 \leq \infty$ or $p_1, p_2 = 1 = q_1, q_2$, $p_1, p_2 = \infty = q_1, q_2$, Avcı and Gürkanlı defined the space $A_{p_1,q_1}^{p_2,q_2}(G)$ by regarding convolution operator's allowance which is acting on Lorentz spaces and showed some topological properties of $A_{p_1,q_1}^{p_2,q_2}(G)$ spaces. Again, under some assumptions, they found $L(p_1, q_1) \otimes_{L_1(G)} L(p_2, q_2) \cong A_{p_1,q_1}^{p_2,q_2}(G)$ and some important results in [1]. Also in [5], the space of multipliers from Beurling algebra to a subspace of a weighted Lorentz space is examined by relative completion method.

Throughout the paper, $C_c(G)$ and $C_0(G)$ will denote the space of complex-valued continuous functions on G with compact support and the space of complex-valued continuous functions on G vanishing at infinity, respectively. Also, L_y (R_y) will stand for the left (right) translation operators which are given by $L_y f(x) = f(x-y)$ $(R_y f(x) = f(x+y))$ for all $x, y \in G$.

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Certain well-known terms such as multiplier, module homomorphism, (semi) homogeneous Banach space, rearrangement invariant Banach function space etc. are used frequently in the paper. We will not give their definitons and properties explicitly. One can find more about these terms in [2,3,12]. For the convenience of the reader, we will now review briefly what we need from the theory of Lorentz spaces.

Let (G, Σ, μ) be a positive measure space and let f be a complex-valued, measurable function on G. Then the *rearrangement function* of f on $(0, \infty)$ is defined by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) = \mu\{x \in G : \ |f(x)| > y\} \leq t\ \},\ t \geq 0$$

where $\inf \phi = \infty$. Also the *average(maximal) function* of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \ \ ,\ t > 0.$$

If the functions are defined as

$$\|f\|^*_{p,q} = \|f\|^*_{p,q,\mu} = \left(\frac{q}{p}\int_0^\infty [f^*(t)]^q\, t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}} \text{ for } p, q \in (0, \infty)$$

and

$$\|f\|^*_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) \text{ for } 0 < p, q = \infty,$$

then the Lorentz spaces denoted by $L(p,q)(G)$ is defined to be the vector space of all (equivalence classes of) measurable functions f on G such that $\|f\|^*_{p,q} < \infty$. We know that, for $1 \leq p \leq \infty$, $\|f\|^*_{p,p} = \|f\|_p$ and $L_p(G) = L(p,p)(G)$. It is also known that the usage of f^{**} instead of f^* causes a norm $\|\cdot\|_{p,q}$ on $L(p,q)(G)$ for $1 < p < \infty$ and $1 \leq q \leq \infty$ with

$$\|f\|^*_{p,q} \leq \|f\|_{p,q} \leq \frac{p}{p-1}\|f\|^*_{p,q} \tag{1}$$

for each $f \in L(p,q)(G)$.

The space $\left(L(p,q)(G), \|\cdot\|_{pq}\right)$ is a reflexive rearrangment-invariant Banach function spaces with its associate space $\left(L(p',q')(G), \|\cdot\|_{p'q'}\right)$ where $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$ [2,10]. We also know that

$$L(p,q_1)(G) \subset L(p,q_2)(G) \tag{2}$$

if $p \in (0, \infty)$, $0 < q_1 \leq q_2 \leq \infty$ and

$$L(p_2, q_2)(G) \subset L(p_2, \infty)(G) \subset L(p_1, q_1)(G) \tag{3}$$

provided that $\mu(G) < \infty$ and $p_1 \leq p_2$ [10]. For further properties of Lorentz spaces, we refer to [2,4,10,16].

Let $\wp(p, q, r, s, G)$ be the set of all complex-valued functions f which can be written as

$$f = f_1 + f_2 \quad \text{with} \quad (f_1, f_2) \in L(p, q)(G) \times L(r, s)(G).$$

If we define a norm on $\wp(p, q, r, s, G)$ by

$$\|f\|_{\wp} = \inf\left(\|f_1\|_{p,q} + \|f_2\|_{r,s}\right), \tag{4}$$

where the infimum is taken over all such decompositions of f, then $\wp(p, q, r, s, G)$ is a Banach space under this norm. This can be derived from [10] and [14]. Similarly, if $D(p, q, r, s, G)$ denotes the set of all complex-valued functions defined on G which are in $L(p, q)(G) \cap L(r, s)(G)$, then we can introduce a norm by

$$\|g\|_D = \max\left(\|g\|_{p,q}\ ,\ \|g\|_{r,s}\right). \tag{5}$$

Hence $D(p, q, r, s, G)$ is also a Banach space with the norm $\|\cdot\|_D$ due to [10, 14]. It is not hard to see that $D(p, q, r, s, G)$ is a Banach $L^1(G)-$module where $1 < p, r < \infty$, $1 \leq q, s < \infty$.

