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ON THE CONVERGENCE OF ITERATIVE SEQUENCES FOR A FAMILY OF NONEXPANSIVE MAPPINGS AND INVERSE-STRONGLY MONOTONE MAPPINGS

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Abstract

The purpose of this paper is to introduce a general iterative process for the problem of finding a common element in the set of common fixed points of an infinite family of nonexpansive mappings and in the set of solutions of variational inequalities for inverse-strongly monotone mappings.

1. Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty, closed and convex subset of H and $A : K \to H$ be a nonlinear mapping. We denote by P_K be the metric projection of H onto the closed convex subset K. The classical variational inequality problem is to find $u \in K$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in K. \tag{1.1}$$

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In this paper, we use $VI(K,A)$ to denote the solution set of the variational inequality (1.1). For a given $z \in H, u \in K$ satisfies the inequality $\langle u-z, v-u\rangle \geq 0$, $\forall v \in K$, if and only if $u = P_K z$. It is known that projection operator P_K is nonexpansive. It is also known that P_K satisfies

$$\langle x-y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2, \quad \forall x, y \in H.$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. The element $u \in K$ is a solution of the variational inequality problem (1.1) if and only if $u \in K$ satisfies the relation $u = P_K(I - \lambda A)u$, where $\lambda > 0$ is a constant.

Recall that the following definitions.

(1) A mapping $A : K \to H$ is said to be *inverse-strongly monotone* if there exists a positive real number μ such that

$$\langle x-y, Ax - Ay\rangle \geq \mu\|Ax - Ay\|^2, \quad \forall x, y \in K.$$

For such a case, A is called *μ-inverse-strongly monotone.*

(2) A mapping $S : K \to K$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x-y\|, \quad \forall x, y \in K.$$

In this paper, we use $F(S)$ to denote the fixed point set of S.

(3) A mapping $f : K \to K$ is said to be a *contraction* if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x-y\|, \quad \forall x, y \in K.$$

(4) A set-valued mapping $T : H \to 2^H$ is said to be *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let A be a monotone map of K into H and let $N_K v$ be the normal cone to K at $v \in K$, i.e., $N_K v = \{w \in H : \langle v-u, w\rangle \geq 0,\ \forall u \in K\}$ and define

$$Tv = \begin{cases} Av + N_K v, & v \in K, \\ \emptyset, & v \notin K. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(K,A)$; see [24].

The classical variational inequality and fixed point problems have been studied based on iterative methods by many authors; see [3-14,18-23,27,30,31]

For finding a common element of the set of fixed points of a nonexpansive mapping S and the solution of the variational inequalities for a μ-inverse-strongly monotone mapping, Takahashi and Toyoda [27] introduced the following iterative process

$$x_1 \in K, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_K(x_n - \lambda_n A x_n), \quad n \geq 1, \tag{1.2}$$

where A is a μ-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\mu)$. They showed that, if $F(S) \cap VI(K,A)$ is nonempty, then the sequence $\{x_n\}$ generated in (1.2) converges weakly to some $z \in F(S) \cap VI(K,A)$.

Recently, Iiduka and Takahashi [8] proposed another iterative scheme as following

$$x_{n+1} = \alpha_n x + (1-\alpha_n) SP_K(x_n - \lambda_n A x_n), \quad n \geq 1, \tag{1.3}$$

where $x_1 = x \in K$, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\mu)$. They proved that the sequence $\{x_n\}$ converges strongly to $z \in F(S) \cap VI(K,A)$.

Very recently, Chen et al. [3] studied the following iterative process

$$x_1 \in K, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) SP_K(x_n - \lambda_n A x_n), \quad n \geq 1, \tag{1.4}$$

where A is an inverse-strongly monotone mapping and also obtained a strong convergence theorem by so-called viscosity approximation method which first introduced by Moudafi [13] in the framework of Hilbert spaces.

On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$, Korpelevich [10] introduced the following so-called extra-gradient method

$$\begin{cases} x_0 = x \in K, \\ y_n = P_K(x_n - \lambda A x_n), \\ x_{n+1} = P_K(x_n - \lambda A y_n), \quad n \geq 0, \end{cases} \tag{1.5}$$

where $\lambda \in \left(0, \frac{1}{k}\right)$.

