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Iterative methods for k-strict pseudo-contractive mappings in Hilbert spaces

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Abstract

In this paper, we investigate two iterative methods for k-strict pseudo-contractive mappings in a real Hilbert space. We prove that the proposed iterative algorithms converge strongly to some fixed point of a strict pseudo-contractive mapping.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping. We use $Fix(T)$ to denote the set of fixed points of T. Recall that T is said to be a *strict pseudo-contractive mapping* if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \text{for all } x, y \in C. \tag{1.1}$$

For such case, we also say that T is a k-strict pseudo-contractive mapping. When $k = 0$, T is said to be *nonexpansive*, and it is said to be *pseudo-contractive* if $k = 1$. T is said to be *strongly pseudo-contractive* if there exists a constant $\alpha \in (0, 1)$ such that $\langle Tx - Ty, x - y\rangle \leq \alpha\|x - y\|^2$ for all $x, y \in C$. Clearly, the class of k-strict pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings.

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We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strict pseudo-contractive mappings (see, e.g., [3-5]). It is clear that, in a real Hilbert space H, (1.1) is equivalent to

(1.2)

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|(I-T)x - (I-T)y\|^2 \quad \text{for all } x, y \in C.$$

Recall also that a mapping $f : C \to C$ is called *contractive* if there exists a constant $\alpha \in [0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\| \quad \text{for all } x, y \in C.$$

Iterative methods for nonexpansive mappings have been extensively investigated; see [1, 2, 6-15, 17-18, 20-22, 24-36] and the references therein. However iterative methods for strict pseudo-contractive mappings are far less developed than those for nonexpansive mappings though Browder and Petryshyn [5] initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1.1) impedes the convergence analysis for iterative methods used to find a fixed point of the strict pseudo-contractive mapping T. However, on the other hand, strict pseudo-contractive mappings have more powerful applications than nonexpansive mappings do in solving inverse problems; see Scherzer [23]. Therefore it is interesting to develop the iterative methods for strict pseudo-contractive mappings. As a matter of fact, Browder and Petryshyn [5] show that if a k-strict pseudo-contractive mappings T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula:

$$x_{n+1} = \lambda x_n + (1 - \lambda)Tx_n, \quad n \ge 0,$$

where λ is a constant such that $k < \lambda < 1$, converges weakly to a fixed point of T.

Recently, Marino and Xu [16] have extended Browder and Petryshyn's result by proving that the sequence $\{x_n\}$ generated by the following Mann's method:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)Tx_n, \quad n \ge 0$$

converges weakly to a fixed point of T, provided the control sequence $\{\lambda_n\}$ satisfies the conditions that $k < \lambda_n < 1$ for all n and $\sum_{n=0}^{\infty}(\lambda_n - k)(1 - \lambda_n) = \infty$. However, this convergence is in general not strong. So in order to get strong convergence for strict pseudo-contractive mappings, Marino and Xu [16] further suggested the following modified Mann's method based on the CQ method:

$$(1.3)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad\qquad + (1-\alpha_n)(k-\alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

They proved that the sequence $\{x_n\}$ generated by (1.3) is strongly convergent to a fixed point of T for any choice of the control sequence $\{\alpha_n\}$ such that $\alpha_n < 1$ for all n.

We observe that the CQ method (1.3) generates a sequence $\{x_n\}$ by projecting the initial guess x_0 onto the intersection of two appropriately constructed closed convex subsets C_n and Q_n. Hence, how to construct closed convex subsets C_n and Q_n is very crucial for the CQ method.

It is the purpose of this paper to suggest and analyze some iterative methods for strict pseudo-contractive mappings in the sense of Browder-Petryshyn in a real Hilbert space. First, we consider a modified Mann's method which is different from (1.3). Secondly, we study another modified method for strict pseudo-contractive mappings. We prove that the proposed iterative methods converge strongly to some fixed point of a strict pseudo-contractive mapping.

2. Preliminaries

In this section, we collect some facts in a real Hilbert space H which are listed as below.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C,$$

where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping. For given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \to x \text{ weakly}\}$ denote the weak ω-limit set of $\{x_n\}$.

We note the following Lemmas 2.1 and 2.2 are well-known.