Again, it is easy to see that $D(p, q, r, s, G)$ and $\wp(p, q, r, s, G)$ are reflexive rearrangment-invariant Banach function spaces for $1 < p, q, r, s < \infty$ and

$$D(p, q, r, s, G)^* \cong \wp(p', q', r', s', G), \tag{6}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{s} + \frac{1}{s'} = 1$ [10, 14].

2 Multipliers Spaces

In this section, we will introduce the space of multipliers acting on some Lorentz spaces. Before starting to define multipliers spaces, we will give the following theorems whose proofs can be found in [2], [16] and [17] respectively.

Theorem 1. *Let T be a convolution operator and $h = T(f, g) = f * g$. T can be uniquely extended so that if $f \in L(p_1, q_1)(G)$, $1 < p_1 < \infty$ and $g \in L(p_2, q_2)(G)$ where $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $\frac{1}{q_1} + \frac{1}{q_2} \geq 1$, then $h \in L^\infty(G)$ and*

$$\|h\|_\infty \leq C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2},$$

where C is a constant depending on q_1 and q_2.

Theorem 2. *If T is a convolution operator $h = T(f,g) = f * g$ for $f \in L(p_1,q_1)(G)$, $g \in L(p_2,q_2)(G)$ with $\frac{1}{p_1}+\frac{1}{p_2}>1$, then $h \in L(r,s)(G)$, where $\frac{1}{p_1}+\frac{1}{p_2}-1=\frac{1}{r}$ and $s \geq 1$ is any number such that $\frac{1}{q_1}+\frac{1}{q_2}\geq\frac{1}{s}$. Morever*

$$\|h\|_{r,s} \leq 3r\,\|f\|_{p_1,q_1}\,\|g\|_{p_2,q_2}\,. \tag{7}$$

Theorem 3. *if $f \in L(m,q_2)(G) \cap L(n,q_2)(G)$ and $m<n$ then $f \in L(p_2,q_2)(G)$ for all $m<p_2<n$. Morever*

$$\|f\|^*_{p_2,q_2} \leq \left(\|f\|^*_{m,q_2}\right)^{\beta}\left(\|f\|^*_{n,q_2}\right)^{1-\beta}, \tag{8}$$

where $\beta = \left(\frac{1}{p_2}-\frac{1}{n}\right)\left(\frac{1}{m}-\frac{1}{n}\right)^{-1}$.

2.1 Multipliers from $L(p_1,q_1)(G)$ into $\wp(m',q_2',n',q_2')$

By taking Theorem 3 into consideration, define $K(G)$ to be set of all functions h which can be written in the form

$$h = \sum_{i=1}^{\infty} f_i * g_i,$$

where $f_i \in C_c(G) \subset L(p_1,q_1)(G)$, $g_i \in D(m,q_2,n,q_2,G)$ with

$$\sum_{i=1}^{\infty} \|f_i\|_{p_1,q_1}\,\|g_i\|_D < \infty$$

and $m<n$. Here m',q_2',n',q_2' are conjugates of m,q_2,n,q_2 respectively. If we define a norm on $K(G)$ by

$$|||h||| = \inf\left\{\sum_{i=1}^{\infty}\|f_i\|_{p_1,q_1}\,\|g_i\|_D \;:\; h=\sum_{i=1}^{\infty} f_i * g_i,\; h\in K(G)\right\},$$

where the infimum is taken over all such representations of h in $K(G)$, then evidently, the function $|||\cdot|||$ is a norm of $K(G)$ and $K(G)$ is a Banach space under this norm. If we pay attention to Theorem 2 and condition (8), we get

$$\|f*g\|_{r,s} \leq \|f\|_{p_1,q_1}\,\|g\|_{p_2,q_2} \leq \|f\|_{p_1,q_1}\,\|g\|_D\,,$$

for $f \in C_c(G) \subset L(p_1,q_1)(G)$ and $g \in L(m,q_2)(G)\cap L(n,q_2)(G)$ where $m<p_2<n$, $\frac{1}{p_1}+\frac{1}{p_2}>1$, $\frac{1}{p_1}+\frac{1}{p_2}-1=\frac{1}{r}$ and $s\geq 1$ is any number such that $\frac{1}{q_1}+\frac{1}{q_2}\geq\frac{1}{s}$. It is easy to see that $K(G)\subset L(r,s)(G)$ and the topology so defined is not weaker than the topology induced from $L(r,s)(G)$.

