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Strong convergence theorems for equilibrium problems and quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces

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Abstract

In this paper, we introduce two modified Mann-type iterative algorithms for finding a common element of the set of common fixed points of a family of quasi-ϕ-asymptotically nonexpansive mappings and the set of solutions of an equilibrium problem in Banach spaces. Then we study the strong convergence of the algorithms. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Let E be a Banach space and let E^* be the dual space of E. Let C be a nonempty closed convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $\hat{x} \in EP(f)$ if and only if $\langle T\hat{x}, y - \hat{x}\rangle \geq 0$ for all $y \in C$, i.e., $\hat{x}$ is a solution of the variational inequality. Numerous problems in physics, optimization, engineering and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for example, Blum-Oettli [2] and Moudafi [7].

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For solving the equilibrium problem, let us assume that a bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:
$(A1)$ $f(x,x) = 0$ for all $x \in C$;
$(A2)$ f is monotone, that is, $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x,y);$$

$(A4)$ for each $x \in C$, the function $y \mapsto f(x,y)$ is convex and lower semicontinuous.

Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is said to be a fixed point of T provided $Tx = x$. A point $x \in C$ is said to be an asymptotic fixed point of T provided C contains a sequence $\{x_n\}$ which converges weakly to x such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of fixed points of T and the set of asymptotic fixed points of T by $F(T)$ and $F^a(T)$, respectively. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C.$$

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see, for instance, [4, 5, 9, 11] and the references therein.

Very recently, Takahashi and Zembayashi [10] introduced the following iterative process:

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \ \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x, \ \forall n \geq 1, \end{cases} \tag{1.2}$$

where $f : C \times C \to \mathbb{R}$ is a bifunction satisfying $(A1)$-$(A4)$, J is the normalized duality mapping on E and $S : C \to C$ is a relatively nonexpansive mapping. They proved the sequences $\{x_n\}$ defined by (1.2) converge strongly to a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of S provided the control sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions in Banach spaces.

Qin et al. [8] proved strong convergence theorem for finding a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of two quasi-ϕ-nonexpansive mappings.

In 2009, Cho et al. [3] introduced a modified Halpern-type iteration algorithm and proved strong convergence for quasi-ϕ-asymptotically nonexpansive mappings.

Motivated and inspired by the research going on in this direction, we prove strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a family of quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a Banach space with the dual space E^*. We will use the following notations:
(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
(ii) $\langle x, x^* \rangle$ denotes the value of x^* at x for all $x \in E$ and $x^* \in E^*$.
(iii) $S(E)$ denotes the unit sphere of E, that is, $S(E) = \{z \in E : \|z\| = 1\}$.

The normalized duality mapping J on E is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x,\, y \in S(E)$ with $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1-\delta$ for $x,\, y \in S(E)$ with $\|x-y\| \geq \epsilon$. The space E is said to be smooth if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x,\, y \in S(E)$. It is also said to be uniformly smooth if the limit exists uniformly for $x,\, y \in S(E)$. We know that if E is uniformly smooth, strictly convex and reflexive, then the normalized duality mapping J is single-valued, one-to-one, onto and uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x,\, Jy \rangle + \|y\|^2, \quad \forall x,\, y \in E.$$

Following Alber [1], the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional

$\phi(y,x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem:

$$\phi(\bar{x},x) = \inf_{y\in C} \phi(y,x).$$

It follows from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2, \ \forall x,\, y \in E,$$

see [3] for more details. If E is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$ and $\Pi_C = P_C$ is the metric projection of H onto C.

Now, we give some definitions for our main results in this paper.

Let C be a nonempty, closed and convex subset of a smooth Banach E and T a mapping from C into itself.

(1) The mapping T is said to be relatively nonexpansive if

$$F^a(T) = F(T) \ne \emptyset,\ \phi(p,Tx) \le \phi(p,x),\ \forall x \in C,\, p \in F(T).$$

(2) The mapping T is said to be relatively asymptotically nonexpansive if

$$F^a(T) = F(T) \ne \emptyset,\ \phi(p,T^n x) \le k_n \phi(p,x),\ \forall x \in C,\, p \in F(T),$$

where $k_n \ge 1$ is a sequence such that $k_n \to 1$ as $n \to \infty$.