Lemma 2.1. *Let H be a real Hilbert space. Then there hold the following well-known identities:*

(a) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$;

(b) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2$ *for all* $t \in [0,1]$, $x, y \in H$.

Lemma 2.2. *Let* C *be a closed convex subset of a real Hilbert space* H. *Given* $x \in H$ *and* $z \in C$. *Then* $z = P_C x$ *if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leq 0 \quad \text{for all } y \in C.$$

Lemma 2.3. ([16]) *Assume* C *is a closed convex subset of a real Hilbert space* H, *let* $T : C \to C$ *be a* k*-strict pseudo-contractive mapping. Then, the mapping* $I - T$ *is demiclosed at zero. That is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \to x^*$ *weakly and* $(I - T)x_n \to 0$ *strongly, then* $(I - T)x^* = 0$.

Lemma 2.4. ([17]) *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $\{x_n\}$ *be a sequence in* H *and* $u \in H$. *Let* $q = P_C u$. *If* $\{x_n\}$ *is such that* $\omega_w(x_n) \subset C$ *and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \quad \text{for all } n.$$

Then $x_n \to q$.

Lemma 2.5. ([25]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *such that* $x_{n+1} = \sigma_n x_n + (1 - \sigma_n)y_n, n \geq 0$ *where* $\{\sigma_n\}$ *is a sequence in* $[0, 1]$ *such that* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. *Assume* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be a* k*-strict pseudo-contractive mapping with* $Fix(T) \neq \emptyset$. *Then,* $Fix(T)$ *is convex and closed.*

Proof. First, we prove that $Fix(T)$ is convex. For all $p, q \in Fix(T)$ and $\xi \in [0, 1]$, we set $R = \xi p + (1 - \xi)q$. From Lemma 2.1(b) and (1.1), we have

$$\begin{aligned}
&\|R - TR\|^2 \\
&= \|\xi(p - TR) + (1 - \xi)(q - TR)\|^2 \\
&= \xi\|p - TR\|^2 + (1 - \xi)\|q - TR\|^2 - \xi(1 - \xi)\|p - q\|^2 \\
&= \xi\|Tp - TR\|^2 + (1 - \xi)\|Tq - TR\|^2 - \xi(1 - \xi)\|p - q\|^2 \\
&\leq \xi[\|p - R\|^2 + k\|R - TR\|^2] + (1 - \xi)[\|q - R\|^2 + k\|R - TR\|^2] \\
&\quad - \xi(1 - \xi)\|p - q\|^2 \\
&= \xi\|p - R\|^2 + (1 - \xi)\|q - R\|^2 - \xi(1 - \xi)\|p - q\|^2 + k\|R - TR\|^2 \\
&= \|\xi(p - R) + (1 - \xi)(q - R)\|^2 + k\|R - TR\|^2 \\
&= k\|R - TR\|^2,
\end{aligned}$$

which implies that

$$(1-k)\|R-TR\|^2 \le 0,$$

that is

$$R = TR.$$

Therefore, $R \in Fix(T)$ and $Fix(T)$ is convex.

Next, we prove that $Fix(T)$ is closed. First, we note that T is Lipschitz with Lipschitzian constant $L = \frac{1+k}{1-k}$. If we take $p_n \in Fix(T)$ satisfying $p_n \to p$ as $n \to \infty$, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} Tp_n = Tp.$$

Therefore, $p \in Fix(T)$. Hence, $Fix(T)$ is closed. This completes the proof.

Lemma 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strict pseudo-contractive mapping with $Fix(T) \neq \emptyset$. Let $f : C \to C$ be a continuous strongly pseudo-contractive mapping. For any $t \in (0,1)$, let x_t be the unique fixed point of $tf + (1-t)T$. Then as $t \to 0+$, the path $\{x_t\}$ converges strongly to $p = P_{Fix(T)}f(p)$ which is the unique solution of the variational inequality: $\langle f(p)-p, z-p\rangle \le 0$ for all $z \in Fix(T)$.*

