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On partitionable, confidentially connected and unbreakable graphs

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Abstract

Some problems related to security in communication networks lead to consider a new type of connectivity in graphs, namely the confidential connectivity. In this paper we present a characterization of unbreakable graphs using the notion of weak decomposition and we give some applications of minimal unbreakable graphs. In fact, we showed that a graph G is confidentially connected if and only if it does not have a star cutset. We also showed that a minimal imperfect graph does not have a star cutset. We gave a constructive proof of the fact that every (α, ω)-partitionable graph is confidentially connected, for a superclass of minimal imperfect graphs.

1 Introduction

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph ([1]), without loops and multiple edges, having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G. If $U \subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of the graph G, then x and y are adjacent, while x and e as well as y and e are incident. If $xy \in E$, we also use $x \sim y$, and $x \not\sim y$ whenever x, y are not adjacent in G. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote

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by $A \sim B$, while by $A \not\sim B$ we mean that no edge of G joins some vertex of A to a vertex of B and, in this case, we say A and B are *non-adjacent*.

The *neighborhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we denote $N(v)$ and $N[v]$, when G appears clearly from the context. The *degree* of v in G is $d_G(v) = |N_G(v)|$. The neighborhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighborhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A graph is complete if every pair of distinct vertices is adjacent. A *clique* is a subset Q of V with the property that $G(Q)$ is complete. The *clique number* of G, denoted by $\omega(G)$, is the size of the maximum clique. A clique cover is a partition of the vertex set such that each part is a clique. $\theta(G)$ is the size of the smallest possible clique cover of G; it is called the *clique cover number* of G. A stable set is a subset X of vertices where every two vertices are not adjacent. $\alpha(G)$ is the number of vertices of a stable set of maximum cardinality; it is called the *stability number* of G. $\chi(G) = \omega(\overline{G})$ and it is called the *chromatic number* of G.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A graph G is called $perfect$ if $\chi(H) = \omega(H)$ for every induced subgraph H of G, otherwise it is called $imperfect$. A graph G is called *minimally* $imperfect$ if it is not perfect, but all its proper subgraphs are perfect.

A graph G is *partitionable* if there exist the integers α and ω greater than one such that G has exactly $\alpha\omega+1$ vertices and, for each vertex $v \in V$, $G - v$ can be partitioned into both α cliques of size ω and ω stable sets of size α.

A graph G is called (α, ω)-partitionable if for every $v \in V(G)$, $G-v$ admits a partition in α ω-cliques and a partition in ω α-stable sets.

An edge uv of the graph G is called a *wing* if, for some vertices x, y, $\{u, v, x, y\}$ induces a P_4 in G. The *coercion* class C_{uv} of a wing uv is defined by the following conditions: (a) $uv \in C_{uv}$ and (b) if $xy \in C_{uv}$ and xy, $x'y'$ are wings of the same P_4 in G, then $x'y' \in C_{uv}$.

Let $\mathcal{F}$ denote a family of graphs. A graph G is called $\mathcal{F}$-*free* if none of its subgraphs are in $\mathcal{F}$.

The *Zykov sum* of the graphs G_1, G_2 is the graph $G = G_1 + G_2$ having:

$$V(G) = V(G_1) \cup V(G_2),$$
$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

When searching for recognition algorithms, it frequently appears a type of partition for the set of vertices in three classes A, B, C, which we call a *weak decomposition*, such that: A induces a connected subgraph, C is totally adjacent to B, while C and A are totally nonadjacent.

The security represents one of the most important properties in communication networks. The starting point of this paper is a particular security problem arising in message passing in distributed systems. A communication system is *confidential* if it is possible to exchange a message between every pair of nodes so that any other specified node cannot intercept this message. If between two nodes there is a direct link in the network, it is clear that they can communicate in a confidential way. However, if there is no direct link between the two vertices then the exchanged message must follow a path having in its set of internal nodes neither the specified node nor one of its neighbors. Usually, the topology of the communication network is modeled as a simple, undirected graph.

The above type of communication suggests a new type of graph connectivity (see Definition 4 in Section 3), which is interesting by itself and is closely related to some well-known concepts in the theory of perfect graphs.

The structure of the paper is the following. In Section 2 we recall a characterization of the weak components and the existence of the weak decomposition, and give an algorithm to find one. In Section 3 we present a new characterization of the unbreakable graphs. In Section 4 we present some applications of minimal unbreakable graphs.

2 Preliminary results

At first, we recall the notions of weak components and weak decompositions.

