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Coercivity properties for order nonsmooth functionals

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Abstract

A differential coercivity result is established for a class of order nonsmooth functionals fulfilling an appropriate Palais-Smale condition. The core of this approach is an asymptotic type statement involving such functionals, obtained by means of the monotone variational principle in Turinici [An. Şt. UAIC Iaşi, 36 (1990), 329-352].

1 Introduction

Let $(X, \|, \|)$ be a (real) Banach space; and $(X^*, \|.\|)$, its *topological dual* (endowed with the usual norm). Given a (proper) functional $x \mapsto f(x)$ from X to $R \cup \{\infty\}$ we say that it is *coercive*, provided

(a01) $f(u) \to \infty$, provided $u \to \infty$ (in the sense: $\|u\| \to \infty$).

The basic framework for deducing such a property is the differential one. And, in this case, the most natural approach is a recursion to the celebrated 1964 Palais-Smale condition [21]. A typical result in this direction is the 1990 one due to Caklovic, Li and Willem [5]; it states that, whenever

(a02) f is Gateaux differentiable and lower semicontinuous (lsc)

the relation (a01) is deductible under a Palais-Smale requirement like

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(a03) each sequence (v_n) in X with $(f(v_n))$ bounded and
$f'(v_n) \to 0$ (in X^*) has a convergent (in X) subsequence.

Note that (a02) holds whenever $f \in C^1(X)$; hence, their statement includes the 1991 one due to Brezis and Nirenberg [4]. An extension of this result (under the same condition (a02)) was obtained in 1993 by Goeleven [13]. Specifically, the functional considered there is taken as $f = g + h$, where

(a04) g is Gateaux differentiable lsc and h is (proper) convex lsc;

and the Palais-Smale condition (a03) is adapted to this decomposition. Further enlargements of this contribution were given in the 2000 paper by D. Motreanu and V. V. Motreanu [17]; where

(a05) g is locally Lipschitz (hence continuous) on X,

and the Palais-Smale requirement to be used is that in Motreanu and Panagiotopoulos [19, Ch 3].

The basic instrument of all these developments is Ekeland's variational principle [12] (in short: EVP); and as such, the ambient functional f must be lsc (on X). So, we may ask of to what extent is this removable. An appropriate answer is available in a (linear) quasi-order context. Precisely, let K stand for a (closed) *convex cone* in X; and $(\leq)$, its associated *quasi-order*. Then, (a01) is still retainable under the (weaker than lsc) condition

(a06) f is $(\leq)$-lsc over (the whole of) X

(cf. Section 2) and a specific (modulo K) Palais-Smale condition involving the directional derivatives of f; see the 2002 paper by D. Motreanu, V. V. Motreanu and M. Turinici [18] for details. It is our aim in this exposition to show that further refinements of this last result are possible, via Clarke subgradient methods (discussed in Section 3). The specific tool of our investigations is an asymptotic type statement involving such functionals, given in Section 4; and the basic approach to deduce it is the monotone version of EVP obtained by Turinici [25] (cf. Section 2). Some other aspects will be discussed elsewhere.

2 Monotone variational principle

(A) Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive and transitive relation) $(\leq)$ over M; as well as a function $x \mapsto \varphi(x)$ from M to $R_+ := [0, \infty[$. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when: $w \in M$ and $z \leq w$ imply $\varphi(z) = \varphi(w)$. A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [3]:

Proposition 1. *Suppose that*

(b01) $(M, \leq)$ *is sequentially inductive:*
each ascending sequence has an upper bound (modulo $(\leq)$*)*

(b02) φ *is* $(\leq)$*-decreasing* $(x \leq y \Longrightarrow \varphi(x) \geq \varphi(y))$.

Then, for each $u \in M$ *there exists a* $(\leq, \varphi)$*-maximal* $v \in M$ *with* $u \leq v$.

Note that Codom$(\varphi) \subseteq R_+$ is not essential for the conclusion above; see Cârjă, Necula and Vrabie [6, Ch 2, Sect 2.1] for details. Moreover, as established there, Proposition 1 is reducible to the Principle of Dependent Choices (see, e.g., Wolk [27]). Finally, (cf. Zhu and Li [29]) $(R_+, \geq)$ may be substituted by a separable ordered structure $(P, \leq)$ without altering the conclusion above; see also Turinici [26].