Proof. From Lemma 2.6, we know that $Fix(T)$ is closed and convex. Hence, $P_{Fix(T)}$ is well defined. For fixed $u \in C$ arbitrarily, let the path $\{y_t : t \in (0,1)\}$ be defined by $y_t = tu + (1-t)Tx_t$ for all $t \in (0,1)$. Then, from [19], it is clear that the path $\{y_t\}$ converges strongly to $p = P_{Fix(T)}u$. Taking $u = f(p)$. Then, the path $\{y_t\}$ defined by $y_t = tf(p) + (1-t)Ty_t$ converges strongly to $p = P_{Fix(T)}f(p)$. Note that

$$\begin{aligned}
\|x_t - y_t\|^2 &= t\langle f(x_t) - f(p), x_t - y_t\rangle + (1-t)\langle Tx_t - Ty_t, x_t - y_t\rangle \\
&= t\langle f(x_t) - f(y_t), x_t - y_t\rangle + t\langle f(y_t) - f(p), x_t - y_t\rangle \\
&\quad + (1-t)\langle Tx_t - Ty_t, x_t - y_t\rangle \\
&\le t\alpha\|x_t - y_t\|^2 + t\|f(y_t) - f(p)\|\|x_t - y_t\| + (1-t)\|x_t - y_t\|^2,
\end{aligned}$$

that is,

$$\|x_t - y_t\| \le \frac{1}{1-\alpha}\|f(y_t) - f(p)\|.$$

Since $y_t \to p = P_{Fix(T)}f(p)$ and f is continuous, then $x_t \to p = P_{Fix(T)}f(p)$. From Lemma 2.2, it is clear that p is the unique solution of the variational inequality: $\langle f(p)-p, z-p\rangle \le 0$ for all $z \in Fix(T)$. This completes the proof.

Lemma 2.8. ([21]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1-b_n)a_n + c_n$, $n \ge 0$, where $\{b_n\}$ is a sequence in $(0,1)$ and $\{c_n\}$ is a sequence in $\mathbf{R}$ such that*

(a) $\sum_{n=0}^{\infty} b_n = \infty$;
(b) $\limsup_{n\to\infty} c_n/b_n \le 0$ *or* $\sum_{n=0}^{\infty} |c_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

3.1. Projection methods for strict pseudo-contractive mappings

First, we introduce the following modified Mann's method based on the projection methods which is different from (1.3).

Algorithm 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strict pseudo-contractive mapping for some $0 \le k < 1$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1)$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, let the sequence $\{x_n\}$ be generated by the following method:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases} \tag{3.1}$$

Now we prove the following strong convergence theorem concerning the above projection method (3.1).

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strict pseudo-contractive mapping for some $0 \le k < 1$ with $Fix(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the method (3.1). Assume that the sequence $\{\alpha_n\}$ is chosen so that $\alpha_n \in [a, b] \subset [k, 1)$ for all $n \ge 0$. Then $\{x_n\}$ converges strongly to $P_{Fix(T)} x_0$.*

Proof. First, we note that C_n is convex and closed. As a matter of fact, we observe that $\|y_n - z\| \le \|x_n - z\|$ is equivalent to $\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \ge 0$. So C_n is closed and convex. Hence, $\{x_n\}$ is well-defined.

Next we show that $Fix(T) \subset C_n$ for all n. From (1.1) and (1.2), we note that for all $p \in Fix(T)$,

$$\langle Tx_n - p, x_n - p\rangle \le \|x_n - p\|^2 - \frac{1-k}{2}\|x_n - Tx_n\|^2$$

and

$$\|Tx_n - p\|^2 \le \|x_n - p\|^2 + k\|x_n - Tx_n\|^2.$$

Then, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(Tx_n - p)\|^2 \\
&= \alpha_n^2\|x_n - p\|^2 + 2\alpha_n(1-\alpha_n)\langle Tx_n - p, x_n - p\rangle \\
&\quad + (1-\alpha_n)^2\|Tx_n - p\|^2 \\
&\le \alpha_n^2\|x_n - p\|^2 + 2\alpha_n(1-\alpha_n)\left[\|x_n - p\|^2 - \frac{1-k}{2}\|x_n - Tx_n\|^2\right] \\
&\quad + (1-\alpha_n)^2[\|x_n - p\|^2 + k\|x_n - Tx_n\|^2] \\
&= \|x_n - p\|^2 + (1-\alpha_n)(k-\alpha_n)\|x_n - Tx_n\|^2 \\
&\le \|x_n - p\|^2,
\end{aligned}$$

that is, $\|y_n - p\| \le \|x_n - p\|$. So $p \in C_{n+1} \subset C_n$ for all n. This implies that

$$Fix(T) \subset C_n, \quad n \ge 0.$$

From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0 \quad \text{for all } y \in C_n.$$