Recently, Nadezhkina and Takahashi [14], Yao and Yao [30] and Zeng and Yao [31] proposed some new iterative schemes for finding common elements in $F(S) \cap VI(K,A)$ by combining (1.3) and (1.5). In particular, Yao and Yao [30] introduced the following iterative algorithm

$$\begin{cases} x_1 \in K, \\ y_n = P_K(I - \lambda_n A) x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_K(y_n - \lambda_n A y_n), \quad n \geq 1, \end{cases} \tag{1.6}$$

where S is a nonexpansive mapping and A is a inverse-strongly monotone mapping. They proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to some point in $F(S) \cap VI(K, A)$.

Concerning a family of nonexpansive mappings has been considered by many authors; see [2,7,11,12,15,16,18,20,25,29] and the references therein. In this paper, we consider the mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\quad\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\quad\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{1.7}$$

where $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \le \gamma_n \le 1$ and $T_1, T_2, \ldots$ be an infinite family of mappings of K into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n we have the following lemmas which are important to prove our main results.

Lemma 1.1. ([25]) *Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of K into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then for every $x \in K$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 1.1, one can define the mapping W of K into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in K. \tag{1.8}$$

Such a W is called the *W-mapping* generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.

Throughout this paper, we will assume that $0 < \gamma_n \le b < 1$ for all $n \ge 1$.

Lemma 1.2. ([25]) *Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of K into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

In this paper, motivated by research work going on in this direction, we introduce a general iterative process as following

$$\begin{cases} x_1 \in K, \\ y_n = P_K(I - \eta_n B)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_K(I - \lambda_n A)y_n, \quad n \geq 1, \end{cases} \tag{1.9}$$

where A and B are μ_i-inverse-strongly monotone mappings from K into H, respectively for $i = 1, 2$, f is a contraction on K and W_n is a mapping defined by (1.7). It is proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of common fixed points of an infinite nonexpansive mappings and the set of solutions of the variational inequalities for the inverse-strongly monotone mappings, which solves another variation inequality

$$\langle f(q) - q, p - q\rangle \leq 0, \quad \forall p \in \cap_{i=1}^{\infty} F(T_i) \cap VI(K, A) \cap VI(K, B).$$

In order to prove our main results, we also need the following lemmas.

Lemma 1.3. ([28]) *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.4. ([26]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5. ([17]) *Let E be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\gamma\|x - z\|^2 - \alpha\beta\|x - y\|^2 - \beta\gamma\|y - z\|^2.$$

2. Main results

Now, we are ready to give our main results.

Theorem 2.1. *Let K be a nonempty closed convex subset of a real Hilbert space H, $A : K \to H$ be μ_1-inverse-strongly monotone mapping and $B : K \to H$ be μ_2-inverse-strongly monotone mappings. Let $f : K \to K$ be a contraction with the coefficient α, where $0 < \alpha < 1$. Let $\{x_n\}$ be a sequence generated by (1.9), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$, $\{\eta_n\}$ are chosen such that $\{\eta_n\}$, $\{\lambda_n\} \subset [0, 2\min\{\mu_1, \mu_2\}]$. Assume that $F = \cap_{i=1}^{\infty} F(T_i) \cap VI(K,A) \cap VI(K,B) \neq \emptyset$. If the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ are chosen such that*

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} |\eta_{n+1} - \eta_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;
(e) $\{\eta_n\}$, $\{\lambda_n\} \in [u, v]$ *for some* u, v *with* $0 < u < v < 2\min\{\mu_1, \mu_2\}$,

then $\{x_n\}$ converges strongly to $x^* \in F$, where $x^* = P_F f(x^*)$, which solves the following variation inequality

$$\langle f(x^*) - x^*, p - x^* \rangle \leq 0, \quad \forall p \in F.$$

Proof. First, we show that $I - \lambda_n A$ and $I - \eta_n B$ are nonexpansive for all $n \geq 1$. Indeed, we see from condition (e) that

$$\begin{aligned}
&\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \\
&= \|x - y - \lambda_n(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle + \lambda_n^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\mu_1)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2
\end{aligned}$$

from which it follows that $I - \lambda_n A$ is nonexpansive, so is $I - \eta_n B$. Letting $p \in F$, we have

$$\|y_n - p\| = \|P_K(I - \eta_n B)x_n - p\| \leq \|x_n - p\|.$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C(I - \lambda_n A)y_n - p\| \\
&\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|W_n P_C(I - \lambda_n A)y_n - p\| \\
&\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - p\| + \alpha_n \|f(p) - p\|.
\end{aligned}$$

By simple inductions, we have

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|p - f(p)\|}{1-\alpha}\right\},$$

which yields that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$.