Theorem 4. *Let G be a locally compact abelian group. If condition (8) is satisfied and* $\frac{1}{p_1}+\frac{1}{p_2}>1$, $\frac{1}{p_1}+\frac{1}{p_2}-1=\frac{1}{r}$ *and* $s\geq 1$ *is any number such that* $\frac{1}{q_1}+\frac{1}{q_2}\geq\frac{1}{s}$, *then the space of multipliers* $M\left(L\left(p_1,q_1\right)(G),\ \wp\left(m',q_2',n',q_2',G\right)\right)$ *is isometrically isomorphic to* $\left(K\left(G\right)\right)^*$, *the dual space of* $K\left(G\right)$.

Proof. For any $T\in M\left(L\left(p_1,q_1\right)(G),\ \wp\left(m',q_2',n',q_2'\right)\right)$, define

$$t\left(h\right)=\sum_{i=1}^{\infty}Tf_i*g_i\left(0\right),$$

for $h=\sum\limits_{i=1}^{\infty}f_i*g_i$ in $K\left(G\right)$. Firstly, we will show that t is well-defined, i.e. $t\left(h\right)$ is independent of the particular representation of h chosen. To this end, it is sufficent to show that if $h=\sum\limits_{i=1}^{\infty}f_i*g_i=0$ in $K\left(G\right)$ and $\sum\limits_{i=1}^{\infty}\left\|f_i\right\|_{p_1,q_1}\left\|g_i\right\|_D<\infty$, then $\sum\limits_{i=1}^{\infty}Tf_i*g_i\left(0\right)=0$.

It is known by [1] that $L\left(p,q\right)(G)$ has an approximate identity $\left\{e_\alpha\right\}_{\alpha\in I}$ in $L^1\left(G\right)$ with compactly supported such that $\left\|e_\alpha\right\|_1=1$ for each $\alpha\in I$. Then for each $f\in L\left(p_1,q_1\right)(G)$, we have

$$\lim_{\alpha}\left\|e_\alpha*f-f\right\|_{p_1,q_1}=0. \tag{9}$$

Therefore using (9) and the fact that T is a multiplier, we obtain

$$\left|T\left(e_\alpha*f_i\right)*g_i\left(0\right)-Tf_i*g_i\left(0\right)\right|\leq\left\|T\right\|\left\|e_\alpha*f_i-f_i\right\|_{p_1,q_1}\left\|g_i\right\|_D, \tag{10}$$

for all $g_i\in D\left(m,q_2,n,q_2,G\right)$ and so

$$\lim_{\alpha}T\left(e_\alpha*f_i\right)*g_i\left(0\right)=Tf_i*g_i\left(0\right). \tag{11}$$

Also for each $e_\alpha\in C_c\left(G\right)$ and $f_i\in C_c\left(G\right)$, we have

$$T\left(e_\alpha*f_i\right)=Te_\alpha*f_i, \tag{12}$$

by [12] or [7, Lemma 2.1]. Since $h=\sum\limits_{i=1}^{\infty}f_i*g_i=0$ and the series $\sum\limits_{i=1}^{\infty}f_i*g_i$ converges uniformly, we get

$$\begin{aligned}\sum_{i=1}^{\infty}T\left(e_\alpha*f_i\right)*g_i\left(0\right)&=\sum_{i=1}^{\infty}\int_G T\left(e_\alpha\right)\left(-y\right)\left(f_i*g_i\right)\left(y\right)dy \quad (13)\\&=\int_G T\left(e_\alpha\right)\left(-y\right)\sum_{i=1}^{\infty}\left(f_i*g_i\right)\left(y\right)dy=0.\end{aligned}$$

by (12). Now we will show that $\sum\limits_{i=1}^{\infty} T(e_\alpha * f_i) * g_i(0)$ converges uniformly with respect to α. Since