Using $Fix(T) \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - p\rangle \ge 0 \quad \text{for all } p \in Fix(T).$$

So, for $p \in Fix(T)$, we have

$$\begin{aligned}
0 &\le \langle x_0 - x_n, x_n - p\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - p\rangle \\
&\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|.
\end{aligned}$$

Hence,

$$\|x_0 - x_n\| \le \|x_0 - p\| \quad \text{for all } p \in Fix(T).$$

In particular, $\{x_n\}$ is bounded and

$$\|x_0 - x_n\| \le \|x_0 - q\|, \quad \text{where } q = P_{Fix(T)}x_0. \tag{3.2}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0. \tag{3.3}$$

Hence,

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|, \end{aligned}$$

and therefore

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|,$$

which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

From Lemma 2.1(a) and (3.3), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\to 0. \end{aligned}$$

By the fact $x_{n+1} \in C_{n+1} \subset C_n$, we get

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|,$$

which implies that $\|y_n - x_{n+1}\| \to 0$. At the same time, we note that

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &= \|\alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(Tx_n - x_{n+1})\|^2 \\ &= \alpha_n\|x_n - x_{n+1}\|^2 + (1-\alpha_n)\|Tx_n - x_{n+1}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \|Tx_n - x_{n+1}\|^2 &= \|Tx_n - x_n + x_n - x_{n+1}\|^2 \\ &= \|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n, x_n - x_{n+1}\rangle \\ &\quad + \|x_n - x_{n+1}\|^2. \end{aligned} \tag{3.5}$$

Substitute (3.5) into (3.4) to get

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &= \alpha_n\|x_n - x_{n+1}\|^2 + (1-\alpha_n)\|Tx_n - x_n\|^2 \\ &\quad + 2(1-\alpha_n)\langle Tx_n - x_n, x_n - x_{n+1}\rangle + (1-\alpha_n)\|x_n - x_{n+1}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - x_{n+1}\|^2 + 2(1-\alpha_n)\langle Tx_n - x_n, x_n - x_{n+1}\rangle \\ &\quad + (1-\alpha_n)^2\|x_n - Tx_n\|^2. \end{aligned}$$

This together with $x_n - x_{n+1} \to 0$ and $y_n - x_{n+1} \to 0$ imply that

$$\|x_n - Tx_n\| \to 0. \tag{3.6}$$

Now (3.6) and Lemma 2.3 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of T. That is, $\omega_w(x_n) \subset F(T)$. This fact, the inequality (3.2) and Lemma 2.4 ensure the strong convergence of $\{x_n\}$ to $P_{F(T)}x_0$. This completes the proof.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the method* (3.1)*. Assume that the sequence $\{\alpha_n\}$ is chosen so that $\alpha_n \in [a, b] \subset [0, 1)$ for all $n \geq 0$. Then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to $P_{Fix(T)}x_0$.*

3.2. Modified methods for strict pseudo-contractive mappings

Below is another modified method for strict pseudo-contractive mappings.

Algorithm 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strict pseudo-contractive mapping with $0 \leq k < 1$. Let $f : C \to C$ be a contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$. Let the sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in C$ by the following iterative method:*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Tx_n, \quad n \geq 0. \tag{3.7}$$

In particular, if we take $f \equiv u$, then (3.7) reduces to

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Tx_n, \quad n \geq 0. \tag{3.8}$$

Now we state and prove the following strong convergence theorem.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a k-strict pseudo-contractive mapping with $Fix(T) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with contractive coefficient α. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in* $[0, 1]$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $\beta_n \in [a, b] \subset (k, 1)$.

