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THE TOTAL TORSION ELEMENT GRAPH OF A MODULE OVER A COMMUTATIVE RING

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Abstract

The total graph of a commutative ring have been introduced and studied by D. F. Anderson and A. Badawi in [1]. In a manner analogous to a commutative ring, the total torsion element graph of a module M over a commtative ring R can be defined as the undirected graph $T(\Gamma(M))$. The basic properties and possible structures of the graph $T(\Gamma(M))$ are studied. The main purpose of this paper is to extend the definition and some results given in [1] to a more general total torsion element graph case.

1 Introduction

The study of the set of torsion elements of a module over a commutative ring can often be a frustrating one. Almost immediately one runs into the ugly issue of a profound lack of algebraic structure, highlighted by (typically) a lack of closure under addition. This unfortunate lack of algebraic structure is most disturbing in such an important subset within a module over a ring. In this paper, we introduce and study the total torsion element graph of a module over a commutative ring with the tools and methods of graph theory.

The study of algebraic structures using the properties of graphs has become an exciting research topic in the recent years, leading to many fascinating results and questions. Among the most interesting graphs are the zero-divisor

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graphs, because these involve both ring theory and graph theory. By studying these graphs we can gain a broader insight into the concepts and properties that involve both graphs and rings. There are many papers on assigning a graph to a ring and the relations between them. It was Beck (see [3]) who first introduced the notion of a zero-divisor graph for commutative ring. This notion was later redefined by D.F. Anderson and P.S. Livingston in [2]. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [1, 2, 3]). Let R be a commutative ring, $\mathrm{Reg}(R)$ its set of regular elements, $Z(R)$ its set of zero-divisors, and $\mathrm{Nil}(R)$ its ideal of nilpotent elements. The total graph of R denoted by $T(\Gamma(R))$ was introduced by D.F. Anderson and A. Badawi in [1], as the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. They study the three (induced) subgraphs $\mathrm{Nil}(\Gamma(R))$, $Z(\Gamma(R))$, and $\mathrm{Reg}(\Gamma(R))$, with vertices $\mathrm{Nil}(R)$, $Z(R)$, and $\mathrm{Reg}(R)$, repectively [1].

Throughout this paper all rings are commutative with non-zero identity and all modules unitary. Let M be a module over a ring R. We use $T(M)$ to denote the set of torsion elements of M; (that is, $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$; we use $T(M)^*$ to denote the set of non-zero torsion elements of M. So, if R is an integral domain, then $T(M)$ is a submodule of M. We will use $Tof(M) = M - T(M)$ to denote the set of non-torsion elements of M. In the present paper, we introduce and investigate the total torsion element graph of M, denoted by $T(\Gamma(M))$, as the (undirected) graph with all elements of M as vertices, and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $m + n \in T(M)$ (this definition is the same as that introduced in [1]). Let $\mathrm{Tof}(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $\mathrm{Tof}(M)$, and let $\mathrm{Tor}(\Gamma(M))$ be the (induced) subgraph of $T(\Gamma(M))$ with vertices $T(M)$. This paper is motivated by the results in [1]. The study of $T(\Gamma(M))$ breaks naturally into two cases depending on whether or not $T(M)$ is a submodule of M. For every case, we complettely characterize the girths and diameters of $T(\Gamma(M))$, $Tor(\Gamma(M))$, and $Tof(\Gamma(M))$ (see Sections 2 and 3).

We begin with some notation and definitions. For a graph Γ by $E(\Gamma)$ and $V(\Gamma)$ we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. At the other extreme, we say that Γ is totally disconnected if no two vertices of Γ are adjacent. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, a) = 0$ and $d(a, b) = \infty$). The diameter of graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,m}$ a star graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (respectively, Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (respectively, Γ_2). A general reference for graph theory is [4]. Throughout this paper we shall assume unless otherwise stated, that M is a module over a commutative ring R.

2 The case when $T(M)$ is a submodule of M

In this section, we investigate some properties of the total torsion element graph of a module M over a ring R such that $T(M)$ is a submodule of M. Our starting point is the following theorem.

Theorem 2.1. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following hold:*

(1) $Tor(\Gamma(M))$ is a complete (induced) subgraph of $T(\Gamma(M))$ and $Tor(\Gamma(M))$ is disjoint from $Tof(\Gamma(M))$.