$$\left|\sum_{i=1}^{\infty} T(e_\alpha * f_i) * g_i(0)\right| \leq \sum_{i=1}^{\infty} \|T(e_\alpha * f_i)\|_{\wp} \|g_i\|_D \tag{14}$$
$$\leq \|T\| \sum_{i=1}^{\infty} \|e_\alpha * f_i\|_{p_1,q_1} \|g_i\|_D$$
$$\leq \|T\| \sum_{i=1}^{\infty} \|f_i\|_{p_1,q_1} \|g_i\|_D < \infty,$$

we have

$$\lim_{\alpha} \sum_{i=1}^{\infty} T(e_\alpha * f_i) * g_i(0) = \sum_{i=1}^{\infty} Tf_i * g_i(0) = 0, \tag{15}$$

by using (11) and (13). Thus t is well-defined.

It is obvious that the mapping $T \to t$ is linear and an isometry. Indeed,

$$\begin{aligned} |t(h)| &\leq \sum_{i=1}^{\infty} |Tf_i * g_i(0)| \\ &\leq \sum_{i=1}^{\infty} \|Tf_i\|_{\wp} \|g_i\|_D \\ &\leq \|T\| \sum_{i=1}^{\infty} \|f_i\|_{p_1,q_1} \|g_i\|_D \end{aligned}$$

implies that

$$|t(h)| \leq \|T\| \, |||h||| .$$

Hence $\|t\| \leq \|T\|$. On the other hand, according to (6) we obtain

$$\begin{aligned} \|T\| &= \sup\left\{|Tf * g(0)| : \ \|f\|_{p_1,q_1} \leq 1, \ \|g\|_D \leq 1\right\} \\ &= \sup\left\{|t(f * g)| : \ \|f\|_{p_1,q_1} \leq 1, \ \|g\|_D \leq 1\right\} \leq \|t\| . \end{aligned}$$

Therefore $\|t\| = \|T\|$. Finally we will show that the mapping $T \to t$ is surjective. If we take $t \in (K(G))^*$, $f \in C_c(G) \subset L(p_1, q_1)(G)$ and define

$$g \to t(f * g),$$

for all $g \in D(m, q_2, n, q_2, G)$, then we get

$$|t(f * g)| \leq \|t\| \cdot \|f\|_{p_1,q_1} \|g\|_D . \tag{16}$$

This implies that the mapping gives a bounded linear functional on $D(m, q_2, n, q_2, G)$. Hence there is a unique element, denoted by Tf, in $\wp(m', q_2', n', q_2', G)$ by (6) such that

$$Tf * g(0) = t(f * g), \tag{17}$$

for all $g \in D(m, q_2, n, q_2, G)$ and $\|Tf\|_{\wp} \leq \|t\| \|f\|_{p_1,q_1}$ by (16) and (17). Hence T is a continuous (and bounded) operator from $C_c(G)$ into $\wp(m', q_2', n', q_2', G)$ and can be extended uniquely as a bounded linear operator on $L(p_1, q_1)(G)$. It remains to show that this extended bounded linear operator T is actually a multiplier. Indeed, for any $f \in L(p_1, q_1)(G)$, $g \in D(m, q_2, n, q_2, G)$ and $y \in G$, we see that $L_y f \in L(p_1, q_1)(G)$ and $L_y g \in D(m, q_2, n, q_2, G)$. Therefore,

$$TL_y f * g(0) = t(L_y f * g) = t(f * L_y g) = Tf * L_y g(0) = L_y Tf * g(0)$$

holds for all $g \in D(m, q_2, n, q_2, G)$. Then, we have

$$TL_y f = L_y Tf$$

and $TL_y = L_y T$. This shows that $T \in M(L(p_1, q_1)(G), \wp(m', q_2', n', q_2', G))$.