For initial guess $x_0 \in C$, then the sequence $\{x_n\}$ defined by (3.7) *converges strongly to $p \in Fix(T)$ which solves the variational inequality:*

$$\langle f(p) - p, z - p\rangle \leq 0 \quad \text{for all } z \in Fix(T).$$

Proof. We first show that $\{x_n\}$ is bounded. Indeed, take a point $p \in Fix(T)$ to get

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(f(x_n)-p)+\beta_n(x_n-p)+\gamma_n(Tx_n-p)]\| \\ &\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\| \\ &\quad +\|\beta_n(x_n-p)+\gamma_n(Tx_n-p)\|.\end{aligned} \tag{3.9}$$

From (1.1) and (1.2), we obtain

$$\begin{aligned}&\|\beta_n(x_n-p)+\gamma_n(Tx_n-p)\|^2 \\ &= \beta_n^2\|x_n-p\|^2+\gamma_n^2\|Tx_n-p\|^2+2\beta_n\gamma_n\langle Tx_n-p, x_n-p\rangle \\ &\le \beta_n^2\|x_n-p\|^2+\gamma_n^2[\|x_n-p\|^2+k\|x_n-Tx_n\|^2] \\ &\quad +2\beta_n\gamma_n[\|x_n-p\|^2-\frac{1-k}{2}\|x_n-Tx_n\|^2] \\ &= (\beta_n+\gamma_n)^2\|x_n-p\|^2+[\gamma_n^2k-(1-k)\beta_n\gamma_n]\|x_n-Tx_n\|^2 \\ &= (\beta_n+\gamma_n)^2\|x_n-p\|^2+\gamma_n[(\beta_n+\gamma_n)k-\beta_n]\|x_n-Tx_n\|^2 \\ &\le (\beta_n+\gamma_n)^2\|x_n-p\|^2,\end{aligned}$$

which implies that

$$\|\beta_n(x_n-p)+\gamma_n(Tx_n-p)\| \le (\beta_n+\gamma_n)\|x_n-p\|. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\alpha\|x_n-p\|+\alpha_n\|f(p)-p\|+(\beta_n+\gamma_n)\|x_n-p\| \\ &= [1-(1-\alpha)\alpha_n]\|x_n-p\|+(1-\alpha)\alpha_n\frac{\|f(p)-p\|}{1-\alpha}.\end{aligned}$$

By induction, we obtain, for all $n \ge 0$,

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\}.$$

Hence, $\{x_n\}$ is bounded.

We note that (3.7) can be rewritten as

$$x_{n+1} = \alpha_n f(x_n)+\left(\beta_n-\frac{k\gamma_n}{1-k}\right)x_n+\frac{\gamma_n}{1-k}[kx_n+(1-k)Tx_n].$$

It is clear that $\alpha_n+(\beta_n-\frac{k\gamma_n}{1-k})+\frac{\gamma_n}{1-k}=1$ and $(\beta_n-\frac{k\gamma_n}{1-k}) \in (\frac{a-k}{1-k}, \frac{1}{1-k}) \subset (0,1)$.

Now we define $x_{n+1} = (\beta_n - \frac{k\gamma_n}{1-k})x_n + (1 - \beta_n + \frac{k\gamma_n}{1-k})y_n, n \geq 0$. It follows that

$$\begin{aligned}
& y_{n+1} - y_n \\
&= \frac{x_{n+2} - (\beta_{n+1} - \frac{k\gamma_{n+1}}{1-k})x_{n+1}}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} - \frac{x_{n+1} - (\beta_n - \frac{k\gamma_n}{1-k})x_n}{1 - \beta_n + \frac{k\gamma_n}{1-k}} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + \frac{\gamma_{n+1}}{1-k}[kx_{n+1} + (1-k)Tx_{n+1}]}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} \\
&\quad - \frac{\alpha_n f(x_n) + \frac{\gamma_n}{1-k}[kx_n + (1-k)Tx_n]}{1 - \beta_n + \frac{k\gamma_n}{1-k}} \\
&= \frac{\alpha_{n+1} f(x_{n+1})}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} - \frac{\alpha_n f(x_n)}{1 - \beta_n + \frac{k\gamma_n}{1-k}} \\
&\quad + \frac{\frac{\gamma_{n+1}}{1-k}}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} \big[k(x_{n+1} - x_n) + (1-k)(Tx_{n+1} - Tx_n)\big] \\
&\quad + \left(\frac{\frac{\gamma_{n+1}}{1-k}}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} - \frac{\frac{\gamma_n}{1-k}}{1 - \beta_n + \frac{k\gamma_n}{1-k}} \right)(kx_n + (1-k)Tx_n).
\end{aligned} \tag{3.11}$$