This principle including Ekeland's [12], found some useful applications to convex and nonconvex analysis (cf. the above references). For this reason, it was the subject of many extensions; such as the ones in Altman [1], Anisiu [2] and Szaz [23]. The obtained results are interesting from a technical viewpoint. However, we must emphasize that, whenever a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder's is always possible. This raises the question of to what extent are these enlargements of Proposition 1 effective. As we shall see, the answer is *essentially* negative. This will necessitate some conventions. By a *pseudometric* over M we shall mean any map $d : M \times M \to R_+$. If, in addition, d is *reflexive* $[d(x, x) = 0, \forall x \in M]$, *triangular* $[d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in M]$ and *symmetric* $[d(x, y) = d(y, x), \forall x, y \in M]$, we say that it is a *semimetric* over M. Suppose that we fixed such an object. Call the point $z \in M$, $(\leq, d)$*-maximal*, in case: $w \in M$ and $z \leq w$ imply $d(z, w) = 0$. Note that, if (in addition) d is *sufficient* $[d(x, y) = 0$ implies $x = y]$, this property becomes: $w \in M, z \leq w \Longrightarrow z = w$ (and reads: z is *strongly* $(\leq)$*-maximal*). So, existence results involving such points may be viewed as "metrical" versions of the Zorn maximality principle. To get sufficient conditions for these, one may proceed as below. Let (x_n) be an ascending sequence in M. The d-Cauchy property for it is introduced in the usual way: $\forall \varepsilon > 0, \exists n(\varepsilon)$ such that $n(\varepsilon) \leq p \leq q \Longrightarrow d(x_p, x_q) \leq \varepsilon$. Also, call this sequence d*-asymptotic*, when $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$. Clearly, each (ascending) d-Cauchy sequence is d-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent:

(b03) each ascending sequence is d-Cauchy

(b04) each ascending sequence is d-asymptotic.

By definition, either of these will be referred to as $(M, \le)$ is *regular* (modulo d). Moreover, this property implies its relaxed version

(b05) $(M, \le)$ is weakly regular (modulo d): $\forall x \in M, \forall \varepsilon > 0$,
$\exists y = y(x, \varepsilon) \ge x$ such that $y \le u \le v \Longrightarrow d(u, v) \le \varepsilon$.

The following ordering principle is available (cf. Kang and Park [15]):

Proposition 2. *Assume that (b01) and (b05) are true. Then, for each $u \in M$ there exists a $(\le, d)$-maximal $v \in M$ with $u \le v$.*

As a direct consequence of this, we have (cf. Turinici [24]):

Proposition 3. *Assume that $(M, \le)$ is sequentially inductive and regular (modulo d). Then, the conclusion of Proposition 2 is retainable.*

Now (see the above reference) Prop 1 $\Longrightarrow$ Prop 2. On the other hand, Prop 2 $\Longrightarrow$ Prop 3 in a trivial way. Finally, Prop 3 $\Longrightarrow$ Prop 1; just take

$$d(x, y) = |\varphi(x) - \varphi(y)|, \quad x, y \in M \qquad \text{(where } \varphi \text{ is the above one).}$$

Summing up, all these variants of the Brezis-Browder ordering principle (Proposition 1) are nothing but logical equivalents of it.

(B) A basic application of these facts is to "monotone" variational principles. Let M be a nonempty set. Take a quasi-order $(\le)$ and a metric $d : M \times M \to R_+$ over it; the resulting triple will be termed a *quasi-ordered metric space*. Call the subset Z of M, $(\le)$-*closed* when the limit of each ascending (modulo $(\le)$) sequence in Z belongs to Z. Clearly, any closed part of M is $(\le)$-closed too; but the converse is not in general true. (Just take $M = R$ (endowed with the usual order/metric); and choose $Z =]0, 1]$). Further, call the quasi-order $(\le)$, *self-closed* provided $M(x, \le) := \{u \in M; x \le u\}$ is $(\le)$-closed, for each $x \in M$; or, equivalently: the limit of each ascending sequence is an upper bound of it (modulo $(\le)$). Finally, call the ambient metric d, $(\le)$-*complete* provided each ascending (modulo $(\le)$) d-Cauchy sequence converges. As before, if d is complete, then it is $(\le)$-complete too. The reciprocal is not in general true; take $M =]0, 1]$ endowed with the standard order/metric.