(3) The mapping T is said to be ϕ-nonexpansive if

$$\phi(Tx,Ty) \le \phi(x,y),\ \forall x,\, y \in C.$$

(4) The mapping T is said to be quasi-ϕ-nonexpansive if

$$F(T) \ne \emptyset,\ \phi(p,Tx) \le \phi(p,x),\ \forall x \in C,\, p \in F(T).$$

(5) The mapping T is said to be ϕ-asymptotically nonexpansive if there exists some real sequence $\{k_n\}$ with $k_n \ge 1$ and $k_n \to 1$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \le k_n \phi(x,y),\ \forall x,\, y \in C.$$

(6) The mapping T is said to be quasi-ϕ-asymptotically nonexpansive if there exists some real sequence $\{k_n\}$ with $k_n \ge 1$ and $k_n \to 1$ as $n \to \infty$ such that

$$F(T) \ne \emptyset,\ \phi(p,T^n x) \le k_n \phi(p,x),\ \forall x \in C,\, p \in F(T).$$

(7) The mapping T is said to be asymptotically regular on C if, for any bounded subset K of C,

$$\limsup_{n\to\infty} \{\|T^{n+1}x - T^n x\| : x \in K\} = 0.$$

(8) The mapping T is said to be closed on C if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Remark 2.1 The class of quasi-ϕ-nonexpansive mappings and quasi-ϕ- asymptotically nonexpansive mappings are more general than the class of relatively nonexpansive mappings and relatively asymptotically nonexpansive mappings, respectively. The quasi-ϕ-nonexpansive mappings and quasi-ϕ-asymptotically nonexpansive mappings do not require $F(T) = F^a(T)$.

Remark 2.2 A ϕ-asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ is a quasi-ϕ-asymptotically nonexpansive mapping, but the converse may be not true.

In order to the main results of this paper, we need the following lemmas.

Lemma 2.3([1, 6]) *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \ \forall x \in C, \, y \in E.$$

Lemma 2.4([1, 6]) *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, \, Jx - Jz \rangle \leq 0, \ \forall y \in C.$$

Lemma 2.5([6]) *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6([12, 13]) *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all x, $y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.7([2]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.8([10]) *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in E$. Then, the following hold:
(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$;*
(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

Lemma 2.9([10]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.10([3]) *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty, closed and convex subset of E and T a closed quasi-ϕ-asymptotically nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

3. Strong convergence theorems

First, we propose a modified Mann-type iterative algorithm for finding a common element of the set of common fixed points of a countable infinite family of quasi-ϕ-asymptotically nonexpansive mappings and the set of solutions of an equilibrium problem in Banach spaces.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_i\}_{i\in I} : C \to C$ a family of closed quasi-ϕ-asymptotically nonexpansive mappings with sequences $\{k_{n,i}\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) such that $F = (\bigcap_{i\in I} F(T_i)) \bigcap EP(f) \neq \emptyset$. Assume that T_i is asymptotically regular on C for each $i \in I$ and F is bounded. For each $i \in I$, let $\{\alpha_{n,i}\}$ be a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$*

and $\{r_{n,i}\}$ *a sequence in* $[a,\infty)$ *for some* $a>0$. *Define a sequence* $\{x_n\}$ *in* C *in the following manner:*

$$\begin{cases} x_0 \in C \ \text{chosen arbitrarily}, \\ C_{1,i}=C, \ C_1=\bigcap_{i\in I} C_{1,i}, \ x_1=\Pi_{C_1}x_0, \\ y_{n,i}=J^{-1}(\alpha_{n,i}Jx_n+(1-\alpha_{n,i})JT_i^n x_n), \\ u_{n,i}\in C \ \text{such that} \ f(u_{n,i},y)+\frac{1}{r_{n,i}}\langle y-u_{n,i}, Ju_{n,i}-Jy_{n,i}\rangle \geq 0, \ \ \forall y\in C, \\ C_{n+1,i}=\{z\in C:\phi(z,u_{n,i})\leq \phi(z,x_n)+(1-\alpha_{n,i})(k_{n,i}-1)L_n\}, \\ C_{n+1}=\bigcap_{i\in I} C_{n+1,i}, \\ Q_1=C, \\ Q_{n+1}=\{z\in Q_n:\langle x_n-z, Jx_1-Jx_n\rangle\geq 0\}, \\ x_{n+1}=\Pi_{C_{n+1}\cap Q_{n+1}}x_1 \end{cases} \tag{3.1}$$

for every $n\geq 0$, *where* J *is the normalized duality mapping on* E *and* $L_n=\sup\{\phi(p,x_n):p\in F\}<\infty$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_1$.