It follows from (3.11) that

$$\begin{aligned}
& \|y_{n+1} - y_n\| \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} \|f(x_{n+1})\| + \frac{\alpha_n}{1 - \beta_n + \frac{k\gamma_n}{1-k}} \|f(x_n)\| \\
&\quad + \frac{\frac{\gamma_{n+1}}{1-k}}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} \big\|k(x_{n+1} - x_n) + (1-k)(Tx_{n+1} - Tx_n)\big\| \\
&\quad + \left| \frac{\frac{\gamma_{n+1}}{1-k}}{1 - \beta_{n+1} + \frac{k\gamma_{n+1}}{1-k}} - \frac{\frac{\gamma_n}{1-k}}{1 - \beta_n + \frac{k\gamma_n}{1-k}} \right| \|kx_n + (1-k)Tx_n\|.
\end{aligned} \tag{3.12}$$

Again from Lemma 2.1(b) and (1.1), we have

$$\begin{aligned}
& \|k(x_{n+1} - x_n) + (1-k)(Tx_{n+1} - Tx_n)\|^2 \\
&= k\|x_{n+1} - x_n\|^2 + (1-k)\|Tx_{n+1} - Tx_n\|^2 \\
&\quad - k(1-k)\|(I-T)x_{n+1} - (I-T)x_n\|^2 \\
&\leq k\|x_{n+1} - x_n\|^2 + (1-k)[\|x_{n+1} - x_n\|^2 + k\|(I-T)x_{n+1} - (I-T)x_n\|^2] \\
&\quad - k(1-k)\|(I-T)x_{n+1} - (I-T)x_n\|^2 \\
&= \|x_{n+1} - x_n\|^2,
\end{aligned}$$

that is,

$$\|k(x_{n+1} - x_n) + (1-k)(Tx_{n+1} - Tx_n)\| \le \|x_{n+1} - x_n\|. \tag{3.13}$$

At the same time, we observe that

$$\begin{aligned}\frac{\frac{\gamma_{n+1}}{1-k}}{1-\beta_{n+1}+\frac{k\gamma_{n+1}}{1-k}} &= \frac{\gamma_{n+1}}{1-k+(1-\alpha_{n+1})k-\beta_{n+1}} \\ &= \frac{1-\alpha_{n+1}k-\beta_{n+1}+(k-1)\alpha_{n+1}}{1-\alpha_{n+1}k-\beta_{n+1}} \\ &= 1-\frac{(1-k)\alpha_{n+1}}{1-\alpha_{n+1}k-\beta_{n+1}}.\end{aligned}$$

Hence, $\frac{\frac{\gamma_{n+1}}{1-k}}{1-\beta_{n+1}+\frac{k\gamma_{n+1}}{1-k}} \in [0,1]$ and $\frac{\frac{\gamma_{n+1}}{1-k}}{1-\beta_{n+1}+\frac{k\gamma_{n+1}}{1-k}} \to 1$ as $n \to \infty$. Therefore,

$$\frac{\frac{\gamma_{n+1}}{1-k}}{1-\beta_{n+1}+\frac{k\gamma_{n+1}}{1-k}} - \frac{\frac{\gamma_n}{1-k}}{1-\beta_n+\frac{k\gamma_n}{1-k}} \to 0.$$

Combining (3.12) and (3.13) yields

$$\begin{aligned}\|y_{n+1}-y_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}+\frac{k\gamma_{n+1}}{1-k}}\|f(x_{n+1})\| + \frac{\alpha_n}{1-\beta_n+\frac{k\gamma_n}{1-k}}\|f(x_n)\| \\ &+ \|x_{n+1}-x_n\| + \left|\frac{\frac{\gamma_{n+1}}{1-k}}{1-\beta_{n+1}+\frac{k\gamma_{n+1}}{1-k}} - \frac{\frac{\gamma_n}{1-k}}{1-\beta_n+\frac{k\gamma_n}{1-k}}\right| M,\end{aligned}$$

where M is a constant such that $\sup_n\{\|kx_n + (1-k)Tx_n\|\} \le M$. Since $\alpha_n \to 0$, the last inequality implies