We are now in position to state the announced result. Take a function $\varphi : M \to R \cup \{\infty\}$ fulfilling

(b06) φ is inf-proper ($\mathrm{Dom}(\varphi) \ne \emptyset$ and $\varphi_* := \inf[\varphi(M)] > -\infty$)

(b07) φ is $(\le)$-lsc over M:
$[\varphi \le t] := \{x \in X; \varphi(x) \le t\}$ is $(\le)$-closed, $\forall t \in R$.

Proposition 4. *Let* $(\leq)$ *be self-closed and* d *be* $(\leq)$*-complete. Then, for each* $u \in \mathrm{Dom}(\varphi)$*, there exists* $v \in \mathrm{Dom}(\varphi)$ *with*

$$u \leq v, d(u,v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v)\text{)} \tag{2.1}$$

$$d(v,x) > \varphi(v) - \varphi(x), \text{ for each } x \in M(v, \leq) \setminus \{v\}. \tag{2.2}$$

The original argument is that appearing in Turinici [25]. For the sake of completeness, we shall provide it, with some modifications.

Proof. **(of Proposition 4)** Denote $M[u] = \{x \in M; u \leq x, \varphi(u) \geq \varphi(x)\}$. Clearly, $\emptyset \neq M[u] \subseteq \mathrm{Dom}(\varphi)$; moreover, by (b07) (and the choice of $(\leq)$)

$$M[u] \text{ is } (\leq)\text{-closed; hence } d \text{ is } (\leq)\text{-complete on } M[u]. \tag{2.3}$$

Let $(\preceq)$ stand for the relation (over M): $x \preceq y$ iff $x \leq y,\ d(x,y)+\varphi(y) \leq \varphi(x)$. It is not hard to see that $(\preceq)$ acts as and *order* (antisymmetric quasi-order) on $\mathrm{Dom}(\varphi)$; so, it remains as such on $M[u]$. We claim that conditions of Proposition 3 are fulfilled on $(M[u], d; \preceq)$. In fact, let (x_n) be an ascending (modulo $(\preceq)$) sequence in $M[u]$:

(b08) $x_n \leq x_m$ and $d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, if $n \leq m$.

The sequence $(\varphi(x_n))$ is descending and (by (b06)) bounded from below; hence a Cauchy one. This, along with (b08), shows that (x_n) is an ascending (modulo $(\preceq)$) d-Cauchy sequence; wherefrom $(M[u], \preceq)$ is regular (modulo d). Moreover, the obtained properties give us (by (2.3)) some $y \in M[u]$ with $x_n \to y$. Again with (b08) one derives (via (b07) and the choice of $(\leq)$)

$$x_n \leq y, d(x_n, y) \leq \varphi(x_n) - \varphi(y), \text{ (i.e.: } x_n \preceq y\text{), for all } n.$$

In other words, $y \in M[u]$ is an upper bound (modulo $(\preceq)$) of (x_n); and this shows that $(M[u], \preceq)$ is sequentially inductive. By Proposition 3 it then follows that, for the starting $u \in M[u]$ there exists $v \in M[u]$ with **j)** $u \preceq v$ and **jj)** v is $(\preceq, d)$-maximal in $M[u]$. The former of these is just (2.1). And the latter one gives at once (2.2); because it reads: $x \in M[u]$ and $v \preceq x$ imply $v = x$. □

A basic particular case of this corresponds to the choice $(\leq) = M \times M$ (=the trivial quasi-order on M). Then, (b07) may be written as

(b09) φ is lsc over M: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \to x$;

and Proposition 4 is nothing but Ekeland's variational principle [12] (EVP). On the other hand, (b07) also holds under (b02) and the self-closeness of $(\leq)$. For this reason, Proposition 4 will be called the *monotone* version of EVP; note that, by the remarks above, it may be derived from Proposition 1 as well. Further aspects may be found in Hyers, Isac and Rassias [14, Ch 5].