(2) If N is a submodule of M, then $T(\Gamma(N))$ is the (induced) subgraph of $T(\Gamma(M))$.

(3) If $(0:_R M) \neq \{0\}$, then $T(\Gamma(M))$ is a complete graph.

Proof. (1) follows directly from the definitions. (2) follows from the fact that $T(N) \subseteq T(M)$ and $T(N) = N \cap T(M)$. (3) Is clear. □

Theorem 2.2. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following hold:*

(1) Assume that G is an induced subgraph of $Tof(\Gamma(M))$ and let m and m' be distinct vertices of G that are connected by a path in G. Then there exists a path in G of length 2 between m and m'. In particular, if $Tof(\Gamma(M))$ is connected, then $\mathrm{diam}(Tof(\Gamma(M))) \leq 2$.

(2) Let m and m' be distinct elements of $Tof(M)$ that are connected by a path. If $m + m' \notin T(M)$ (that is, if m and m' are not adjacent), then $m - (-m) - m'$ and $m - (-m') - m'$ are paths of length 2 between m and m' in $Tof(\Gamma(M))$.

Proof. (1) It is enough to show that if m_1, m_2, m_3, and m_4 are distinct vertices of G and there is a path $m_1 - m_2 - m_3 - m_4$ from m_1 to m_4, then m_1 and m_4 are adjacent. So $m_1 + m_2, m_2 + m_3, m_3 + m_4 \in T(M)$ gives $m_1 + m_4 = (m_1 + m_2) - (m_2 + m_3) + (m_3 + m_4) \in T(M)$ since $T(M)$ is a submodule of M. Thus m_1 and m_4 are adjacent. So if $Tof(\Gamma(M))$ is connected, then $\mathrm{diam}(Tof(\Gamma(M))) \leq 2$.

(2) Since $m, m' \in Tof(M)$ and $m + m' \notin T(M)$, there exists $u \in Tof(M)$ such that $m - u - m'$ is a path of length 2 by part (1) above. Thus $m + u, u + m' \in T(M)$, and hence $m - m' = (m + u) - (u + m') \in T(M)$. Also, since $m + m' \notin T(M)$, we must have $m \neq -m$ and $m' \neq -m$. Thus $m - (-m) - m'$ and $m - (-m') - m'$ are paths of length 2 between m and m' in $Tof(\Gamma(M))$. □

Theorem 2.3. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following statements are equivalent.*

(1) $Tof(\Gamma(M))$ is connected.

(2) Either $m + m' \in T(M)$ or $m - m' \in T(M)$ for all $m, m' \in Tof(M)$.

(3) Either $m + m' \in T(M)$ or $m + 2m' \in T(M)$ for all $m, m' \in Tof(M)$. In particular, either $2m \in T(M)$ or $3m \in T(M)$ (but not both) for all $m \in Tof(M)$.

Proof. (1) $\Rightarrow$ (2) Let $m, m' \in Tof(M)$ be such that $m + m' \notin T(M)$. If $m = m'$, then $m - m' \in T(M)$. Otherwise, $m - (-m') - m'$ is a path from m to m' by Theorem 2.2 (2), and hence $m - m' \in T(M)$.

(2) $\Rightarrow$ (3) Let $m, m' \in Tof(M)$, and suppose that $m + m' \notin T(M)$. By assumption, since $(m + m') - m' = m \notin T(M)$, we conclude that $m + 2m' = (m + m') + m' \in T(M)$. In particular, if $m \in Tof(M)$, then either $2m \in T(M)$ or $3m \in T(M)$. Both $2m$ and $3m$ cannot be in $T(M)$ since then $m = 3m - 2m \in T(M)$, a contradiction.