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Apply Lemma 2.5 to get

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\left(1-\beta_n+\frac{k\gamma_n}{1-k}\right)\|y_n-x_n\| = 0.$$

Note that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_{n+1}-x_n\| + \|x_{n+1}-Tx_n\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\|f(x_n)-Tx_n\| + \beta_n\|x_n - Tx_n\|,\end{aligned}$$

that is,

$$\|x_n - Tx_n\| \leq \frac{1}{1-\beta_n}(\|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - Tx_n\|) \to 0.$$

Next we prove that

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \leq 0 \quad \text{for all } p \in Fix(T).$$

Let z_t be the unique solution to the equation

$$z_t = tf(z_t) + (1-t)Tz_t.$$

Since f is contractive, it must be strict pseudo-contractive. From Lemma 2.7, we know that $z_t \to p \in Fix(T)$ which solves the variational inequality

$$\langle f(p) - p, z - p\rangle \leq 0 \quad \text{for all } z \in Fix(T).$$

We can write

$$z_t - x_n = (1-t)(Tz_t - x_n) + t(f(z_t) - x_n).$$

It follows that

$$\begin{aligned}
\|z_t - x_n\|^2 &= (1-t)\langle Tz_t - x_n, z_t - x_n\rangle + t\langle f(z_t) - x_n, z_t - x_n\rangle \\
&= (1-t)(\langle Tz_t - Tx_n, z_t - x_n\rangle + \langle Tx_n - x_n, z_t - x_n\rangle) \\
&\quad + t\langle f(z_t) - z_t, z_t - x_n\rangle + t\langle z_t - x_n, z_t - x_n\rangle \\
&\leq (1-t)\|z_t - x_n\|^2 + (1-t)\|Tx_n - x_n\|\|z_t - x_n\| \\
&\quad + t\langle f(z_t) - z_t, z_t - x_n\rangle + t\|z_t - x_n\|^2 \\
&= \|z_t - x_n\|^2 + (1-t)\|Tx_n - x_n\|\|z_t - x_n\| \\
&\quad + t\langle f(z_t) - z_t, z_t - x_n\rangle,
\end{aligned}$$

and hence

$$\langle f(z_t) - z_t, x_n - z_t\rangle \leq \frac{1-t}{t}\|Tx_n - x_n\|\|z_t - x_n\|.$$

Note that $\|x_n - Tx_n\| \to 0$, $z_t \to p$ and $\{z_t\}$ and $\{x_n\}$ are all bounded. By using the standard proof, it is easy to obtain

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \leq 0 \quad \text{for all } p \in Fix(T). \tag{3.14}$$

Finally we claim that $x_n \to p$ in norm. Indeed, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle + \langle \beta_n(x_n - p) + \gamma_n(Tx_n - p), x_{n+1} - p\rangle \\
&\le \alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle + \|\beta_n(x_n - p) + \gamma_n(Tx_n - p)\|\|x_{n+1} - p\| \\
&\le \alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + \alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\quad + (\beta_n + \gamma_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\le \alpha_n\alpha\|x_n - p\|\|x_{n+1} - p\| + \alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\quad + (1 - \alpha_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\le [1 - (1 - \alpha)\alpha_n]\frac{1}{2}(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + \alpha_n\langle f(p) - p, x_{n+1} - p\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\|x_{n+1} - p\|^2 \le{}& [1 - (1 - \alpha)\alpha_n]\|x_n - p\|^2 \\
&+ 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - p\|^2 \le{}& [1 - (1 - \alpha)\alpha_n]\|x_n - p\|^2 \\
&+ (1 - \alpha)\alpha_n\left\{\frac{2}{1 - \alpha}\langle f(p) - p, x_{n+1} - p\rangle\right\}.
\end{aligned} \tag{3.15}$$

So combining Lemma 2.8 with (3.14) and (3.15) we conclude that $\|x_n - p\| \to 0$. This completes the proof.