Next, we show the sequence $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Note that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P_K(I - \eta_{n+1}B)x_{n+1} - P_K(I - \eta_n B)x_n\| \\ &\le \|(I - \eta_{n+1}B)x_{n+1} - (I - \eta_n B)x_n\| \\ &= \|(I - \eta_{n+1}B)x_{n+1} - (I - \eta_{n+1}B)x_n \\ &\quad + (I - \eta_{n+1}B)x_n - (I - \eta_n B)x_n\| \\ &\le \|x_{n+1} - x_n\| + |\eta_{n+1} - \eta_n|M_1,\end{aligned} \tag{2.1}$$

where M_1 is an appropriate constant such that $M_1 = \sup_{n\ge1}\{\|Bx_n\|\}$. Putting $\rho_n = P_K(I - \lambda_n A)y_n$, we have

$$\begin{aligned}\|\rho_{n+1} - \rho_n\| &= \|P_K(I - \lambda_{n+1}A)y_{n+1} - P_K(I - \lambda_n A)y_n\| \\ &\le \|(I - \lambda_{n+1}A)y_{n+1} - (I - \lambda_n A)y_n\| \\ &= \|(I - \lambda_{n+1}A)y_{n+1} - (I - \lambda_{n+1}A)y_n \\ &\quad + (I - \lambda_{n+1}A)y_n - (I - \lambda_n A)y_n\| \\ &= \|y_{n+1} - y_n\| + |\lambda_{n+1} - \lambda_n|\|Ay_n\|.\end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2), we arrive at

$$\|\rho_{n+1} - \rho_n\| \le \|x_{n+1} - x_n\| + (|\eta_{n+1} - \eta_n| + |\lambda_{n+1} - \lambda_n|)M_2, \tag{2.3}$$

where M_2 is an appropriate constant such that $M_2 \ge \max\{\sup_{n\ge1}\|Ay_n\|, M_1\}$. Define a sequence $\{z_n\}$ by

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}, \quad \forall n \ge 1. \tag{2.4}$$

It follows that

$$\begin{aligned}& z_{n+1} - z_n \\ &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}W_{n+1}\rho_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n W_n \rho_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - W_{n+1}\rho_{n+1}) + \frac{\alpha_n}{1 - \beta_n}(W_n\rho_n - f(x_n)) \\ &\quad + W_{n+1}\rho_{n+1} - W_n\rho_n.\end{aligned}$$

This implies that

$$\begin{aligned}&\|z_{n+1}-z_n\|\\&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\|+\|W_{n+1}\rho_{n+1}\|)+\frac{\alpha_n}{1-\beta_n}(\|W_n\rho_n\|+\|f(x_n)\|)\\&\quad+\|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|.\end{aligned} \tag{2.5}$$

Since T_i and $U_{n,i}$ are nonexpansive, we obtain from (1.7) that

$$\begin{aligned}\|W_{n+1}\rho_n-W_n\rho_n\| &= \|\gamma_1T_1U_{n+1,2}\rho_n-\gamma_1T_1U_{n,2}\rho_n\|\\&\le \gamma_1\|U_{n+1,2}\rho_n-U_{n,2}\rho_n\|\\&= \gamma_1\|\gamma_2T_2U_{n+1,3}\rho_n-\gamma_2T_2U_{n,3}\rho_n\|\\&\le \gamma_1\gamma_2\|U_{n+1,3}\rho_n-U_{n,3}\rho_n\|\\&\le \cdots\\&\le \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\\&\le M_3\prod_{i=1}^{n}\gamma_i,\end{aligned} \tag{2.6}$$