Proof. We break the proof into eight steps.

Step 1. $\Pi_F x_1$ is well defined for $x_1\in C$.

By lemma 2.10 we know that $F(T_i)$ is a closed convex subset of C for every $i\in I$. Hence $F=(\bigcap_{i\in I}F(T_i))\bigcap EP(f)$ is a nonempty closed convex subset of C. Consequently, $\Pi_F x_1$ is well defined for $x_1\in C$.

Step 2. C_n and Q_n are closed and convex for all $n\in N$.

It is obvious that $C_1=C_{1,i}=C$ is closed and convex for every $i\in I$. Since the defining inequality in $C_{n+1,i}$ is equivalent to the inequality:

$$2\langle z, Jx_n-Ju_{n,i}\rangle \leq \|x_n\|^2-\|u_{n,i}\|^2+(1-\alpha_{n,i})(k_{n,i}-1)Q_n$$

for every $i\in I$. This shows that $C_{n+1,i}$ is closed and convex for every $i\in I$. So, we have $C_{n+1}=\bigcap_{i\in I}C_{n+1,i}$ is a closed and convex subset of C for all $n\geq 1$. From the definition of Q_n, it is obvious that Q_n is closed and convex for each $n\geq 1$. Consequently, $\Pi_{C_{n+1}\cap Q_{n+1}}x_1$ is well defined.

Step 3. $F\subset C_n\bigcap Q_n$ for all $n\geq 1$.

For $n=1$, we have $F\subset C=C_1$. Let $p\in F\subset C$ and $i\in I$. Putting $u_{n,i}=T_{r_{n,i}}y_{n,i}$ for all $n\in\mathbb{N}$, we have that $T_{r_{n,i}}$ is relatively nonexpansive from Lemma 2.9. Since T_i is quasi-ϕ–asymptotically nonexpansive, we have

$$\begin{aligned}
&\phi(p,u_{n,i}) = \phi(p,T_{r_{n,i}}y_{n,i}) \le \phi(p,y_{n,i})\\
=&\phi(p,J^{-1}(\alpha_{n,i}Jx_n + (1-\alpha_{n,i})JT_i^n x_n))\\
=&\|p\|^2 - 2\langle p, \alpha_{n,i}Jx_n + (1-\alpha_{n,i})JT_i^n x_n\rangle + \|\alpha_{n,i}Jx_n + (1-\alpha_{n,i})JT_i^n x_n\|^2\\
\le&\|p\|^2 - 2\alpha_{n,i}\langle p, Jx_n\rangle - 2(1-\alpha_{n,i})\langle p, JT_i^n x_n\rangle + \alpha_{n,i}\|x_n\|^2 + (1-\alpha_{n,i})\|T_i^n x_n\|^2\\
=&\alpha_{n,i}\phi(p,x_n) + (1-\alpha_{n,i})\phi(p,T_i^n x_n)\\
\le&\alpha_{n,i}\phi(p,x_n) + (1-\alpha_{n,i})k_{n,i}\phi(p,x_n)\\
=&\phi(p,x_n) + (1-\alpha_{n,i})(k_{n,i}-1)\phi(p,x_n)\\
\le&\phi(p,x_n) + (1-\alpha_{n,i})(k_{n,i}-1)L_n,
\end{aligned} \tag{3.2}$$

which shows that $p \in C_{n+1,i}$ for all $n \ge 1$. It follows that $p \in C_{n+1} = \bigcap_{i\in I} C_{n+1,i}$ for all $n \ge 1$. This proves that $F \subset C_n$ for all $n \ge 1$.