3 Conical subgradients

Let $(X, \|, \|)$ be a (real) normed space. As usually, the metric over X is the one induced by the norm: $d(x, y) = \|x - y\|, x, y \in X$. (Hence, in particular, d is invariant to translations). Likewise, the quasi-order setting to be used is linear. Precisely, let K be a *convex cone* of X $[\alpha K + \beta K \subseteq K, \forall \alpha, \beta \geq 0]$; supposed to be *non-degenerate* ($K \neq \{0\}$). Denote by $(\leq_K)$ its *induced quasi-order* [$x \leq_K y$ if and only if $y - x \in K$]; when no confusion can arise, we simply write it as $(\leq)$. Further, take some function $f : X \to R$. Put, for each $z \in X$,

$$\mathcal{Q}f(z)(h; t) = (1/t)(f(z + th) - f(z)), \quad h \in X, \ t > 0.$$

This is a map from $X \times R^0_+$ to R; referred to as the *incremental quotient* of f at z in the direction h. For practical reasons, the "limit" operators associated to the above one are of interest. Denote, again for each $z \in X$,

(c01) $Df(z)(h) = \limsup_{t \to 0+} \mathcal{Q}f(z)(h; t), \quad h \in X;$

this will be referred to as the (generalized) *directional derivative* of f at z in the direction h. Another construction of this type is the one given as

(c02) $Cf(z)(h) = \limsup_{\substack{y \to z \\ t \to 0+}} \mathcal{Q}f(y)(h; t), \quad h \in X;$

we shall term it, the (generalized) Clarke *directional derivative* for f at z in the direction h (cf. Clarke [7]). These objects exist, as elements of $R \cup \{\pm\infty\}$; and

$$Df(z)(h) \leq Cf(z)(h), \quad \text{for all } h \in X. \tag{3.1}$$

The infinite values of both operators are excluded in case

(c03) f is locally Lipschitz: $\forall z \in X$, $\exists \rho = \rho(z) > 0$, $\exists L = L(z) \geq 0$ such that $|f(u) - f(v)| \leq L||u - v||$, when $u, v \in X(z, \rho)$.

[Here, $X(z, \rho) = \{x \in X; d(z, x) < \rho\}$ is the *open sphere* centered at z, with radius ρ]. It results from this that the (generalized) directional derivative is a *finer* tool than the (generalized) Clarke directional derivative. But, from a practical perspective, the situation is a bit reversed; as results from

Lemma 1. *The following are valid (for all $z \in X$):*

i) $Cf(z)$ is positively homogeneous and subadditive (i.e.: sublinear):

$$Cf(z)(\lambda h) = \lambda Cf(z)(h), \ \forall \lambda \geq 0, \forall h \in X \tag{3.2}$$

$$Cf(z)(h + k) \leq Cf(z)(h) + Cf(z)(k), \ \forall h, k \in X \tag{3.3}$$

ii) $Df(z)$ is positively homogeneous but not subadditive (in general).

Proof. It will suffice verifying the subadditivity. For $y \in X$, $t > 0$, we have

$$\mathfrak{Q}f(y)(h+k;t) = \mathfrak{Q}f(u)(k;t) + \mathfrak{Q}f(y)(h;t), \text{ where } u = y + th.$$

Let $\gamma, \delta > 0$ be arbitrary fixed; and $\varepsilon > 0$ be taken as $\varepsilon < \min\{\gamma/(1+\|h\|), \delta\}$. By the preceding relation

$$Cf(z)(h+k) \leq \sup_{\substack{\|y-z\|<\varepsilon \\ 0<t<\varepsilon}} \mathfrak{Q}f(y)(h+k;t) \leq$$
$$\sup_{\substack{\|u-z\|<\gamma \\ 0<t<\gamma}} \mathfrak{Q}f(u)(k;t) + \sup_{\substack{\|v-z\|<\delta \\ 0<s<\delta}} \mathfrak{Q}f(v)(h;s), \quad \forall \gamma, \delta > 0.$$

Passing to infimum over $\gamma > 0$ and $\delta > 0$ yields the desired conclusion. □

As a consequence of this, it would be possible that the (extended) functional $Cf(z)$ be supported over K by a linear continuous functional $x^* \in X^*$

(c04) $x^*(h) \leq Cf(z)(h)$, for all $h \in K$;

referred to as a Clarke K-*subgradient* of f at z; the class of all these will be denoted $\partial_K f(z)$. Note that (c04) is equivalent with $Cf(z)$ being supported over K by continuous superlinear functionals $\psi : X \to R$:

(c05) $\psi(h) \leq Cf(z)(h)$, for all $h \in K$.