where $M_3\ge 0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\le M_3$ for all $n\ge 1$. Substituting (2.3) and (2.6) into (2.5), we arrive at

$$\begin{aligned}&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\|+\|W_{n+1}\rho_{n+1}\|)+\frac{\alpha_n}{1-\beta_n}(\|W_n\rho_n\|+\|f(x_n)\|)\\&\quad+M_4\left(|\eta_{n+1}-\eta_n|+|\lambda_{n+1}-\lambda_n|+\prod_{i=1}^{n}\gamma_i\right).\end{aligned}$$

where M_4 is an appropriate constant such that $M_4=\max\{M_2,M_3\}$. From the conditions (b), (c) and (d), we obtain that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

From the condition (c) and applying Lemma 1.4, we obtain that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{2.7}$$

Consequently, we obtain from (2.4) and the condition (c) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{2.8}$$

Next, we show that

$$\lim_{n\to\infty} \|Wx_n - x_n\| = 0. \tag{2.9}$$

For any $p \in F$, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|P_K(I - \eta_n B)x_n - p\|^2 \\ &\le \|(x_n - p) - \eta_n(Bx_n - Bp)\|^2 \\ &= \|x_n - p\|^2 - 2\eta_n\langle x_n - p, Bx_n - Bp\rangle + \eta_n^2\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2 - 2\eta_n\mu_2\|Bx_n - Bp\|^2 + \eta_n^2\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2 + \eta_n(\eta_n - 2\mu_2)\|Bx_n - Bp\|^2. \end{aligned} \tag{2.10}$$

On the other hand, we have

$$\begin{aligned} \|\rho_n - p\|^2 &= \|P_K(I - \lambda_n A)y_n - p\|^2 \\ &\le \|(I - \lambda_n A)y_n - p\|^2 \\ &= \|y_n - p - \lambda_n(Ay_n - Ap)\|^2 \\ &= \|y_n - p\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Ap\rangle + \lambda_n^2\|Ay_n - Ap\|^2 \\ &\le \|y_n - p\|^2 - 2\lambda_n\mu_1\|Ay_n - Ap\|^2 + \lambda_n^2\|Ay_n - Ap\|^2 \\ &\le \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\mu_1)\|Ay_n - Ap\|^2. \end{aligned} \tag{2.11}$$

It follows from Lemma 1.5 that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(W_n\rho_n - p)\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|\rho_n - p\|^2. \end{aligned} \tag{2.12}$$

Substituting (2.11) into (2.12), we arrive at

$$\|x_{n+1}-p\|^2 \le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2+\gamma_n\lambda_n(\lambda_n-2\mu_1)\|Ay_n-Ap\|^2. \tag{2.13}$$

It follows from condition (e) that

$$\begin{aligned} &\gamma_n u(2\mu_1 - v)\|Ay_n - Ap\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|. \end{aligned}$$

From the conditions (b) and (c), we obtain from (2.8) that

$$\lim_{n\to\infty} \|Ay_n - Ap\| = 0. \tag{2.14}$$

Using (2.12) again, we have

$$\|x_{n+1}-p\|^2 \le \alpha_n\|f(x_n)-p\|^2 + \beta_n\|x_n-p\|^2 + \gamma_n\|y_n-p\|^2, \tag{2.15}$$

which combines with (2.10) yields that

$$\|x_{n+1}-p\|^2 \le \alpha_n\|f(x_n)-p\|^2 + \|x_n-p\|^2 + \gamma_n\eta_n(\eta_n-2\mu_2)\|Bx_n-Bp\|^2.$$

From condition (e), we arrive at

$$\begin{aligned}
&\gamma_n u(2\mu_2-v)\|Bx_n-Bp\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2 + \|x_n-p\|^2 - \|x_{n+1}-p\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2 + (\|x_n-p\| + \|x_{n+1}-p\|)\|x_n-x_{n+1}\|.
\end{aligned}$$