Definition 1. ([14], [16]) *A set $A \subset V(G)$ is called a weak set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weak set, maximal with respect to set inclusion, then $G(A)$ is called a weak component. For simplicity, the weak component $G(A)$ will be denoted with A.*

Definition 2. ([14], [16]) *Let $G = (V, E)$ be a connected non-complete graph. If A is a weak set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weak decomposition of G with respect to A.*

Below we recall a characterization of the weak decomposition of a graph.

The name "*weak component*" is justified by the following result.

Theorem 1. ([14], [16]) *Every connected non-complete graph $G = (V, E)$ admits a weak component A so that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 2. ([14], [16]) *Let $G = (V, E)$ be a connected non-complete graph and $A \subset V$. Then A is a weak component of G if and only if $G(A)$ is* *connected and* $N(A) \sim \overline{N}(A)$.

The next result, that follows from the Theorem 1, ensures the existence of a weak decomposition in a connected non-complete graph.

Corollary 1. *If* $G = (V, E)$ *is a connected non-complete graph, then* V *admits a weak decomposition* (A, B, C), *such that* $G(A)$ *is a weak component and* $G(V - A) = G(B) + G(C)$.

The Theorem 2 provides an $O(n + m)$ algorithm for building a weak decomposition for a non-complete connected graph.

Algorithm for the weakly decomposition of a graph ([14])
Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \not\sim R = \overline{N}(A)$.
begin
$A :=$ any set of vertices such that $A \cup N(A) \neq V$
$N := N(A)$
$R := V - A \cup N(A)$
while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) *do*
begin
$A := A \cup \{n\}$
$N := (N - \{n\}) \cup (N(n) \cap R)$
$R := R - (N(n) \cap R)$
end
end

3 A new characterization of the unbreakable graphs using the weak decomposition

In this section we recall that the class of (α, ω)-partitionable graphs is a subclass of the class of graphs without star cutset and we give a new characterization of the unbreakable graphs.

Definition 3. ([2]) *A graph* $G = (V, E)$ *is called unbreakable if it has at least three vertices and neither* G *nor* $\overline{G}$ *has a star cutset. The subset* $A \subset V$ *is called a cutset if* $G - A$ *is not connected. If, in addition, some* $v \in A$ *is adjacent to every vertex in* $A - \{v\}$, *then* A *is called a star cutset and* v *is called the center of* A.

Significant results on unbreakable graphs have been obtained in the last decade. In ([7]), R.B. Hayward proved that, in an unbreakable graph, every vertex belongs either to some C_k or to $\overline{C}_k$, where $k \geq 5$. In ([10]), S. Olariu gave some results concerning unbreakable graphs, using the notion of *coercion class*. These results generalize several previously known results about unbreakable graphs.

Definition 4. *A graph $G = (V, E)$ with at least three vertices is confidentially connected if for any three distinct vertices $v, x, y \in V$, there exists a path P_{xy} in G such that $N_G[v] \cap V(P_{xy}) \subseteq \{x, y\}$.*

In ([4], [14]), the authors proved that a graph G is confidentially connected if and only if G does not have a star cutset.

Below we remind a central result about perfect graphs.

Star Cutset Lemma. ([2]) *No minimal imperfect graph has a star cutset.*

M.W. Padberg ([11]) proved that every minimal imperfect graph is partitionable.

In ([4], [14]) we gave a constructive proof of the fact that every (α, ω)-partitionable graph is confidentially connected.

Thus, a graph G is unbreakable if and only if G and $\overline{G}$ are confidentially connected. This means that an important class of graphs, as that of (α, ω)-partitionable graphs, is included in that of confidentially connected graphs.

Using Theorem 2 we give, in Theorem 3 below, a necessary and sufficient condition for a connected and non-complete graph to be unbreakable. A similar result is stated in [3] and proved in [4] and [14], but for confidentially connected graphs.

Theorem 3. *A connected non-complete graph $G = (V, E)$ is unbreakable if and only if $\{\overline{N}_G(v) | v \in V\}$ is the family of the weak components of G, while $\{\overline{N}_{\overline{G}}(v) | v \in V\}$ is the family of the weak components of $\overline{G}$.*

Proof. Let G be unbreakable. Then, neither G nor $\overline{G}$ have a star cutset. Because G is unbreakable if and only if $\overline{G}$ is unbreakable, we can just put $\overline{G}$ as G. Therefore, it is sufficient to prove the direct implication only for G.

Claim 1. For any three distinct vertices $v, x, y \in V$, there exists a chordless path P_{xy} in G such that $N_G[v] \cap V(P_{xy}) \subseteq \{x, y\}$.
It follows immediately from the definition of a unbreakable graph.