(Here, superlinear means: $(-\psi)$ is sublinear). Precisely, one has:

Lemma 2. *Let $\psi : X \to R$ be a continuous superlinear functional satisfying (c05). There exists then some $x^* \in \partial_K f(z)$ with*

$$\psi(x) \leq x^*(x) \leq -\psi(-x), \quad \textit{for all } x \in X. \tag{3.4}$$

Proof. Let the functional θ be introduced as:

$$\theta(x) = \inf\{Cf(z)(h) - \psi(x+h); h \in K\}, \quad x \in X.$$

By the imposed conditions, $Cf(z)(h) \geq \psi(h) \geq \psi(x+h) + \psi(-x), h \in K$; and this (by the definition of θ) yields

$$\psi(-x) \leq \theta(x) \leq Cf(z)(0) - \psi(x) = -\psi(x), \quad x \in X. \tag{3.5}$$

This shows that θ has finite values; moreover, by (3.2)+(3.3) and the choice of ψ, the functional θ is sublinear. By the standard Hahn-Banach theorem (see, e.g., Cristescu [10, Ch 1, Sect 1]) there must be a linear functional y^* with $y^*(x) \leq \theta(x)$, for all $x \in X$. Combining with (3.5) yields, on the one hand $y^*(x) \leq -\psi(x), x \in X$; wherefrom $y^* \in X^*$ (by the continuity of ψ). On the other hand, $\theta(-h) \leq Cf(z)(h)$ [hence $-y^*(h) \leq Cf(z)(h)$], for each $h \in K$. It suffices now putting $x^* = -y^*$ to get the desired conclusion. □

Now, a natural question is to indicate some concrete circumstances under which the premises of Lemma 2 be fulfilled. These are deductible from the variational principles in Section 2; and correspond to the choice $\psi(.) = -\eta\|.\|$ (for some $\eta > 0$). The main advantage of this approach is the possibility of expressing these relations by means of some (extended) positive indicators. This will necessitate some conventions. Let $h \mapsto \Lambda(h)$ stand for a positively homogeneous map from X to $R \cup \{\pm\infty\}$. Denote

$$S_K(\Lambda) = \inf\{\Lambda(h/\|h\|); h \in K \setminus \{0\}\}, \quad S_K^{(+)}(\Lambda) = \max\{-S_K(\Lambda); 0\}.$$

The extended positive number $S_K^{(+)}(\Lambda)$ has the minimal property

$$\eta \geq 0, (\Lambda(h) \geq -\eta\|h\|, \forall h \in K) \Longrightarrow S_K^{(+)}(\Lambda) \leq \eta. \tag{3.6}$$

Further, for each subset P^* of X^*, put

$$\Theta(P^*) = \inf\{\|x^*\|; x^* \in P^*\}, \text{ if } P^* \neq \emptyset; \ \ \Theta(\emptyset) = \infty.$$

Now, let M be some part of X with nonempty interior.

Lemma 3. *Assume that the point $z \in \mathrm{int}(M)$ fulfills (for some $\eta > 0$)*

(c06) $\eta\|z - x\| > f(z) - f(x)$, for all $x \in M(z, \leq) \setminus \{z\}$.

Then, we necessarily have the conclusions

$$Cf(z)(h) \geq Df(z)(h) \geq -\eta\|h\|, \quad \text{for all } h \in K \tag{3.7}$$

$$\|x^*\| \leq \eta, \quad \text{for at least one } x^* \in \partial_K f(z); \tag{3.8}$$

wherefrom (by the above conventions)

$$S_K^{(+)}(Cf(z)) \leq S_K^{(+)}(Df(z)) \leq \eta; \quad S_K^{(+)}(Cf(z)) \leq \Theta(\partial_K f(z)) \leq \eta. \tag{3.9}$$

Proof. By the admitted hypotheses one gets, for each $h \in K$,

$$\mathfrak{Q}f(z)(h;t) \geq -\eta\|h\|, \text{ whenever } t > 0 \text{ fulfills } z + th \in M.$$

This, by a limit process yields (3.7), if one takes (3.1) into account. Moreover, by Lemma 2, it is also clear that (3.8) holds. The first half of (3.9) is a simple consequence of (3.7) and the definition of the indicator $S_K^{(+)}(.)$. And, for the second one, it will suffice noting that, for each $x^* \in \partial_K f(z)$,

$$Cf(z)(h) \geq x^*(h) \geq -\|x^*\|\|h\|, \forall h \in K \text{ (hence } S_K^{(+)}(Cf(z)) \leq \|x^*\|).$$

This completes the argument. □

The last conclusion in the statement tells us that the (local) indicators $S_K^{(+)}(Df(z))$ and $\Theta(\partial_K f(z))$ give, practically, the same amount of information about the variational point z. However, for technical reasons, the latter of these is more appropriate in applications. In particular, when $K = X$, the concept of Clarke K-subgradient is just the traditional one; cf. Rockafellar [22]. Further aspects may be found in Lebourg [16].