It follows from the condition (b) and (2.8) that

$$\lim_{n\to\infty}\|Bx_n-Bp\| = 0. \tag{2.16}$$

On the other hand, we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|P_K(I-\eta_nB)x_n - P_K(I-\eta_nB)p\|^2\\
&\le \langle (I-\eta_nB)x_n-(I-\eta_nB)p, y_n-p\rangle\\
&= \frac{1}{2}\{\|(I-\eta_nB)x_n-(I-\eta_nB)p\|^2 + \|y_n-p\|^2\\
&\qquad - \|(I-\eta_nB)x_n-(I-\eta_nB)p-(y_n-p)\|^2\}\\
&\le \frac{1}{2}\{\|x_n-p\|^2 + \|y_n-p\|^2 - \|(x_n-y_n)-\eta_n(Bx_n-Bp)\|^2\}\\
&= \frac{1}{2}\{\|x_n-p\|^2 + \|y_n-p\|^2 - \|x_n-y_n\|^2 - \eta_n^2\|Bx_n-Bp\|^2\\
&\qquad + 2\eta_n\langle x_n-y_n, Bx_n-Bp\rangle\},
\end{aligned}$$

which yields that

$$\|y_n-p\|^2 \le \|x_n-p\|^2 - \|x_n-y_n\|^2 + 2\eta_n\|x_n-y_n\|\|Bx_n-Bp\|. \tag{2.17}$$

In a similar way, we can prove that

$$\|\rho_n-p\|^2 \le \|x_n-p\|^2 - \|\rho_n-y_n\|^2 + 2\lambda_n\|\rho_n-y_n\|\|Ay_n-Ap\|. \tag{2.18}$$

Substitute (2.18) into (2.12) yields that

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le{}& \alpha_n\|f(x_n)-p\|^2 + \|x_n-p\|^2 - \gamma_n\|\rho_n-y_n\|^2\\
&+ 2\gamma_n\lambda_n\|\rho_n-y_n\|\|Ay_n-Ap\|.
\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|\rho_n - y_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2\gamma_n\lambda_n\|\rho_n - y_n\|\|Ay_n - Ap\| \\ &\le \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad + 2\gamma_n\lambda_n\|\rho_n - y_n\|\|Ay_n - Ap\|.\end{aligned}$$

In view of the conditions (b) and (c), we see from (2.8) and (2.14) that

$$\lim_{n\to\infty}\|\rho_n - y_n\| = 0. \tag{2.19}$$

Similarly, substituting (2.17) into (2.15), we can prove that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{2.20}$$

On the other hand, we have

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + \gamma_n(W_n\rho_n - x_n).$$

It follows that

$$\gamma_n\|W_n\rho_n - x_n\| \le \|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - x_n\|.$$

In view of conditions (b) and (c), we see from (2.8) that

$$\lim_{n\to\infty}\|W_n\rho_n - x_n\| = 0. \tag{2.21}$$

Observe that

$$\begin{aligned}\|W_nx_n - x_n\| &\le \|W_nx_n - W_n\rho_n\| + \|W_n\rho_n - x_n\| \\ &\le \|x_n - \rho_n\| + \|W_n\rho_n - x_n\| \\ &\le \|x_n - y_n\| + \|y_n - \rho_n\| + \|W_n\rho_n - x_n\|.\end{aligned}$$

It follows from (2.19)-(2.21) that

$$\lim_{n\to\infty}\|W_nx_n - x_n\| = 0. \tag{2.22}$$

From Remark 3.3 of [29], see also [7], we have $\|Wx_n - W_nx_n\| \to 0$ as $n \to \infty$. It follows that (2.9) holds. Observe that P_Ff is a contraction. Indeed, for all $x, y \in C$, we have

$$\|P_Ff(x) - P_Ff(y)\| \le \|f(x) - f(y)\| \le \alpha\|x - y\|.$$

Banach's contraction mapping principle guarantees that $P_F f$ has a unique fixed point, say $x^* \in C$. That is, $x^* = P_F f(x^*)$.

Next, we show that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^* \rangle \le 0. \tag{2.23}$$

To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle f(x^*) - x^*, x_{n_i} - x^* \rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to $\bar{x}$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup \bar{x}$. From (2.19) and (2.20), we also have $y_{n_i} \rightharpoonup \bar{x}$ and $\rho_{n_i} \rightharpoonup \bar{x}$, respectively.