□

2.2 Multipliers from $D(m, q_2, n, q_2, G)$ to $L(p_1, q_1)(G)$

Let $f \in D(m, q_2, n, q_2, G)$, with $m < n$. Then $f \in L(p_2, q_2)(G)$ for all $m < p_2 < n$ and $\|f\|^*_{p_2,q_2} \leq \left(\|f\|^*_{m,q_2}\right)^{\beta} \left(\|f\|^*_{n,q_2}\right)^{1-\beta}$ where $\beta = \left(\frac{1}{p_2} - \frac{1}{n}\right)\left(\frac{1}{m} - \frac{1}{n}\right)^{-1}$ by Theorem 3. Define the space $A(G)$ to be the set of all functions $h(x)$ of the form

$$h = \sum_{i=1}^{\infty} f_i * g_i \quad ; \quad f_i \in C_c(G) \subset D(m, q_2, n, q_2, G), \; g_i \in L(p_1', q_1')(G)$$

with $\sum\limits_{i=1}^{\infty} \|f_i\|_D \|g_i\|_{p_1',q_1'} < \infty$ where $\frac{1}{p_1} + \frac{1}{p_1'} = \frac{1}{q_1} + \frac{1}{q_1'} = 1$. Now define $h \to |||h|||$ by

$$|||h||| = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_{p_1',q_1'} \; : \; h = \sum_{i=1}^{\infty} f_i * g_i, \; h \in A(G) \right\},$$

where the infimum is taken over all such representations of h in $A(G)$. It is easy to see that, the function $|||\cdot|||$ is a norm and $A(G)$ is a Banach space with this norm. Since

$$\|f * g\|_{r,s} \leq \|f\|_{p_2,q_2} \|g\|_{p_1',q_1'} \leq \|f\|_D \|g\|_{p_1',q_1'} < \infty,$$

for $f \in C_c(G) \subset D(m, q_2, n, q_2, G)$, $g \in L(p_1', q_1')(G)$ with(8), $\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{r} > 0$ and $1 - \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$, we have $A(G) \subset L(r, s)(G)$.

Theorem 5. *Let G be a locally compact abelian group, condition (8) be satisfied and $\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{r} > 0$ and $1 - \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$. Then the space of multipliers $M(D(m, q_2, n, q_2, G), L(p_1, q_1)(G))$ is isometrically isomorphic to $(A(G))^*$, the dual space of $A(G)$.*

Proof. Using the same method as in the proof of Theorem 4, we can conclude our assertion. □

2.3 Multipliers from $D(p_1, q_1, p_2, q_2, G)$ to $\wp(m_1', n_1', m_2', n_2', G)$

Suppose that $\frac{1}{p_i} + \frac{1}{m_i} > 1$, $\frac{1}{p_i} + \frac{1}{m_i} - 1 = \frac{1}{r_i}$ and $s_i \geq 1$ are numbers such that $\frac{1}{q_i} + \frac{1}{n_i} \geq \frac{1}{s_i}$ for $i = 1, 2$. Also let m_i', n_i' be conjugate numbers of m_i, n_i respectively for $i = 1, 2$. If $D(r_1, s_1, r_2, s_2, G)$ denotes the set of all complex-valued functions defined on G which are in $L(r_1, s_1)(G) \cap L(r_2, s_2)(G)$, then we can introduce a norm by

$$\|f\|_{r_1,s_1}^{r_2,s_2} = \max\left(\|f\|_{r_1,s_1},\ \|f\|_{r_2,s_2}\right).$$

$D(r_1, s_1, r_2, s_2, G)$ is also a Banach space with this norm.

To obtain the space of multipliers from $D(p_1, q_1, p_2, q_2, G)$ to $\wp(m_1', n_1', m_2', n_2', G)$ as a dual space, we define the space $K(G)$ to be the set of all functions h which can be written in the form

$$h = \sum_{i=1}^{\infty} f_i * g_i,$$

where $f_i \in C_c(G) \subset D(p_1, q_1, p_2, q_2, G)$ and $g_i \in D(m_1, n_1, m_2, n_2, G)$ with $\sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_D < \infty$. It is not hard to see that $C_c(G)$ is dense in $D(p_1, q_1, p_2, q_2, G)$. Define a function $h \to |||h|||$ by

$$|||h||| = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_D\right\},$$

where the infimum is taken over all such representations of h. It is easy to verify that $|||\cdot|||$ defines a norm on $K(G)$ and that the latter is a Banach space.

Now, let $f \in C_c(G) \subset D(p_1, q_1, p_2, q_2, G)$ and $g \in D(m_1, n_1, m_2, n_2, G)$. It follows from (7) that $f * g \in L(r_1, s_1)(G)$,

$$\|f * g\|_{r_1,s_1} \leq \|f\|_{p_1,q_1} \|g\|_{m_1,n_1} \leq \|f\|_D \|g\|_D$$

and $f * g \in L(r_2, s_2)(G)$,

$$\|f * g\|_{r_2,s_2} \le \|f\|_{p_2,q_2} \|g\|_{m_2,n_2} \le \|f\|_D \|g\|_D$$

so that

$$\|f * g\|_{r_1,s_1}^{r_2,s_2} \le \|f\|_D \|g\|_D .$$

From this, it is clear that $K(G) \subset D(r_1, s_1, r_2, s_2, G)$ and that the topology on $K(G)$ is not weaker than the topology induced by $\left(D(r_1, s_1, r_2, s_2, G),\ \|\cdot\|_{r_1,s_1}^{r_2,s_2}\right)$.