As a direct corollary of Theorem 3.2, we obtain the following.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a k-strict pseudo-contractive mapping with $Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0, 1]$ satisfying the following conditions:*
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $\beta_n \in [a, b] \subset (k, 1)$.
For initial guess $x_0 \in C$ and fixed $u \in C$, then the sequence $\{x_n\}$ defined by (3.8) converges strongly to $p = P_{Fix(T)}u$.

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References

[1] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.,* 67 (2007), 2350–2360.

[2] H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.,* 202 (1996), 150–159.

[3] F. E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, *Proc. Natl. Acad. Sci. USA,* 53 (1965), 1272–1276.

[4] F. E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Ration. Mech. Anal.,* 24 (1967), 82–90.

[5] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hiblert space, *J. Math. Anal. Appl.,* 20 (1967), 197–228.

[6] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems,* 20 (2004), 103–120.

[7] S. S. Chang, Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.,* 323 (2006), 1402–1416.

[8] C. E. Chidume and C. O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, *J. Math. Anal. Appl.,* 318 (2006), 288–295.

[9] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.,* 73 (1967), 957–961.

[10] J. S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.,* 302 (2005), 509–520.

[11] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal.,* 61 (2005), 51–60.

[12] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal.,* 64 (2006), 1140–1152.

[13] P. L. Lions, Approximation de points fixes de contractions, *C. R. Acad. Sci. Paris Ser. A-B* 284 (1977), 1357–1359.

[14] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.,* 4 (1953), 506–510.

[15] G. Marino and H. K. Xu, Convergence of generalized proximal point algorithms, *Commun. Pure Appl. Anal.,* 3 (2004), 791–808.

[16] G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudocontractions in Hilbert spaces, *J. Math. Anal. Appl.,* 329 (2007), 336–349.

[17] C. Matinez-Yanes and H. K. Xu, Strong convergence of the CQ method for fixed point processes, *Nonlinear Anal.,* 64 (2006), 2400–2411.

[18] S. Y. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx. Theory* 134 (2005), 257–266.

[19] C. H. Morales and J. S. Jung, Convergence of paths for pseudocontractive mappings in Banach spaces, *Proc. Amer. Math. Soc.,* 128 (2000), 3411–3419.

[20] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.,* 279 (2003), 372–379.

[21] J. G. O'Hara, P. Pillay and H. K. Xu, Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces, *Nonlinear Anal.,* 54 (2003), 1417–1426.

[22] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.,* 67 (1979), 274–276.

[23] O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, *J. Math. Anal. Appl.,* 194 (1991), 911–933.

[24] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.,* 125 (1997), 3641–3645.

[25] T. Suzuki, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.,* 135 (2007), 99–106.

[26] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.,* 58 (1992), 486–491.

[27] D. P. Wu, S. S. Chang and G. X. Yuan, Approximation of common fixed points for a family of finite nonexpansive mappings in Banach space, *Nonlinear Anal.,* 63 (2005), 987–999.

[28] H. K. Xu, Strong convergence of an iterative method for nonexpansive mappings and accretive operators, *J. Math. Anal. Appl.,* 314 (2006), 631–643.

[29] H. K. Xu, Strong convergence of approximating fixed point sequences for nonexpansive mappings, *Bull. Austral. Math. Soc.,* 74 (2006), 143–151.

[30] Y. Yao, A general iterative method for a finite family of nonexpansive mappings, *Nonlinear Anal.,* 66 (2007), 2676–2687.

[31] Y. Yao and M. A. Noor, On viscosity iterative methods for variational inequalities, *J. Math. Anal. Appl.,* 325 (2007), 776–787.

[32] Y. Yao and J. C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.,* 186 (2007), 1551–1558.

[33] H. Zegeye and N. Shahzad, Viscosity approximation methods for a common fixed point of a family of quasi-nonexpansive mappings, *Nonlinear Anal.,* 68 (2008), 2005–2012.

[34] L. C. Zeng, N. C. Wong and J. C. Yao, Strong convergence theorems for strictly pseudo-contractive mappings of Browder-Petryshyn type, *Taiwanese J. Math.,* 10 (2006), 837–849.

[35] L.C. Zeng and J. C. Yao, Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, *Nonlinear Anal.,* 64 (2006), 2507–2515.

[36] L. C. Zeng and J. C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwanese J. Math.,* 10 (2006), 1293–1303.

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