Next, we have $\bar{x} \in F$. Indeed, let us first show that $\bar{x} \in VI(K, A)$. Put

$$Tv = \begin{cases} Av + N_K v, & v \in K, \\ \emptyset, & v \notin K. \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_K v$ and $\rho_n \in K$, we have

$$\langle v - \rho_n, w - Av \rangle \ge 0.$$

On the other hand, we see from $\rho_n = P_K(I - \lambda_n A)y_n$ that

$$\langle v - \rho_n, \rho_n - (I - \lambda_n A)y_n \rangle \ge 0$$

and hence

$$\left\langle v - \rho_n, \frac{\rho_n - y_n}{\lambda_n} + Ay_n \right\rangle \ge 0.$$

It follows that

$$\begin{aligned} \langle v - \rho_{n_i}, w \rangle &\ge \langle v - \rho_{n_i}, Av \rangle \\ &\ge \langle v - \rho_{n_i}, Av \rangle - \left\langle v - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\ &\ge \left\langle v - \rho_{n_i}, Av - \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} - Ay_{n_i} \right\rangle \\ &= \langle v - \rho_{n_i}, Av - A\rho_{n_i} \rangle + \langle v - \rho_{n_i}, A\rho_{n_i} - Ay_{n_i} \rangle \\ &\quad - \left\langle v - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\ge \langle v - \rho_{n_i}, A\rho_{n_i} - Ay_{n_i} \rangle - \left\langle v - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{\lambda_{n_i}} \right\rangle, \end{aligned}$$

which implies that $\langle v-\bar{x}, w\rangle \geq 0$. We have $\bar{x} \in T^{-1}0$ and hence $\bar{x} \in VI(K, A)$. In a similar way, we can show $\bar{x} \in VI(K, B)$.

Next, let us show $\bar{x} \in \cap_{i=1}^{\infty} F(T_i)$. Since Hilbert spaces are Opial's spaces, we obtain from (2.9) that

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| &< \liminf_{i\to\infty} \|x_{n_i} - W\bar{x}\| \\ &= \liminf_{i\to\infty} \|x_{n_i} - Wx_{n_i} + Wx_{n_i} - W\bar{x}\| \\ &\leq \liminf_{i\to\infty} \|Wx_{n_i} - W\bar{x}\| \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\|,\end{aligned}$$

which derives a contradiction. Thus, we have $\bar{x} \in F(W) = \cap_{i=1}^{\infty} F(T_i)$. On the other hand, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{n\to\infty} \langle f(x^*) - x^*, x_{n_i} - x^*\rangle \\ &= \langle f(x^*) - x^*, \bar{x} - x^*\rangle \leq 0.\end{aligned}$$

That is, (2.23) holds. It follows that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_{n+1} - x^*\rangle \leq 0. \tag{2.24}$$

Finally, we show that $x_n \to x^*$ as $n \to \infty$. Note that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \langle \alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(W_n\rho_n - x^*), x_{n+1} - x^*\rangle \\ &= \alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle + \beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle \\ &\quad + \gamma_n\langle W_n\rho_n - x^*, x_{n+1} - x^*\rangle \\ &= \alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle + \gamma_n\langle W_n\rho_n - x^*, x_{n+1} - x^*\rangle \\ &\leq \alpha_n\|f(x_n) - f(x^*)\|\|x_{n+1} - x^*\| + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \beta_n\|x_n - x^*\|\|x_{n+1} - x^*\| + \gamma_n\|W_n\rho_n - x^*\|\|x_{n+1} - x^*\| \\ &\leq \alpha_n\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \beta_n\|x_n - x^*\|\|x_{n+1} - x^*\| + \gamma_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &= (1 - \alpha_n(1-\alpha))\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq \frac{1 - \alpha_n(1-\alpha)}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\quad + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\leq \frac{1 - \alpha_n(1-\alpha)}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2 \\ &\quad + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.\end{aligned}$$

It follows that

$$\|x_{n+1} - x^*\|^2 \leq [1 - \alpha_n(1-\alpha)]\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.$$

From (2.24) and applying Lemma 1.3, we can obtain the desired conclusion immediately. This completes the proof.