Theorem 6. *Let G be a locally compact abelian group and $\frac{1}{p_i} + \frac{1}{m_i} > 1$, $\frac{1}{p_i} + \frac{1}{m_i} - 1 = \frac{1}{r_i}$ and $s_i \ge 1$ are any numbers such that $\frac{1}{q_i} + \frac{1}{n_i} \ge \frac{1}{s_i}$ for $i = 1, 2$. The space of multipliers from $D(p_1, q_1, p_2, q_2, G)$ into $\wp(m_1', n_1', m_2', n_2', G)$ is isometrically isomorphic to $(K(G))^*$, the dual space of $K(G)$.*

Proof. We use the same method employed in the proof of the theorem 4. □

Remark 7. ***a)*** *If $p_1 = m_1$ and $q_1 = n_1$ then Theorem 6 coincides with Corollary 3.6 in* [1].
b) *If $\mu(G) < \infty$, then we can induce the problem to the usual Lebesgue spaces as in* [9].

2.4 Multipliers on $D(p, q, r, s, G)$

Let $1 < p, q, r, s < \infty$ and p', q', r', s' be conjugate numbers of p, q, r, s respectively. Define the space $A(G)$ to be the set of all functions $h(x)$ of the form

$$h = \sum_{i=1}^{\infty} f_i * g_i \quad ; \quad f_i \in D(p, q, r, s, G),\ g_i \in C_c(G) \subset \wp(p', q', r', s', G),$$

with $\sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_\wp < \infty$ and define $h \to |||h|||$ by

$$|||h||| = \inf\left\{ \sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_\wp \ :\ h = \sum_{i=1}^{\infty} f_i * g_i,\ h \in A(G) \right\},$$

where the infimum is taken over all such representations of h in $A(G)$. The function $|||\cdot|||$ is a norm of $A(G)$ and since

$$\|f * g\|_\infty \le \|f\|_D \|g\|_\wp ,$$

for $f \in D(p,q,r,s,G)$ and $g \in C_c(G) \subset \wp(p',q',r',s',G)$ where $\frac{1}{p}+\frac{1}{p'}=1, \frac{1}{q}+\frac{1}{q'} \geq 1, \frac{1}{r}+\frac{1}{r'}=1$ and $\frac{1}{s}+\frac{1}{s'} \geq 1$, it is easy to see that $A(G)$ is a dense linear subspace of $C_0(G)$ and is a Banach space with respect to the norm $|||\cdot|||$. Also the topology so defined is not weaker than the uniform norm topology.

Theorem 8. *Let G be a locally compact abelian group. The multiplier space $M(D(p,q,r,s,G))$ is isometrically isomorphic to $(A(G))^*$, the conjugate space of $A(G)$.*

Proof. For any $T \in M(D(p,q,r,s,G))$, define

$$\mu(h) = \sum_{i=1}^{\infty} Tf_i * g_i(0),$$

for $h = \sum_{i=1}^{\infty} f_i * g_i$ in $A(G)$. Firstly, we will show that μ is well-defined, i.e. $\mu(h)$ is independent of the particular representation of h chosen. To this end, it is sufficent to show that if $h = \sum_{i=1}^{\infty} f_i * g_i = 0$ in $A(G)$ and $\sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_\wp < \infty$, then $\sum_{i=1}^{\infty} Tf_i * g_i(0) = 0$.