(3) $\Rightarrow$ (1) Let $m, m' \in Tof(M)$ be distinct elements of M such that $m + m' \notin T(M)$. By hypothesis, since $T(M)$ is a submodule of M and $m + 2m' \in T(M)$, we get $2m' \notin T(M)$. Thus $3m' \in T(M)$ by hypothesis. Since $m + m' \notin T(M)$ and $3m' \in T(M)$, we conclude that $m \neq 2m'$, and hence $m - 2m' - m'$ is a path from m to m' in $Tof(\Gamma(M))$. Thus $Tof(\Gamma(M))$ is connected. □

Example 2.4. *Let $R = Z_4$ denote the ring of integers modulo 4 and let $M = Z_8$ be the ring of integers modulo 8. Then M is an R-module with the usual operations, and $T(M) = \{\bar{0}, \bar{4}\}$ is a submodule of M. An inspection will show that $1_R + 1_R \in Z(R)$, $M/T(M) = \{T(M), \bar{1} + T(M), \bar{2} + T(M), \bar{3} + T(M)\}$ and $Tof(M) = (\bar{1} + T(M)) \cup (\bar{2} + T(M)) \cup (\bar{3} + T(M))$. Moreover, since $\bar{5} + \bar{2}, \bar{5} - \bar{2} \notin T(M)$, we conclude that $Tof(\Gamma(M))$ is not connected by Theorem 2.3.*

Our next theorem gives a complete description of $T(\Gamma(M))$. We allow α and β to be infinite, then of course $\beta - 1 = (\beta - 1)/2 = \beta$.

Theorem 2.5. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M.*

(1) If $2 = 1_R + 1_R \in Z(R)$, then $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K^{α}'s.

(2) If $2 = 1_R + 1_R \notin Z(R)$, then $Tof(\Gamma(M))$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.

Proof. (1) We first note that $m + T(M) \subseteq Tof(M)$ for every $m \notin T(M)$ since $T(M)$ is a submodule of M. Now Assume that $2 \in Z(R)$ and let $m \in Tof(M)$. Since $(m+m_1)+(m+m_2) = 2m+m_1+m_2 \in Tof(M)$ for all $m_1, m_2 \in T(M)$ (note that since $(2r)m = r(2m) = 0$ for some non-zero element r of R, we get $2m \in T(M)$), we must have the coset $m + T(M)$ is a complete subgraph of $Tof(\Gamma(M))$. Now we show that distinct cosets form disjoint subgraphs of $Tof(\Gamma(M))$ since if $m+m_1$ and $m'+m_2$ are adjacent for some $m, m' \in Tof(M)$ and $m_1, m_2 \in T(M)$, then $m+m' = (m+m_1)+(m'+m_2)-(m_1+m_2) \in T(M)$, and hence $m-m' = (m+m') - 2m' \in T(M)$ since $T(M)$ is a submodule of M and $2m \in T(M)$. Therefore, $m + T(M) = m' + T(M)$, a contradiction. Thus $Tof(\Gamma(M))$ is the union of $\beta-1$ disjoint (induced) subgraphs $m+T(M)$, each of which is a K^{α}, where $\alpha = |T(M)| = |m + T(M)|$.

(2) Let $m \in Tof(M)$. Then no two distinct elements in $m + T(M)$ are adjacent. Suppose not. Let $(m+m_1)+(m+m_2) = 2m+(m_1+m_2) \in T(M)$ for $m_1, m_2 \in T(M)$. This implies $2m \in T(M)$. Then $r(2m) = (2r)m = 0$ for some non-zero element r of R. This implies $2r = 0$ since $m \notin T(M)$, so $2 \in Z(R)$, which is a contradiction. Also, the two cosets $m + T(M)$ and $-m + T(M)$ are disjoint (since $2m \notin T(M)$, and each element of $m + T(M)$ is adjacent to each element of $-m + T(M)$. Therefore, $(m + T(M)) \cup (-m + T(M))$ is a complete bipartite (induced) subgrah of $Tof(\Gamma(M))$. Moreover, if $m + x_1$ is adjacent to $m' + x_2$ for some $m, m' \in Tof(M)$ and $x_1, x_2 \in T(M)$, then $m + m' \in T(M)^*$, and hence $m + T(M) = -m' + T(M)$. Thus $Tof(\Gamma(M))$ is the union of $(\beta-1)/2$ disjoint (induced) subgraphs $(m+T(M)) \cup (-m+T(M))$, each of which is a $K^{\alpha,\alpha}$, where $\alpha = |T(M)| = |m + T(M)|$. □

Theorem 2.6. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M with $M - T(M) \neq \emptyset$. Then*

(1) $Tof(\Gamma(M))$ is complete if and only if either $|M/T(M)| = 2$ or

$$|M/T(M)| = |M| = 3$$

.

(2) $Tof(\Gamma(M))$ is connected if and only if either $|M/T(M)| = 2$ or

$$|M/T(M)| = 3$$

.