Let $A = B$ and $f(x) = x_1$ for all $x \in K$ in Theorem 2.1. We can obtain the following result easily.

Corollary 2.2. *Let K be a nonempty closed convex subset of a real Hilbert space H and $A : K \to H$ be μ-inverse-strongly monotone mappings. Let $\{x_n\}$ be a sequence generated by the following iterative process*

$$\begin{cases} x_1 \in K, \\ y_n = P_K(I - \eta_n A)x_n, \\ x_{n+1} = \alpha_n x_1 + \beta_n x_n + \gamma_n W_n P_K(I - \lambda_n A)y_n, \quad n \geq 1, \end{cases}$$

where W_n is defined by (1.8), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$, $\{\eta_n\}$ are chosen such that $\{\eta_n\}, \{\lambda_n\} \subset [0, 2\min\{\mu_1, \mu_2\}]$. Assume that $F = \cap_{i=1}^{\infty} F(T_i) \cap VI(K, A) \neq \emptyset$. If the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ are chosen such that

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty} |\eta_{n+1} - \eta_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(e) $\{\eta_n\}, \{\lambda_n\} \in [u, v]$ *for some* u, v *with* $0 < u < v < 2\min\{\mu_1, \mu_2\}$,

then $\{x_n\}$ converges strongly to $x^* \in F$, where $x^* = P_F x_1$, which solves the following variation inequality

$$\langle x_1 - x^*, p - x^*\rangle \leq 0, \quad \forall p \in F.$$

Remark 2.3. Corollary 2.2 mainly improves the corresponding result of Yao and Yao [30] from a single nonexpansive mapping to an infinite family nonexpansive mappings.

As some applications of our main results, we next consider another class of important nonlinear operator: strict pseudo-contractions.

Recall that a mapping $S : K \to K$ is said to be a *k-strict pseudo-contraction* if there exists a constant $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in K.$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings.

Put $A = I - S$, where $S : K \to K$ is a k-strict pseudo-contraction. Then A is $\frac{1-k}{2}$-inverse-strongly monotone; see [1].

Theorem 2.4. *Let K be a nonempty closed convex subset of a real Hilbert space H, $S_1 : K \to K$ be a k_1-strict pseudo-contraction and $S_2 : K \to K$ be a k_2-strict pseudo-contraction. Let $f : K \to K$ be a contraction with the coefficient α $(0 < \alpha < 1)$. Let $\{x_n\}$ be a sequence generated by the following iterative process*

$$\begin{cases} x_1 \in K, \\ y_n = (1-\eta_n)x_n + \eta_n S_2 x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n((1-\lambda_n)y_n + \lambda_n S_1 y_n), \quad n \geq 1, \end{cases}$$

where W_n is defined by (1.8), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$, $\{\eta_n\}$ are chosen such that $\{\eta_n\}$, $\{\lambda_n\} \subset [0, 2\min\{(1-k_1),(1-k_2)\}]$. Assume that $F = \cap_{i=1}^{\infty} F(T_i) \cap F(S_1) \cap F(S_2) \neq \emptyset$. If the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ are chosen such that

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty} |\eta_{n+1} - \eta_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(e) $\{\eta_n\}$, $\{\lambda_n\} \in [u, v]$ *for some* u, v *with* $0 < u < v < 2\min\{\mu_1, \mu_2\}$,

then $\{x_n\}$ converges strongly to $x^* \in F$, where $x^* = P_F f(x^*)$, which solves the following variation inequality

$$\langle f(x^*) - x^*, p - x^* \rangle \leq 0, \quad \forall p \in F.$$

Proof. Put $A = I - S_1$ and $B = I - S_2$. Then A is $\frac{1-k_1}{2}$-inverse-strongly monotone and B is $\frac{1-k_2}{2}$-inverse-strongly monotone, respectively. We have $F(S_1) = VI(K, A)$, $F(S_2) = VI(K, B)$, $P_K(I - \lambda_n A)y_n = (1-\lambda_n)y_n + \lambda_n S_1 y_n$ and $P_K(I - \eta_n B)x_n = (1-\eta_n)x_n + \eta_n S_2 x_n$. It is easy to conclude from Theorem 2.1 the desired conclusion.

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