Let $\{e_\alpha\}_{\alpha\in I}$ be an approximate identity for $L^1(G)$ with $\|e_\alpha\|_1 = 1$ for each $\alpha \in I$. Since $L^1(G) * L(p,q)(G) = L(p,q)(G)$ for $1 < p < \infty,\ 1 \leq q < \infty$ by [4], we have $e_\alpha * f \in D(p,q,r,s,G)$ for each α and

$$\lim_{\alpha} \|e_\alpha * f - f\|_D = 0, \tag{18}$$

for all $f \in D(p,q,r,s,G)$. Therefore using (18) and the fact that T is a multiplier, we obtain

$$|T(e_\alpha * f_i) * g_i(0) - Tf_i * g_i(0)| \leq \|T\| \, \|e_\alpha * f_i - f_i\|_D \|g_i\|_\wp \tag{19}$$

and

$$\lim_{\alpha} T(e_\alpha * f_i) * g_i(0) = Tf_i * g_i(0). \tag{20}$$

Also for each $f \in D(p,q,r,s,G)$ and $g \in C_c(G)$, we have

$$T(f * g) = Tf * g, \tag{21}$$

by Lemma 2.1 in [7]. Since $h = \sum_{i=1}^{\infty} f_i * g_i = 0$, the series $\sum_{i=1}^{\infty} f_i * g_i$ converges uniformly and using equality (21), we get

$$\begin{aligned}\sum_{i=1}^{\infty} T(e_\alpha * f_i) * g_i(\cdot) &= \sum_{i=1}^{\infty} T(e_\alpha * f_i * g_i)(\cdot) \\ &= T\left(e_\alpha * \sum_{i=1}^{\infty}(f_i * g_i)\right)(\cdot) = 0 \qquad (22)\end{aligned}$$

and then, for any large integer N,

$$\begin{aligned}\left|\sum_{i=1}^{\infty} T(e_\alpha * f_i) * g_i(0)\right| &\leq \left|\sum_{i=1}^{\infty} Tf_i * g_i(0) - \sum_{i=1}^{\infty} T(e_\alpha * f_i) * g_i(0)\right| \qquad (23) \\ &\leq \left|\sum_{i=1}^{N} Tf_i * g_i(0) - \sum_{i=1}^{N} T(e_\alpha * f_i) * g_i(0)\right| \\ &\quad + 2\|T\| \sum_{i=N+1}^{\infty} \|f_i\|_D \|g_i\|_{\wp},\end{aligned}$$

the right hand side of (23) can be made arbitrary small by taking a sufficiently large positive integer N, and then passing to the limit with respect to α, we see that

$$\lim_{\alpha} \sum_{i=1}^{\infty} T(e_\alpha * f_i) * g_i(0) = \sum_{i=1}^{\infty} Tf_i * g_i(0) = 0. \qquad (24)$$

Thus μ is well-defined. It is obvious that the mapping $T \to \mu$ is linear. Now we will show that it is an isometry. Indeed,

$$\begin{aligned}|\mu(h)| &\leq \sum_{i=1}^{\infty} |Tf_i * g_i(0)| \\ &\leq \sum_{i=1}^{\infty} \|Tf_i\|_D \|g_i\|_{\wp} \\ &\leq \|T\| \sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_{\wp}\end{aligned}$$

implies that

$$|\mu(h)| \leq \|T\| \, |||h||| \, .$$

Hence $\|\mu\| \leq \|T\|$. On the other hand by (6), we have

$$\begin{aligned} \|T\| &= \sup\left\{|Tf*g(0)| : \ \|f\|_D \leq 1, \ \|g\|_{\wp} \leq 1\right\} \\ &= \sup\left\{|\mu(f*g)| : \ \|f\|_D \leq 1, \ \|g\|_{\wp} \leq 1\right\} \\ &\leq \sup\{|\mu(f*g)| : \ \||f*g\|| \leq 1\} \\ &\leq \|\mu\|. \end{aligned}$$

Therefore $\|\mu\| = \|T\|$. Finally we will show that the mapping $T \to \mu$ is surjective. Suppose that $\mu \in (A(G))^*$, $f \in D(p,q,r,s,G)$ and define

$$g \to \mu(f*g) = u(g)$$

on $C_c(G) \subset \wp(p',q',r',s',G)$. Then

$$|u(g)| = |\mu(f*g)| \leq \|\mu\| \cdot \|f\|_D \|g\|_{\wp}.$$

This implies that the mapping u can be extended to a bounded linear functional on $\wp(p',q',r',s',G)$ by Hahn-Banach Theorem and

$$|\mu(f*g)| \leq \|\mu\| \cdot \|f\|_D \|g\|_{\wp}$$

for all $f \in D(p,q,r,s,G)$ and $g \in \wp(p',q',r',s',G)$. It follows from (6) that there is a unique element, denoted by Tf, in $D(p,q,r,s,G)$ such that