Next, we show by induction that $F \subset Q_n$ for all $n \ge 1$. For $n = 1$, we have $F \subset C = Q_1$. Assume that $F \subset Q_n$ for some $n > 1$. We show $F \subset Q_{n+1}$. Since $x_n = \Pi_{C_n \cap Q_n} x_1$, by Lemma 2.4, we have

$$\langle x_n - z,\ Jx_1 - Jx_n\rangle \ge 0,\ \forall z \in C_n \bigcap Q_n.$$

Since $F \subset C_n \bigcap Q_n$ by the induction assumptions, we have

$$\langle x_n - z,\ Jx_1 - Jx_n\rangle \ge 0,\ \forall z \in F.$$

This implies that $F \subset Q_{n+1}$. So, we get $F \subset Q_n$ for all $n \ge 1$. Therefore we have $F \subset C_n \bigcap Q_n$ for all $n \ge 1$. This means that the iteration algorithm (3.1) is well defined.

Step 4. $\lim_{n\to\infty} \phi(x_n, x_1)$ exists and $\{x_n\}$ is bounded.

Noticing that $x_n = \Pi_{Q_{n+1}} x_1$ and $x_{n+1} = \Pi_{C_{n+1}\cap Q_{n+1}} x_1 \in Q_{n+1}$, we have

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1)$$

for all $n \ge 1$. We, therefore, obtain that $\{\phi(x_n, x_1)\}$ is nondecreasing. From Lemma 2.3, it follows that

$$\phi(x_n, x_1) = \phi(\Pi_{Q_{n+1}} x_1, x_1) \le \phi(p, x_1) - \phi(p, x_n) \le \phi(p, x_1)$$

for all $p \in F$ and $n \ge 1$. This shows that the sequence $\{\phi(x_n, x_1)\}$ is bounded. Therefore, the limit of $\{\phi(x_n, x_1)\}$ exists and $\{x_n\}$ is bounded. Moreover, for each $i \in I$, $\{y_{n,i}\}$ and $\{u_{n,i}\}$ are bounded.

Step 5. $x_n \to w \in C$.

By the construction of Q_n, we know that $Q_{m+1} \subset Q_n$ and $x_m = \Pi_{Q_{m+1}} x_1 \in Q_n$ for any positive integer $m \geq n$. Notice that

$$\begin{aligned}\phi(x_m, x_n) = \phi(x_m, \Pi_{Q_{n+1}} x_1) &\leq \phi(x_m, x_1) - \phi(\Pi_{Q_{n+1}} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1).\end{aligned} \tag{3.3}$$

In view of step 4 we deduce that $\phi(x_m, x_n) \to 0$ as m, $n \to \infty$. It follows from Lemma 2.5 that $\|x_m - x_n\| \to 0$ as m, $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence of C. Since E is a Banach space and C is closed subset of E, we have

$$x_n \to w \in C \ (n \to \infty).$$

Step 6. $w \in \bigcap_{i \in I} F(T_i)$.

By taking $m = n + 1$ in (3.3), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.4}$$

From Lemma 2.5, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Noticing that $x_{n+1} \in C_{n+1}$, for any $i \in I$, we obtain

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) + (1 - \alpha_{n,i})(k_{n,i} - 1)L_n.$$

From (3.4) and $\lim_{n\to\infty} k_{n,i} = 1$ for any $i \in I$, we know

$$\lim_{n\to\infty} \phi(x_{n+1}, u_{n,i}) = 0, \ \forall i \in I. \tag{3.6}$$

Thus

$$\lim_{n\to\infty} \|x_{n+1} - u_{n,i}\| = 0, \ \forall i \in I. \tag{3.7}$$

Notice that

$$\|x_n - u_{n,i}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_{n,i}\|$$

for all $n \geq 1$ and $i \in I$. It follows from (3.5) and (3.7) that

$$\lim_{n\to\infty} \|x_n - u_{n,i}\| = 0, \ \forall i \in I. \tag{3.8}$$

From $x_n \to w$ $(n \to \infty)$, we know

$$\lim_{n\to\infty} \|w - u_{n,i}\| = 0, \ \forall i \in I. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded sets, from (3.8), we have

$$\lim_{n\to\infty} \|Jx_n - Ju_{n,i}\| = 0, \ \forall i \in I. \tag{3.10}$$

Let $r_i = \sup\{\|x_n\|, \|T_i^n x_n\| : n \in \mathbb{N}\}$ for each $i \in I$. Since E is uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. Therefore, from Lemma 2.6, for each $i \in I$, there exists a strictly increasing, continuous, and convex function $g_i : [0, 2r_i] \to \mathbb{R}$ such that $g_i(0) = 0$ and