(3) $Tof(\Gamma(M))$ (and hence $Tor(\Gamma(M))$ and $T(\Gamma(M))$ is totally disconnected if and only if $T(M) = \{0\}$ and $2 \in Z(R)$.

Proof. Let $|M/T(M)| = \beta$ and $|T(M)| = \alpha$.

(1) Let $Tof(\Gamma(M))$ be complete. Then by Theorem 2.5, $Tof(\Gamma(M))$ is a single K^{α} or $K^{1,1}$. If $2 \in Z(R)$, then $\beta - 1 = 1$. Thus $\beta = 2$, and hence $|M/T(M)| = 2$. If $2 \notin Z(R)$, then $\alpha = 1$ and $(\beta - 1)/2 = 1$. Thus $T(M) = \{0\}$ and $\beta = 3$; hence $|M/T(M)| = |M| = 3$. Conversely, suppose first that $M/T(M) = \{T(M), x+T(M)\}$, where $x \notin T(M)$. Then $x+T(M) = -x+T(M)$ gives $2x \in T(M)$, and hence $(2r)x = 0$ for some non-zero element r of R; so $2r = 0$. Thus $2 \in Z(R)$. Let $m, m' \in Tof(M)$. Then $m + x, m' + x \in T(M)$ (since $m + x + T(M), m' + x + T(M) \neq x + T(M)$); so $m + m' = (m + x) + (m' + x) - 2x \in T(M)$ since $2 \in Z(R)$ and $T(M)$ is a submodule of M. Thus $Tof(\Gamma(M))$ is complete. Next, suppose that $|M/T(M)| = |M| = 3$; we show that $2 \notin Z(R)$. Suppose not. There exists a non-zero element r of R such that $r + r = 0$; so $rm + rm = 0$ for every element m of M, which is a contradiction since M is a cyclic group with order of 3. Thus $2 \notin Z(R)$, and hence $Tof(\Gamma(M)))$ is complete. Thus, every case leads to $Tof(\Gamma(M))$ is complete.

(2) Let $Tof(\Gamma(M))$ be connected. Then by Theorem 2.5, $Tof(\Gamma(M))$ is a single K^{α} or $K^{\alpha,\alpha}$. Thus by Theorem 2.5, either $\beta - 1 = 1$ if $2 \in Z(R)$ or $(\beta - 1)/2 = 1$ if $2 \notin Z(R)$; hence $\beta = 2$ or $\beta = 3$, respectively. Thus $|M/T(M)| = 2$ or $|M/T(M)| = 3$. Conversely, by part (1) above we may assume that $|M/T(M)| = 3$. We show first that $2 \notin Z(R)$. Suppose that $2 \in Z(R)$ and let $M/T(M) = \{T(M), x+T(M), y+T(M)\}$, where $x, y \notin T(M)$. Since $M/T(M)$ is a cyclic group with order of 3, we conclude that $x+y+T(M) = T(M)$; hence x and y are adjacent, a contradiction since $Tof(\Gamma(M))$ is the union of 2 disjoint (induced) subgraphs $x+T(M)$ and $y+T(M)$. Thus $2 \notin Z(R)$. By hypothesis, $M/T(M) = \{T(M), x+T(M), 2x+T(M)\}$, where $x \notin T(M)$ and $3x \in T(M)$. Let $m, m' \in Tof(M)$. Without loss of generality that we may assume that $x + T(M) \neq m + T(M)$ and $m + m' \notin T(M)$. Then $2x + T(M) = m + T(M)$. If $x+T(M) = m'+T(M)$, then $m+m'+T(M) = 3x+T(M) = T(M)$, which is a contradiction. So we may assume that $2x + T(M) = m' + T(M)$. Then $m - (m+m'-6x) - m'$ is a path in $Tof(\Gamma(M))$ since $(2m-4x)+(m'-2x) \in T(M)$ and $(m - 2x) + (2m' - 4x) \in T(M)$. Thus $Tof(\Gamma(M))$ is connected .

(3) $Tof(\Gamma(M))$ is totally disconnected if and only if it is a disjoint union of K^1's. So by Theorem 2.5, $|T(M)| = 1$ and $|M/T(M)| = 1$, and the proof is complete. □

By the proof of the Theorem 2.6, the next theorem gives a more explicit description of the diameter of $Tof(\Gamma(M))$.