$$Tf*g(0) = \mu(f*g) = u(g),$$

for $g \in C_c(G) \subset \wp(p',q',r',s',G)$ and $\|Tf\|_D \leq \|\mu\| \|f\|_D$. Hence T is a continuous operator on $D(p,q,r,s,G)$. It remains to show that this bounded linear operator T is actually a multiplier on $D(p,q,r,s,G)$. Indeed, for any $f \in D(p,q,r,s,G)$, $g \in \wp(p',q',r',s',G)$ and $y \in G$, we have $L_y f \in D(p,q,r,s,G)$ and $L_y g \in \wp(p',q',r',s',G)$. Therefore,

$$TL_y f * g(0) = t(L_y f * g) = t(f * L_y g) = Tf * L_y g(0) = L_y Tf * g(0)$$

holds for all $g \in \wp(p',q',r',s',G)$. Since we have

$$TL_y f = L_y Tf \in D(p,q,r,s,G) \cong \wp^*(p',q',r',s',G)$$

for every $f \in D(p,q,r,s,G)$, $TL_y = L_y T$ can be written. Therefore $T \in M(D(p,q,r,s,G))$. □

Remark 9. ***a)*** *If $p = q$ and $r = s$ then $D(p,q,r,s,G) = L^p(G) \cap L^r(G)$ and Theorem 8 coincides with Theorem 3.2 in* [11].
b) *If $p = q = r = s$ then $D(p,q,r,s,G) = L^p(G)$ and Theorem 8 coincides with Theorem 1 in* [8].
c) *If $\mu(G) < \infty$, then we can induce the problem to usual Lebesgue spaces as in* [12].

In [17], it was found that $B(G) = L^1(G) \cap L(p,q)(G)$ is a Segal Algebra with the norm $\|\cdot\|_B$ defined by

$$\|\cdot\|_B = \|\cdot\|_1 + \|\cdot\|_{p,q}\,. \tag{25}$$

Also, it is showed that $M(B(G))$, the multipliers space of this Segal algebra is isometrically isomorphic to the multipliers space of certain Banach algebras of operators in [6].

Seperately, using the argument like in Theorem 8, mutadis mutandis, we can characterize the multipliers space of $B(G)$ for $1 < p, q < \infty$. If we use the following norm

$$\|\cdot\|^B = \max\left\{\|\cdot\|_1 \ , \ \|\cdot\|_{p,q}\right\}$$

which is equivalent to the norm showed in (25), we can define the multipliers space of $B(G)$. Let r, s be the conjugate numbers of p and q respectively and define the space

$$S^0_{r,s}(G) = \{g : \ g = g_1 + g_2 \text{ with } (g_1, g_2) \in C_0(G) \times L(r,s)(G)\}$$

with the norm by

$$\|g\|_S = \inf\left\{\|g_1\|_\infty + \|g_2\|_{r,s} : \ g = g_1 + g_2, \ (g_1, g_2) \in C_0(G) \times L(r,s)(G)\right\}$$

where the infimum is taken over all decompositions of g. Following Theorem 5 in [13], it is easy to see that

$$\left(S^0_{r,s}(G)\right)^* \cong B(G) \quad \text{where} \quad \frac{1}{p} + \frac{1}{r} = 1, \frac{1}{q} + \frac{1}{s} = 1.$$

Define the space $A^1_{p,q}(G)$ to be the set of all functions u of the form:

$$u = \sum_{i=1}^{\infty} f_i * g_i \ , \ f_i \in B(G)\, , \ g_i \in C_c(G) \subset S^0_{r,s}(G) \ \text{ with } \ \sum_{i=1}^{\infty} \|f_i\|^B \|g_i\|_S < \infty\, .$$

If we equip the space $A^1_{p,q}(G)$ with the norm

$$\|u\|^1_{p,q} = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|^B \|g_i\|_S : \ u = \sum_{i=1}^{\infty} f_i * g_i \text{ in } A^1_{p,q}(G)\right\},$$

where the infimum being taken over all $f_i \in B(G)$ and $g_i \in C_c(G) \subset S^0_{r,s}(G)$ for the representation of $u \in A^1_{p,q}(G)$, then by using the same argument of the Theorem 8, we have the following theorem.

Theorem 10. *The multipliers space $M(B(G))$ is isometrically isomorphic to $\left(A^1_{p,q}(G)\right)^*$, the dual space of $A^1_{p,q}(G)$.*

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