4 Asymptotic minimum properties

With this information, we may now return to the questions of Section 1. Let $(X, ||.||)$ be a real normed space; and K, some (non-degenerated) convex cone of it. Denote by $(\leq)$ its associated quasi-order; and let d stand for the metric over X induced by $||.||$. For an easy reference, we list our basic hypotheses. These will be started with

(d01) K is $(\leq)$-closed and d is $(\leq)$-complete.

Further, take some map $\Gamma : X \to R_+$ with the properties

(d02) Γ is almost (λ, μ)-Lipschitz ($||x - y|| \leq \lambda \Longrightarrow |\Gamma(x) - \Gamma(y)| \leq \mu$) for certain $\lambda, \mu > 0$ with $\lambda \leq 1 \leq \mu$

(d03) $\Gamma(X)$ has arbitrarily large points: $\sup[\Gamma(X)] = \infty$.

A useful consequence of these refers to the level sets $[\Gamma \geq \sigma]$, $\sigma \geq 0$; precisely,

$$\mathrm{cl}([\Gamma \geq \rho]) \subseteq X([\Gamma \geq \rho], \lambda) \subseteq [\Gamma \geq \rho - \mu],\ \forall \rho \geq \mu. \tag{4.1}$$

Here, "cl" denotes the closure operator; and $X(A, \lambda) = \{x \in X; \mathrm{dist}(x, A) < \lambda\}$, $A \subseteq X$; where "dist" is the (metrical) point to set distance. In fact, let $v \in X([\Gamma \geq \rho], \lambda)$ be arbitrary fixed. By definition, there must be $u \in [\Gamma \geq \rho]$ with $d(u, v) < \lambda$; hence $|\Gamma(u) - \Gamma(v)| \leq \mu$ (if we take (d02) into account). But then, $\Gamma(v) \geq \Gamma(u) - \mu \geq \rho - \mu$ (i.e.: $v \in [\Gamma \geq \rho - \mu]$); and the claim follows. Finally, pick some functional $F : X \to R \cup \{\infty\}$ with (cf. Section 2)

(d04) F is inf-proper and $(\leq)$-lsc over all of X.

By the remark we just made, $m(\Gamma, F)(\sigma) := \inf[F([\Gamma \geq \sigma])]$ is finite, for each $\sigma \geq 0$. Moreover, $\sigma \mapsto m(\Gamma, F)(\sigma)$ is increasing on $R_+^0 :=]0, \infty[$; wherefrom

$$\liminf_{\Gamma(u) \to \infty} F(u) := \sup_{\sigma > 0} m(\Gamma, F)(\sigma)\ [= \lim_{\sigma \to \infty} m(\Gamma, F)(\sigma)]$$

exists, as an element of $R \cup \{\infty\}$, in view of

$$F_* \leq m(\Gamma, F)(\sigma) \leq \alpha(\Gamma, F) := \liminf_{\Gamma(u) \to \infty} F(u) \leq \infty, \quad \forall \sigma > 0. \tag{4.2}$$

(Here, as usually, $F_* := \inf[F(X)]$). When $\alpha(\Gamma, F) = \infty$, the functional F will be referred to as Γ-*coercive*. It is our aim in the following to get sufficient conditions in order that such a property be attained. These will be phrased in terms of the differential objects introduced in Section 3 above. The following asymptotic type statement is a basic step to the answer we are looking for.

Theorem 1. *Suppose that*

(d05) $\alpha(\Gamma, F) < \infty$ *(hence (cf. (4.2))* $\alpha(\Gamma, F)$ *is finite).*

There exists then a sequence (v_n) *in* $\Gamma^{-1}(R^0_+)$ *with*

$$\Gamma(v_n) \to \infty \text{ (hence } \Gamma(y_n) \to \infty\text{, for each subsequence } (y_n) \text{ of } (v_n)) \quad (4.3)$$

$$F(v_n) \to \alpha(\Gamma, F) \text{ and } S_K^{(+)}(DF(v_n)) \to 0,\ \Theta(\partial_K F(v_n)) \to 0. \quad (4.4)$$

Proof. **(I)** Let the parameter η be taken according to

(d06) $0 < \eta < \dfrac{\lambda}{2\mu}$; hence (cf. (d02)) $\dfrac{1}{\eta} > \mu > \dfrac{\lambda}{2} > \eta$.