Theorem 2.7. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M.*

(1) $\operatorname{diam}(Tof(\Gamma(M))) = 0$ *if and only if* $T(M) = \{0\}$ *and* $|M| = 2$.

(2) $\operatorname{diam}(Tof(\Gamma(M))) = 1$ *if and only if either* $T(M) \neq \{0\}$ *and*

$$|M/T(M)| = 2$$

or $T(M) = \{0\}$ *and* $|M| = 3$.

(3) $\operatorname{diam}(Tof(\Gamma(M))) = 2$ *if and only if* $T(M) \neq \{0\}$ *and* $|M/T(M)| = 3$.

(4) Otherwise, $\operatorname{diam}(Tof(\Gamma(M))) = \infty$.

Proposition 2.8. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then* $\operatorname{gr}(Tof(\Gamma(M))) = 3, 4$ *or* ∞. *In particular,* $\operatorname{gr}(Tof(\Gamma(M))) \leq 4$ *if* $Tof(\Gamma(M)))$ *contains a cycle.*

Proof. Let $Tof(\Gamma(M))$ contains a cycle. Then since $Tof(\Gamma(M))$ is disjoint union of either complete or complete bipartite graphs by Theorem 2.5, it must contain either a 3-cycle or a 4- cycle. Thus $\operatorname{gr}(Tof(\Gamma(M))) \leq 4$. □

Theorem 2.9. *Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M.*

(1) (a) $\operatorname{gr}(Tof(\Gamma(M))) = 3$ *if and only if* $2 \in Z(R)$ *and* $|T(M)| \geq 3$.

(b) $\operatorname{gr}(Tof(\Gamma(M))) = 4$ *if and only if* $2 \notin Z(R)$ *and* $|T(M)| \geq 2$.

(c) Otherwise, $\operatorname{gr}(Tof(\Gamma(M))) = \infty$.

(2) (a) $\operatorname{gr}T(\Gamma(M))) = 3$ *if and only if* $|T(M)| \geq 3$.

(b) $\operatorname{gr}T(\Gamma(M))) = 4$ *if and only if* $2 \notin Z(R)$ *and* $|T(M)| = 2$.

(c) Otherwise, $\operatorname{gr}T(\Gamma(M))) = \infty$.

Proof. Apply Theorem 2.5, Proposition 2.8, and Theorem 2.1 (1). □

Remark 2.10. *(1) If M is a free module over an integral domain R, then M is torsion-free; so $T(M) = \{0\}$. Also, note that $2 \in Z(R)$ if and only if* $\operatorname{char}R = 2$. *Therefore, if M is a torsion-free R-module and* $\operatorname{char}R = 2$, *then $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K^1's, and if* $\operatorname{char}R \neq 2$, *then $Tof(\Gamma(M))$ is the union of $(\beta - 1)/2$ disjoint $K^{1,1}$'s.*

(2) Let M be an R-module and N a submodule of M. Call N a pure submodule of M if $IN = N \cap IM$ for each ideal I of R. An R-module M is pure multiplication module provided for each proper pure submodule N of M, $N = IM$ for some ideal I of R. Now let M be a pure multiplication module over an integral domain R. Then either M is torsion or torsion-free (see [5, Proposition 2.3]). Then either $T(\Gamma(M)) = Tor(\Gamma(M))$ or by part (1) above,

either $Tof(\Gamma(M))$ is the union of $\beta - 1$ disjoint K^1's, or $Tof(\Gamma(M))$ is the union of $(\beta - 1)/2$ disjoint $K^{1,1}$'s.

(3) Assume that R is a principal ideal domain which is not a field and let M be an R-module of finite length. Then $(0 :_R M) \neq 0$ by [6, 7.46]; so $T(\Gamma(M)))$ is complete by Theorem 2.1 (3).

3 The case when $T(M)$ is not a submodule of M

In this section, we study the total torsion element graph of a module M over a commutative ring R such that $T(M)$ is not a submodule of M. Let $R = Z_4$ denote the ring of integers modulo 4 and let $M = Z_{12}$ be the ring of integers modulo 8. Then M is an R-module with the usual operations, and $T(M) = \{\bar{0}, \bar{4}, \bar{6}, \bar{8}\}$ is not a submodule of M. Clearly, $Tor(\Gamma(M))$ is connected with $\text{diam}(Tor(\Gamma(M))) = 2$. Moreover, since $\bar{6} + \bar{10} = \bar{4} \in T(M)$, we conclude that the subgraphs $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ of $T(\Gamma(M))$ are not disjoint.