$$\|tx^* + (1-t)y^*\|^2 \le t\|x^*\|^2 + (1-t)\|y^*\|^2 - t(1-t)g_i(\|x^* - y^*\|)$$

for all x^*, $y^* \in B^*_{r_i}$ and $t \in [0, 1]$. Let $i \in I$ and $p \in F$, we have

$$\begin{aligned}
&\phi(p, u_{n,i}) \\
=&\phi(p, T_{r_{n,i}} y_{n,i}) \\
\le&\phi(p, y_{n,i}) \\
=&\phi(p, J^{-1}(\alpha_{n,i} Jx_n + (1-\alpha_{n,i})JT_i^n x_n)) \\
=&\|p\|^2 - 2\alpha_{n,i}\langle p, Jx_n\rangle - 2(1-\alpha_{n,i})\langle p, JT_i^n x_n\rangle \\
&+ \|\alpha_{n,i} Jx_n + (1-\alpha_{n,i})JT_i^n x_n)\|^2 \\
\le&\|p\|^2 - 2\alpha_{n,i}\langle p, Jx_n\rangle - 2(1-\alpha_{n,i})\langle p, JT_i^n x_n\rangle \\
&+ \alpha_{n,i}\|x_n\|^2 + (1-\alpha_{n,i})\|T_i^n x_n\|^2 - \alpha_{n,i}(1-\alpha_{n,i})g_i(\|Jx_n - JT_i^n x_n\|) \\
=&\alpha_{n,i}\phi(p, x_n) + (1-\alpha_{n,i})\phi(p, T_i^n x_n) - \alpha_{n,i}(1-\alpha_{n,i})g_i(\|Jx_n - JT_i^n x_n\|) \\
\le&\phi(p, x_n) + (1-\alpha_{n,i})(k_{n,i} - 1)L_n - \alpha_{n,i}(1-\alpha_{n,i})g_i(\|Jx_n - JT_i^n x_n\|).
\end{aligned} \tag{3.11}$$

Therefore, for each $i \in I$, we have

$$\begin{aligned}
&\alpha_{n,i}(1-\alpha_{n,i})g_i(\|Jx_n - JT_i^n x_n\|) \\
\le&\phi(p, x_n) - \phi(p, u_{n,i}) + (1-\alpha_{n,i})(k_{n,i} - 1)L_n.
\end{aligned} \tag{3.12}$$

On the other hand, for each $i \in I$, we have

$$\begin{aligned}
&|\phi(p, x_n) - \phi(p, u_{n,i})| \\
=&|\|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle p, Jx_n - Ju_{n,i}\rangle| \\
\le&|\|x_n\| - \|u_{n,i}\||(\|x_n\| + \|u_{n_i}\|) + 2\|Jx_n - Ju_{n,i}\|\|p\| \\
\le&\|x_n - u_{n,i}\|(\|x_n\| + \|u_{n_i}\|) + 2\|Jx_n - Ju_{n,i}\|\|p\|.
\end{aligned}$$

It follows from (3.8) and (3.10) that

$$\lim_{n\to\infty}(\phi(p, x_n) - \phi(p, u_{n,i})) = 0, \ \forall i \in I. \tag{3.13}$$

Since $\lim_{n\to\infty} k_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$ for each $i \in I$, from (3.12) and (3.13) we have

$$\lim_{n\to\infty} g_i(\|Jx_n - JT_i^n x_n\|) = 0, \ \forall i \in I.$$

Therefore, from the property of g_i, we obtain

$$\lim_{n\to\infty} \|Jx_n - JT_i^n x_n\| = 0, \ \forall i \in I. \tag{3.14}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0, \ \forall i \in I.$$

Noting that $x_n \to w$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|T_i^n x_n - w\| = 0, \ \forall i \in I. \tag{3.15}$$

Since

$$\|T_i^{n+1} x_n - w\| \le \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - w\|,$$

it follows from the asymptotic regularity of T_i and (3.15) that

$$\lim_{n\to\infty} \|T_i^{n+1} x_n - w\| = 0, \ \forall i \in I.$$

That is, $T_i(T_i^n x_n) \to w$ as $n \to \infty$ for each $i \in I$. From the closedness of T_i, we get $T_i w = w$ for each $i \in I$. So, $w \in \bigcap_{i\in I} F(T_i)$.