By (4.2), there exists $r(\eta)$ with

$$r(\eta) \geq 1/\eta; \text{ and } m(\Gamma, F)(r) > \alpha(\Gamma, F) - \eta^2,\ \forall r \geq r(\eta). \quad (4.5)$$

Having these precise, we claim that there exists $v_\eta \in X$ so that

$$\begin{array}{l}\Gamma(v_\eta) \geq r(\eta),\ |F(v_\eta) - \alpha(\Gamma, F)| < \eta^2; \text{ as well as} \\ S_K^{(+)}(DF(v_\eta)) \leq \eta,\ \ \Theta(\partial_K F(v_\eta)) \leq \eta.\end{array} \quad (4.6)$$

In fact, by (4.5), $\alpha(\Gamma, F) - \eta^2 < m(\Gamma, F)(4r(\eta)) < \alpha(\Gamma, F) + \eta^2$; wherefrom $F(u_\eta) < \alpha(\Gamma, F) + \eta^2$, for some $u_\eta \in [\Gamma \geq 4r(\eta)]$. From (d01)+(d04), Proposition 4 is applicable to $[M = \mathrm{cl}[\Gamma \geq 2r(\eta)]$; $(d, \leq)$=as before; $\varphi = (1/\eta)F]$. So, for the starting point $u_\eta \in M$ there must be another one $v_\eta \in M$ with

$$u_\eta \leq v_\eta,\ \eta d(u_\eta, v_\eta) \leq F(u_\eta) - F(v_\eta) \text{ (hence } F(u_\eta) \geq F(v_\eta)) \quad (4.7)$$

$$\eta d(v_\eta, x) > F(v_\eta) - F(x), \text{ for all } x \in M(v_\eta, \leq) \setminus \{v_\eta\}. \quad (4.8)$$

We claim that v_η is our desired point for (4.6). In fact, (4.1) gives

$$v_\eta \in [\Gamma \geq 2r(\eta) - \mu] \subseteq [\Gamma \geq r(\eta)] \quad (4.9)$$

if one takes (d06) and (4.5) into account; wherefrom, the first part of (4.6) is clear. Combining with the second half of both (4.5) and (4.7) yields $\alpha(\Gamma, F) - \eta^2 < F(v_\eta) \leq F(u_\eta) < \alpha(\Gamma, F) + \eta^2$; which tells us that the second part of

(4.6) holds too. This, again coupled with (4.7) (the second half) yields (via (d06)) $d(u_\eta, v_\eta) \leq (1/\eta)2\eta^2 = 2\eta < \lambda$; wherefrom, by (4.1),

$$v_\eta \in X(u_\eta, \lambda) \subseteq X([\Gamma \geq 4r(\eta)], \lambda) \subseteq [\Gamma \geq 4r(\eta) - \mu]; \tag{4.10}$$

which "improves" (4.9); so, again by (4.1) (and (d06)), $X(v_\eta, \lambda) \subseteq [\Gamma \geq 4r(\eta) - 2\mu] \subseteq [\Gamma \geq 2r(\eta)] \subseteq M$. Summing up, v_η is an interior point of M, fulfilling the variational conditions (4.8). This, along with Lemma 3, assures the third and fourth part of (4.6); hence the claim.

(II) Let (η_n) be a descending to zero sequence in $]0, \lambda/2\mu[$ and put $r_n = r(\eta_n)$[=the quantity of (4.5)], $n \geq 0$. Note that, by this choice, $r_n \geq 1/\eta_n$, for all n; hence $r_n \to \infty$ as $n \to \infty$. Moreover, the developments in **(I)** give us a sequence $(v_n = v_{\eta_n})$ in Dom(F) fulfilling (for each n)

$$\Gamma(v_n) \geq r_n, |F(v_n) - \alpha(\Gamma, F)| < \eta_n^2, S_K^{(+)}(DF(v_n)) \leq \eta_n, \Theta(\partial_K F(v_n)) \leq \eta_n.$$

But, from this, (4.3)+(4.4) are clear. The proof is thereby complete. □

Some remarks are in order. The portion of (4.4) involving the operator $\Lambda \mapsto S_K^{(+)}(\Lambda)$ shows that Theorem 1 refines a statement due to D. Motreanu, V. V. Motreanu and M. Turinici [18]. On the other hand, the portion of (4.4) involving the operator $P^* \mapsto \Theta(P^*)$ tells us that Theorem 1 also includes the asymptotic type statement in D. Motreanu and V. V. Motreanu [17], based on (a05). Note finally that "functional" enlargements of these are possible under the lines in Zhong [28]; further aspects will be discussed elsewhere.