Theorem 3.1. *Let M be a module over a commutative ring R such that $T(M)$ is not a submodule of M. Then the following hold:*

(1) $Tor(\Gamma(M))$ is connected with $\text{diam}(Tor(\Gamma(M))) = 2$.

(2) Some vertex of $Tor(\Gamma(M))$ is adjacent to a vertex of $Tof(\Gamma(M))$. In particular, the subgraphs $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$ of $T(\Gamma(M))$ are not disjoint.

(3) If $Tof(\Gamma(M))$ is connected, then $T(\Gamma(M))$ is connected.

Proof. (1) Let $m \in T(M)^*$. Then m is adjacent to 0. Thus $m - 0 - n$ is a path in $Tor(\Gamma(M))$ of length two between any two distinct $m, n \in T(M)^*$. Moreover, there exist nonadjacent $m, n \in T(M)^*$ since $T(M)$ is not a submodule of M; thus $\text{diam}(Tor(\Gamma(M))) = 2$.

(2) By assumption, there exist distinct $m, n \in T(M)^*$ such that $m + n \notin T(M)^*$; so $m + n \in Tof(M)$. Then $-m \in T(M)$ and $m + n \in Tof(M)$ are adjacent vertices in $T(\Gamma(M))$ since $-m + (m + n) = n \in T(M)$. Finally, the "'in particular"' statement is clear.

(3) By part (1) above, it suffices to show that there is a path from m to n in $T(\Gamma(M))$ for any $m \in T(M)$ and $n \in Tof(M)$. By part (2) above, there exist adjacent vertices u and v in $Tor(\Gamma(M))$ and $Tof(\Gamma(M))$, respectively. Since $Tor(\Gamma(M))$ is connected, there is a path from m to u in $Tor(\Gamma(M))$; and since $Tof(\Gamma(M))$ is connected, there is a path from v to n in $Tof(\Gamma(M))$. Then there is a path from m to n in $T(\Gamma(M))$ since u and v are adjacent in $T(\Gamma(M))$. Thus $T(\Gamma(M))$ is connected. □

Theorem 3.2. *Let M be a module over a commutative ring R such that $T(M)$ is not a submodule of M. Then $T(\Gamma(M))$ is connected if and only if $M =< T(M) >$ (that is, $M =< a_1, ..., a_k >$ for some $a_1, ..., a_k \in T(M)$).*

Proof. Suppose that $T(\Gamma(M))$ is connected, and let $m \in M$. Then there exist a path $0-m_1-...-m_n-m$ from 0 to m in $T(\Gamma(M))$. Thus $m_1, m_1+m_2, ..., m_n+m \in T(M)$. Hence $m \in< m_1, m_1+m_2, ..., m_{n-1}+m_n, m_n+m >\subseteq< T(M) >$; thus $M =< T(M) >$. Conversely, suppose that $M =< T(M) >$. We show that for each $0 \neq m \in M$, there exists a path in $T(\Gamma(M))$ from 0 to m. By assumption, there are elements $m_1, ..., m_n \in T(M)$ such that $m = m_1+...+m_n$. Set $x_0 = 0$ and $x_k = (-1)^{n+K}(m_1+...+m_k)$ for each integer k with $1 \leq k \leq n$. Then $x_k + x_{k+1} = (-1)^{n+k+1}m_{k+1} \in T(M)$ for each integer k with $0 \leq k \leq n-1$, and thus $0 - x_1 - x_2 - ... - x_{n-1} - x_n = m$ is a path from 0 to m in $T(\Gamma(M))$ of length at most m. Now let $0 \neq u, w \in M$. Then by the preceding argument, there are paths from u to 0 and 0 to w in $T(\Gamma(M)))$; hence there is a path from u to w in $T(\Gamma(M))$. Thus, $T(\Gamma(M))$ is connected. □