Step 7. $w \in F$.

For each $i \in I$, from $y_{n,i} = J^{-1}(\alpha_{n,i} Jx_n + (1-\alpha_{n,i})JT_i^n x_n)$, we have

$$\begin{aligned}\|Jy_{n,i} - Jx_n\| =& \|\alpha_{n,i} Jx_n + (1-\alpha_{n,i})JT_i^n x_n - Jx_n\| \\ =& (1-\alpha_{n,i})\|JT_i^n x_n - Jx_n\|.\end{aligned}$$

It follows from (3.14) that

$$\lim_{n\to\infty} \|Jy_{n,i} - Jx_n\| = 0, \ \forall i \in I. \tag{3.16}$$

Noting that

$$\|Ju_{n,i} - Jy_{n,i}\| \le \|Ju_{n,i} - Jx_n\| + \|Jx_n - Jy_{n,i}\|,$$

from (3.10) and (3.16) we obtain

$$\lim_{n\to\infty} \|Ju_{n,i} - Jy_{n,i}\| = 0, \ \forall i \in I. \tag{3.17}$$

From the assumption $r_{n,i} \ge a$, we get

$$\lim_{n\to\infty} \frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_{n,i}} = 0, \ \forall i \in I. \tag{3.18}$$

For each $i \in I$, noting that $u_{n,i} = T_{r_{n,i}} y_{n,i}$, we obtain

$$f(u_{n,i}, y) + \frac{1}{r_{n,i}} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0, \ \forall y \in C.$$

From $(A2)$, we have

$$\begin{aligned} \|y - u_{n,i}\| \frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_{n,i}} &\geq \frac{1}{r_{n,i}} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \\ &\geq -f(u_{n,i}, y) \\ &\geq f(y, u_{n,i}), \ \forall y \in C. \end{aligned}$$

Letting $n \to \infty$, from (3.9), (3.18) and $(A4)$, we have

$$0 \geq f(y, w), \ \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $f(y_t, w) \leq 0$. So from $(A1)$ and $(A4)$ we have

$$0 \leq f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, w) \leq tf(y_t, y)$$

and hence $0 \leq f(y_t, y)$. Letting $t \downarrow 0$, from $(A3)$, we have $0 \leq f(w, y)$ for all $y \in C$. This implies that $w \in EP(f)$. Therefore, in view of step 6 we have $w \in F$.

Step 8. $w = \Pi_F x_1$.

From $x_n = \Pi_{Q_{n+1}} x_1$, we get

$$\langle x_n - z, Jx_1 - Jx_n \rangle \geq 0, \ \forall z \in Q_{n+1}.$$

Since $F \subset Q_n$ for all $n \geq 1$, we arrive at

$$\langle x_n - p, Jx_1 - Jx_n \rangle \geq 0, \ \forall p \in F.$$

Letting $n \to \infty$, we have

$$\langle w - p, Jx_1 - Jw \rangle \geq 0, \ \forall p \in F,$$

and hence $w = \Pi_F x_1$ by Lemma 2.4. This completes the proof. □

Next, we consider a simpler algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a family of quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces.

Theorem 3.2 *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_i\}_{i\in I} : C \to C$ a family of closed quasi-ϕ-asymptotically nonexpansive mappings with sequences $\{k_{n,i}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) such that $F = (\bigcap_{i\in I} F(T_i)) \bigcap EP(f) \neq \emptyset$. Assume that T_i is asymptotically regular on C for each $i \in I$ and F is bounded. For each $i \in I$, let $\{\alpha_{n,i}\}$ be a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\{r_{n,i}\}$ a sequence in $[a, \infty)$ for some $a > 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ C_{1,i} = C,\ C_1 = \bigcap_{i\in I} C_{1,i},\ x_1 = \Pi_{C_1} x_0, \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JT_i^n x_n), \\ u_{n,i} \in C \text{ such that } f(u_{n,i}, y) + \frac{1}{r_{n,i}} \langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i} \rangle \geq 0,\ \ \forall y \in C, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(z, u_{n,i}) \leq \phi(z, x_n) + (1 - \alpha_{n,i})(k_{n,i} - 1)L_n\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i} \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \end{cases}$$

for every $n \in \mathbb{N}$, where J is the normalized duality mapping on E and $L_n = \sup\{\phi(p, x_n) : p \in F\} < \infty$. Then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. Following the lines of the proof of Theorem 3.1, we can show that:

(1) F is a nonempty closed convex subset of C and hence $\Pi_F x_1$ is well defined for $x_1 \in C$.