5 Order coercivity result

We are now in position to give the promised answer to our coercivity question. The "hybrid" condition below is to be considered

(e01) each sequence (v_n) in $\Gamma^{-1}(R_+^0)$ for which $(F(v_n))$ converges
and $\min\{S_K^{(+)}(DF(v_n)), \Theta(\partial_K F(v_n))\} \to 0$ as $n \to \infty$
has a subsequence (y_n) with $(\Gamma(y_n))$ bounded (in R_+).

This will be referred to as a Palais-Smale condition (modulo K) upon F.

Theorem 2. *Suppose that (in addition) F satisfies a Palais-Smale condition (modulo K). Then, F is Γ-coercive.*

Proof. If, by absurd, this cannot happen, the relation (d05) must be true. By Theorem 1, we have promised a sequence (v_n) in $\Gamma^{-1}(R_+^0)$ with the properties (4.3)+(4.4). Combining with the imposed Palais-Smale condition (modulo K)

one deduces that (v_n) must have a subsequence (y_n) with $(\Gamma(y_n))$ bounded (in R_+). On the other hand, $\Gamma(y_n) \to \infty$, by (4.3). The obtained contradiction shows that (d05) cannot be accepted; hence the conclusion. $\square$

Now, evidently,

$$\min\{S_K^{(+)}(DF(v_n)), \Theta(\partial_K F(v_n))\} \to 0 \text{ when}$$
$$\text{either } S_K^{(+)}(DF(v_n)) \to 0 \text{ or } \Theta(\partial_K F(v_n)) \to 0.$$

The version of Theorem 2 with (e01) expressed via $S_K^{(+)}(DF(v_n)) \to 0$ refines (under $\Gamma = ||.||$) the related contribution due to D. Motreanu, V. V. Motreanu and M. Turinici [18]. Likewise, the version of the same with (e01) expressed in terms of $\Theta(\partial_K F(v_n)) \to 0$ includes (again under $\Gamma = ||.||$) the "amorphous" $(K = X)$ coercivity result in D. Motreanu and V. V. Motreanu [17], based on (a05). The inclusion between these is strict. This is shown in

Example 1. Take $X = R$, $K = R_+$ and consider the function

$$\begin{array}{ll} F(t) = t, & \text{if } t \geq 0 \\ F(t) = -t + n, & \text{if } -n-1 < t < -n \ (n \in N) \\ F(t) = 2n - (1/2), & \text{if } t = -n \ (n \in N^*). \end{array}$$

This function is bounded from below on K. Moreover, since F is decreasing on $R_- :=]-\infty, 0]$ and continuous on R_+, it is also $(\leq)$-lsc on R. The restriction to K of the Clarke derivative of F is expressed as (by positive homogeneity)

$$CF(z)(1) = 1, \text{ if } z \geq 0; \;\; Cf(z)(1) = -1, \text{ if } z < 0.$$

This in turn yields the expression of the Clarke K-subgradient of f:

$$\partial_K F(z) =]-\infty, 1], \text{ if } z \geq 0; \;\; \partial_K F(z) =]-\infty, -1], \text{ if } z < 0;$$

wherefrom, its associated indicator is

$$\Theta(\partial_K F(z)) = 0, \text{ if } z \geq 0; \;\; \Theta(\partial_K F(z)) = 1, \text{ if } z < 0.$$

As a consequence, F satisfies the Palais-Smale condition relative to this indicator (under $\Gamma = ||.||$). Hence, Theorem 2 is applicable to these data; wherefrom F is coercive. On the other hand, no "amorphous" technique (relative to $K = X$) can establish this; because F is not lsc on R; hence the claim.

Note finally that the same argument allows us to get "order" extensions of the coercivity statements in Costa and Silva [9]. Further aspects will be delineated elsewhere.

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