Theorem 3.3. *Let M be a module over a commutative ring R such that $T(M)$ is not a submodule of M, and let $M =< T(M) >$ (that is, $T(\Gamma(M))$ is connected). Let $n \geq 2$ be the least integer such that $M =< m_1, m_2, ..., m_n >$ for some $m_1, m_2, ..., m_n \in T(M)$. Then $\text{diam}(T(\Gamma(M))) \leq n$. In particular, if M is a cyclic R-module, then $\text{diam}(T(\Gamma(M))) = n$.*

Proof. Let m and m' be distinct elements in M. We show that there exists a path from m to m' in $T(\Gamma(M))$ with length at most n. By hypothesis, we can write $m = \sum_{i=1}^{n} r_i m_i$ and $m' = \sum_{i=1}^{n} s_i m_i$ for some $r_i, s_i \in R$. Define $x_0 = m$ and $x_k = (-1)^k(\sum_{i=k+1}^{n} r_i m_i + \sum_{i=1}^{k} s_i m_i)$, so $x_k+x_{k+1} = (-1)^k m_{k+1}(r_{k+1} - s_{k+1}) \in T(M)$ for each integer k with $1 \leq k \leq n-1$. If we define $x_n = m'$, then $m - x_1 - x_2 - ... - x_{n-1} - m'$ is a path from m to m' in $T(\Gamma(M))$ with length at most n.

Finally, assume that $M =< m >$. Let $0-y_1-y_2-...-y_{m-1}-m$ be a path from 0 to m in $T(\Gamma(M))$ with length m. Thus $y_1, y_1+y_2, ..., y_{m-1}+m \in T(M)$, and hence $m \in< y_1, y_1 + y_2, ..., y_{m-1} + m >\subseteq< T(M) >$. Thus $m \geq n$, as required. □

Theorem 3.4. *Let M be a module over a commutative ring R such that $T(M)$ is not a submodule of M, and let $M =< T(M) >$ (that is, $T(\Gamma(M))$ is connected). Let $n \geq 2$ be the least integer such that $M =< m_1, m_2, ..., m_n >$ for some $m_1, m_2, ..., m_n \in T(M)$.*

(1) If M is a cyclic module with generator m, then $\text{diam}(T(\Gamma(M))) = d(0, m)$.

(2) If $\text{diam}(T(\Gamma(M))) = n$ and M is a cyclic R-module with generator m, then $\text{diam}(Tof(\Gamma(M))) \geq n-2$.

Proof. (1) This follows from Theorem 3.3.

(2) Note that $\text{diam}(T(\Gamma(M))) = d(0, m) = n$ by part (1) above. Let $0 - m_1 - ... - m_{n-1} - m$ be a shortest path from 0 to m in $T(\Gamma(M))$. Clearly,

$m_1 \in T(M)$. If $m_i \in T(M)$ for some i with $2 \le i \le n-1$, then we can construct the path $0 - m_i - ... - m_{n-1} - m$ from 0 to m in $T(\Gamma(M))$ which has length less than n, which is a contradiction. Thus $m_i \in Tof(M)$ for each integer i with $2 \le i \le n-1$. Therefore, $m_2 - m_3 - ... - m_{n-1} - m$ is a shortest path from m_2 to m in $Tof(\Gamma(M))$, and it has length $n-2$. Thus $\mathrm{diam}(Tof(\Gamma(M))) \ge 2$. □

Theorem 3.5. *Let M be a module over a commutative ring R such that $T(M)$ is not a submodule of M.*

(1) Either $\mathrm{gr}(Tor(\Gamma(M))) = 3$ *or* $\mathrm{gr}(Tor(\Gamma(M))) = \infty$.

(2) $\mathrm{gr}(T(\Gamma(M))) = 3$ *if and only if* $\mathrm{gr}(Tor(\Gamma(M))) = 3$.

(3) If $\mathrm{gr}(T(\Gamma(M))) = 4$, *then* $\mathrm{gr}(Tor(\Gamma(M))) = \infty$.

(4) If $\mathrm{Char}(R) = 2$, $Tof(\Gamma(M))$ *contains a cycle, and* $0 \neq N(R) \subsetneqq \mathrm{Nil}(R)$, *then* $\mathrm{gr}(Tof(\Gamma(M))) = 3$, *where* $N(R) = \{x \in R : x^2 = 0\}$.

(5) If $\mathrm{Char}(R) = 2$, $\mathrm{Nil}(R) \neq 0$, *and* $Tof(\Gamma(M))$ *contains a cycle, then* $\mathrm{gr}(Tof(\Gamma(M))) \le 4$.