Claim 2. $\overline{N}_G(a) \neq \emptyset$, for every $a \in V$.
To prove Claim 2 we just write that $V - \{x, y\}$ is a star cutset.

Claim 3. $N_G(\overline{N}_G(a)) = N_G(a)$, for every vertex $a \in V$.

If $\exists b \in N_G(a)$ so that $N_G(b) \cap \overline{N}_G(a) = \emptyset$, then, for $x = b$, $v = a$, $y \in \overline{N}_G(a)$, we obtain a contradiction to *Claim* 1.

Claim ***4***. *$G(\overline{N}(a))$ is connected, for every vertex $a \in V$.*
If $G(\overline{N}(a))$ would be disconnected, then, taking x, y in different connected components of $G(\overline{N}(a))$ and $v = a$, we obtain again a contradiction to *Claim* 1.

Claim ***5***. *The set of the non-neighbors of any vertex in G induces a weakly component in G .*
Since $N_G(a) = N_G(\overline{N}_G(a))$, $\overline{N}_G(\overline{N}_G(a)) = \{a\}$ and $\{a\} \sim N_G(a)$ in G, it follows that $N_G(\overline{N}_G(a)) \sim \overline{N}_G(\overline{N}_G(a))$ in G. Furthermore, as $G(\overline{N}(a))$ is connected, it follows, according to Theorem 2, that $\overline{N}_G(a)$ is a weak component of G.

Claim ***6***. *Every weak component of G has the form $\overline{N}_G(r)$.*
If for any weak component A, $|\overline{N}(A)| > 1$, then we fix a vertex $v \in \overline{N}(A)$. By Theorem 2, v is adjacent to every vertex in $N(A)$, hence $\{v\} \cup N(A)$ forms a star cutset.

The converse implication is trivial by the following argument: if, for some v, $N_G[v]$ is a star cutset, then $G[\overline{N}(v)]$ is not connected, hence $\overline{N}(v)$ is not a weak component, which is a contradiction.

Theorem 3 provides the following recognition algorithm for unbreakable graphs:

Input: A connected non-complete graph $G = (V, E)$.
Output: An answer to the question: "Is G unbreakable"?
begin
1. Generate L_G, the family of the weak components of G as follows:
   $L_G \leftarrow \emptyset$
   while $V \neq \emptyset$ *do*
      determine the weak component A with the weak decomposition algorithm
      $L \leftarrow L \cup \{A\}$
      $V \leftarrow V - A$
   Generate L'_G, the family of the weak components of $\overline{G}$
2. Determine $\overline{N}_G(v)$, $\forall v \in V$
3. *If* $\exists A \in L_G$ such that $A \neq \overline{N}_G(v)$, $\forall v \in V$
      then Return: "G is not unbreakable"
   else
      if $\exists B \in L'_G$ such as $B \neq \overline{N}_{\overline{G}}(v)$, $\forall v \in V$
         then Return: "G is not unbreakable"
         else Return: "G is unbreakable"

end

As Theorem 2 provides an $O(n+m)$-algorithm for building a weak decomposition for a connected non-complete graph, it follows that the step 1 of the above algorithm is $O(n \cdot (n+m))$. Because the steps 2 and 3 perform in smaller time, it follows that the complexity of the recognition algorithm for unbreakable graphs is $O(n \cdot (n+m))$.

4 Some applications of minimal unbreakable graphs

In this section we point out some applications of minimal unbreakable graphs in optimization problems and in chemistry.

Facility location analysis deals with the problem of finding optimal locations for one or more facilities in a given environment (see [9]). Location problems are classical optimization problems with many applications in industry and economy. The spatial location of the facilities often takes place in the context of a given transportation, communication, or transmission system.

The aim of this problem could be to determine a location that minimizes the maximum distance to any other location in the network. Another type of location problems optimizes a "minimum of a sum" criterion, which is used in determining the location for a service facility like a shopping mall, for which we try to minimize the total travel time. The following centrality indices are defined in [9]:

The eccentricity of a vertex u is $e_G(u) = max\{d(u,v) | v \in V\}$.

The radius is $r(G) = min\{e_G(u) | u \in V\}$.

The center of a graph G is $\mathcal{C}(G) = \{u \in V | r(G) = e_G(u)\}$.

We consider the second type of location problems.

Suppose that we want to place a service facility such that the total distance to all customers in the region is minimal. The problem of finding an appropriate location can be solved by computing the set of vertices with minimum total distance.

We denote the sum of the distances from a vertex u to any other vertex in a graph G=(V,E) as the total distance $s(u) = \sum_{v \in V} d(u,v)$. If the minimum total distance of G is denoted by $s(G) = min\{s(u) | u \in V\}$, the median $\mathcal{M}(G)$ of G is given by $\mathcal{M}(G) = \{u \in V | s(G) = s(u)\}$.