(2) C_n is closed and convex for all $n \in N$.

It is obvious that $C_1 = C_{1,i} = C$ is closed and convex for every $i \in I$. Since the defining inequality in $C_{n+1,i}$ is equivalent to the inequality:

$$2\langle z, Jx_n - Ju_{n,i}\rangle \leq \|x_n\|^2 - \|u_{n,i}\|^2 + (1 - \alpha_{n,i})(k_{n,i} - 1)Q_n$$

for every $i \in I$. This shows that $C_{n+1,i}$ is closed and convex for every $i \in I$. So, we have $C_{n+1} = \bigcap_{i\in I} C_{n+1,i}$ is a closed and convex subset of C for all $n \geq 1$. Consequently, $\Pi_{C_{n+1}} x_1$ is well defined.

(3) $F \subset C_n$ for all $n \geq 1$.

It suffices to show that $\forall i \in I$, $F \subset C_{n,i}$ for all $n \geq 1$. This can be proved by induction on n. For $n = 1$, we have $F \subset C = C_{1,i}$. Assume that $F \subset C_{n,i}$ for some $n > 1$. From the induction assumption, (3.2) and the definition of $C_{n+1,i}$, we conclude that $F \subset C_{n+1,i}$ and hence $F \subset C_{n,i}$ for all $n \geq 1$.

(4) $\lim_{n\to\infty} \phi(x_n, x_1)$ exists and $\{x_n\}$ is bounded.

Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$$

for all $n \geq 1$. We, therefore, obtain that $\{\phi(x_n, x_1)\}$ is nondecreasing. From Lemma 2.3, it follows that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1)$$

for all $p \in F$ and $n \geq 1$. This shows that the sequence $\{\phi(x_n, x_1)\}$ is bounded. Therefore, the limit of $\{\phi(x_n, x_1)\}$ exists and $\{x_n\}$ is bounded. Moreover, for each $i \in I$, $\{y_{n,i}\}$ and $\{u_{n,i}\}$ are bounded.

(5) $x_n \to w \in C$.

By the construction of C_n, we know that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_1 \in C_n$ for any positive integer $m \geq n$. Notice that

$$\begin{aligned}\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) &\leq \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1).\end{aligned}$$

In view of (4) we deduce that $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$. It follows from Lemma 2.5 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence of C. We have

$$x_n \to w \in C \ (n \to \infty).$$

(6) By the same method given in Step 6 and Step 7 of the proof of Theorem 3.1 we have $w \in F$.

(7) $w = \Pi_F x_1$.

From $x_n = \Pi_{C_n} x_1$, we get

$$\langle x_n - z, Jx_1 - Jx_n \rangle \geq 0, \ \forall z \in C_n.$$

Since $F \subset C_n$ for all $n \geq 1$, we arrive at

$$\langle x_n - p, Jx_1 - Jx_n \rangle \geq 0, \ \forall p \in F.$$

Hence

$$\langle w - p, Jx_1 - Jw \rangle \geq 0, \ \forall p \in F.$$

It follows that $w = \Pi_F x_1$ by Lemma 2.4. This completes the proof. □

As some corollaries of Theorem 3.1 and Theorem 3.2, we have the following results immediately.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed quasi-ϕ-asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that*