The Wiener index was introduced in 1947 by H. Wiener [17] and is defined as the sum of all distances between all pairs of vertices in G:

$$W(G) = \sum_{u,v \in V} d_G(u,v).$$

We point out that the theoretical framework is especially well elaborated for the Wiener index of trees (see [5]).

The Wiener index is a graph invariant which has found extensive application in chemistry (see [12]).

The distance-counting polynomial was introduced in [8] as:

$$H(G, x) = \sum_k d(G, k)x^k,$$

with $d(G, 0) = |V(G)|$ and $d(G, 1) = |E(G)|$, where $d(G, k)$ is the number of pairs of vertices lying at distance k to each other. This polynomial was called Wiener polynomial (see [6], [13]).

A finite metric space is denoted by (X, d), where X is a finite set of points and d is a metric. Take n=$|X|$. A metric can be stated through C_n^2 nonnegative numbers that give the distance between the unordered pairs of points $\{i, j\}$, which means that we obtain a matrix with n rous and n columns, where, at the intersection of rous i and columns j, the distance between i and j, with $i, j \in X$, appears. We call this matrix, the distances matrix. We have a natural correspondence between metrics and graphs. Given a graph G with n vertices with the lengths on the edges (which may be equal with 1), we can get a natural metric d_G by setting, for every $i, j \in V(G)$, the distance $d_G(i, j)$ as being the length of the shortest path between i and j in G. Conversely, given a metric space (X, d), a weighted graph $G(d)$ can be obtained, generated by the metric, in the following manner: we consider X as the set of vertices of the graph, adding edges between every pair of vertices and considering the length of the edge $\{i, j\}$ as $d(i, j)$. It is clear that the metric $d_{G(d)}$ is identical to the original metric d.

Definition 5. *A unbreakable graph $G = (V, E)$ is called minimal if none of its proper induced subgraphs is unbreakable.*

Theorem 4. ([14], [15]) *G is minimal unbreakable if and only if G is C_k or $\overline{C_k}$ for some $k \geq 5$.*

Our result concerning the center of a minimal unbreakable graph is the following.

Theorem 5. *If G=(V,E) is a minimal unbreakable graph, then the center and the median are equal to V.*

Proof. We consider the distances matrix for C_n, with n even and n odd separately, and for $\overline{C}_n$, with n even and n odd, we obtain: $e_G(v)$= $\frac{n}{2}$ for $G = C_n$, $\forall v \in V$; $e_G(v) = 2$ for $G = \overline{C_n}$, $\forall v \in V$. So, $r(G)$= $\frac{n}{2}$ for $G = C_n$ and $r(G) = 2$ for $G = \overline{C_n}$, and $\mathcal{C}(G) = V$. $s_{\overline{C}_n}(v) = n+1$, $\forall v \in V$; $s(\overline{C}_n) = n+1$; $\mathcal{M}(\overline{C}_n) = V$. $s_{C_n}(v)$=$\frac{n}{2}$ $(\frac{n}{2}+1)$, n odd, $\forall v \in V$; $s_{C_n}(v)$=$\frac{n}{2}$ $\frac{n}{2}$, n even, $\forall v \in V$; $\mathcal{M}(C_n) = V$. So, $\mathcal{M}(G) = V$.

Theorem 6. *If G=(V,E) is a minimal unbreakable graph, then the Wiener polynomial is a polynomial with degree 2 if* $G = \overline{C}_n$ *and with degree* $\frac{n}{2}$ *if* $G = C_n$.
Proof. Having the distances matrix for C_n with n even and n odd and for $\overline{C}_n$ with n even and n odd, we obtain: $H(G,x)=n+\frac{n(n-1)}{2}\, x + nx^2$ for $G = \overline{C}_n$; $H(G,x)=n\sum_{k=0}^{\frac{n}{2}}x^k$ for $G = C_n$; $W(G)=\frac{1}{2}n\ \frac{n}{2}(\frac{n}{2}+1)$ for $G = C_n$ and n odd; $W(G)=\frac{1}{2}n\ \frac{n}{2}\frac{n}{2}$ for $G = C_n$ and n even; $W(G)=\frac{1}{2}n(n+1)$ for $G = \overline{C}_n$.

5 Conclusions and future work

In this paper we study the class of (α,ω)-partitionable graphs as a subclass of graphs without star cutset and we give a new characterization of the unbreakable graphs. In the future, we intend to verify the Normal Graph Conjecture for the class of O-graphs, which is a subclass of the class of (α,ω)-partitionable graphs.

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