$\lim_{n\to\infty} k_n = 1$. *Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) such that $F = F(T)\bigcap EP(f) \neq \emptyset$. Assume that T is asymptotically regular on C and F is bounded. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\}$ a sequence in $[a,\infty)$ for some $a > 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0,\ \ \forall y \in C, \\ C_{n+1} = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n) + (1-\alpha_n)(k_n - 1)L_n\}, \\ Q_1 = C, \\ Q_{n+1} = \{z \in Q_n : \langle x_n - z, Jx_1 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_{n+1}\cap Q_{n+1}} x_1 \end{cases}$$

for every $n \geq 0$, where J is the normalized duality mapping on E and $L_n = \sup\{\phi(p, x_n) : p \in F\} < \infty$. Then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed quasi-ϕ-asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) such that $F = F(T)\bigcap EP(f) \neq \emptyset$. Assume that T is asymptotically regular on C and F is bounded. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\}$ a sequence in $[a,\infty)$ for some $a > 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0,\ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (1-\alpha_n)(k_n - 1)L_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \end{cases}$$

for every $n \in \mathbb{N}$, where J is the normalized duality mapping on E and $L_n = \sup\{\phi(p, x_n) : p \in F\} < \infty$. Then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Corollary 3.5 *Let C be a nonempty, closed and convex subset of a Hilbert* *space H and $\{T_i\}_{i\in I}: C \to C$ a family of closed quasi-ϕ-asymptotically nonexpansive mappings with sequences $\{k_{n,i}\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) such that $F = (\bigcap_{i\in I} F(T_i)) \bigcap EP(f) \neq \emptyset$. Assume that T_i is asymptotically regular on C for each $i \in I$ and F is bounded. For each $i \in I$, let $\{\alpha_{n,i}\}$ be a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\{r_{n,i}\}$ a sequence in $[a,\infty)$ for some $a > 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_{1,i} = C,\ C_1 = \bigcap_{i\in I} C_{1,i},\ x_1 = P_{C_1}x_0, \\ y_{n,i} = \alpha_{n,i}x_n + (1-\alpha_{n,i})T_i^n x_n, \\ u_{n,i} \in C \text{ such that } f(u_{n,i},y) + \frac{1}{r_{n,i}}\langle y-u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \geq 0,\ \forall y \in C, \\ C_{n+1,i} = \{z \in C : \|z-u_{n,i}\| \leq \|z-x_n\| + (1-\alpha_{n,i})(k_{n,i}-1)L_n\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ Q_1 = C, \\ Q_{n+1} = \{z \in Q_n : \langle x_n - z, Jx_1 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}\cap Q_{n+1}}x_1 \end{cases}$$

for every $n \geq 0$, where J is the normalized duality mapping on E and $L_n = \sup\{\|p-x_n\| : p \in F\} < \infty$. Then $\{x_n\}$ converges strongly to $P_F x_1$.

Corollary 3.6 *Let C be a nonempty, closed and convex subset of a Hilbert space H and $\{T_i\}_{i\in I}: C \to C$ a family of closed quasi-ϕ-asymptotically nonexpansive mappings with sequences $\{k_{n,i}\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) such that $F = (\bigcap_{i\in I} F(T_i)) \bigcap EP(f) \neq \emptyset$. Assume that T_i is asymptotically regular on C for each $i \in I$ and F is bounded. For each $i \in I$, let $\{\alpha_{n,i}\}$ be a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\{r_{n,i}\}$ a sequence in $[a,\infty)$ for some $a > 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_{1,i} = C,\ C_1 = \bigcap_{i\in I} C_{1,i},\ x_1 = P_{C_1}x_0, \\ y_{n,i} = \alpha_{n,i}x_n + (1-\alpha_{n,i})T_i^n x_n, \\ u_{n,i} \in C \text{ such that } f(u_{n,i},y) + \frac{1}{r_{n,i}}\langle y-u_{n,i}, u_{n,i} - y_{n,i}\rangle \geq 0,\ \forall y \in C, \\ C_{n+1,i} = \{z \in C_{n,i} : \|z-u_{n,i}\|^2 \leq \|z-x_n\|^2 + (1-\alpha_{n,i})(k_{n,i}-1)L_n\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}}x_1 \end{cases}$$

for every $n \geq 0$, where J is the normalized duality mapping on E and $L_n = \sup\{\|p-x_n\|^2 : p \in F\} < \infty$. Then $\{x_n\}$ converges strongly to $P_F x_1$.

Remark 3.7 Theorem 3.1 and Theorem 3.2 extend the main results of [8, 10] from either equilibrium problems and relatively nonexpansive mappings or equilibrium problems and quasi-ϕ-nonexpansive mappings to equilibrium problems and a countable infinite family of quasi-ϕ-asymptotically nonexpansive mappings.

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