(6) If $\mathrm{Char}(R) \neq 2$, *then* $\mathrm{gr}(Tor(\Gamma(M))) = 3, 4$ *or* ∞.

Proof. (1) If $m + m' \in T(M)$ for some distinct $m, m' \in T(M)^*$, then $0 - m - m' - 0$ is a 3-cycle in $Tor(\Gamma(M))$; so $\mathrm{gr}(Tor(\Gamma(M))) = 3$. Otherwise, $m + m' \in Tof(M)$ for all distinct $m, m' \in T(M)$. Therefore, in this case, each $m \in T(M)^*$ is adjacent to 0, and no two distinct $m, m' \in T(M)^*$ are adjacent. Thus $Tor(\Gamma(M))$ is a star graph with center 0; hence $\mathrm{gr}(Tor(\Gamma(M))) = \infty$.

(2) It suffices to show that $\mathrm{gr}(Tor(\Gamma(M))) = 3$ when $\mathrm{gr}(T(\Gamma(M))) = 3$. If $2m \neq 0$ for some $m \in T(M)^*$, then $0 - m - (-m) - 0$ is a 3-cycle in $Tor(\Gamma(M))$. Otherwise, $2m = 0$ for all $m \in T(M)^*$. We claim that $\mathrm{Char}(R) = 2$. Since $T(M)$ is not a submodule of M, there are distinct elements m, m' of $T(M)^*$ such that $m + m' \in Tof(M)$. Then $2(m + m') = 0$; so $2 = 0$, a contradiction. Thus $\mathrm{Char}(R) = 2$. Let $m - m_1 - m_2 - m$ be a 3-cycle in $T(\Gamma(M))$. Then $t = m + m_1, s = m + m_2, m_1 + m_2 \in T(M)^*$ (clearly, $m + m_1 \neq 0$ and $m + m_2 \neq 0$). Moreover, $t + s = (m + m_1) + (m + m_2) \in T(M)^*$. Thus $0 - t - s - 0$ is a 3-cycle in $Tor(\Gamma(M))$; so $\mathrm{gr}(Tor(\Gamma(M)))) = 3$.

(3) This follows by parts (1) and (2) above.

(4) By hypothesis, there exist a non-zero element r of R and a positive integer n ($n \ge 3$) such that $r^n = 0$, but $r^{n-1} \neq 0$. Clearly, $2m = 0$ for every $m \in M$. Let $m \in Tof(M)$. Since $r^{n-1}(rm) = 0$ and $r + 1$ is a unit of R, we conclude that $rm \in T(M)$ and $m + rm \in Tof(M)$. Moreover, $rm + m \neq r^{n-1}m + m$ (otherwise, $r^2 m = 0$; so $m \in T(M)$, a contradiction). Thus $m - (rm+m) - (r^{n-1}m+m) - m$ is a 3-cycle in $Tof(\Gamma(M))$; so $\mathrm{gr}(Tof(\Gamma(M))) = 3$.

(5) Let $0 \neq r \in \mathrm{Nil}(R)$. By assumption, there is a path $s - m - t$ in $Tof(\Gamma(M))$. If s and t are adjacent vertices in $Tof(\Gamma(M))$, then we are done. So we may assume that s and t are not adjacent in $Tof(\Gamma(M))$. It is easy to

see that $rm, rs, rt \in T(M)$, and $rm+m, rs+s$ and $rt+t$ are distinct elements of $Tof(M)$ (since $1+r$ is a unit in R). Clearly, $2m=0$ for every $m \in M$. We split the proof into four cases.

Case 1. $rm+m \neq t$ and $rt+t \neq m$. If $rm+m+t \in T(M)$, then $(rm+m)-m-t-(rm+m)$ is a 3-cycle in $Tof(\Gamma(M))$. If $rt+t+m \in T(M)$, then $(rt+t)-t-m-(rt+t)$ is a 3-cycle in $Tof(\Gamma(M))$. So we may assume that $rm+m+t, rt+t+m \notin T(M)$. Then $(rm+m)-m-t-(rt+t)-(rm+m)$ is a 4-cycle in $Tof(\Gamma(M))$.

Case 2. $rm+m=t$ and $rt+t \neq m$. Since $rt+t+m = r(t+m) \in T(M)$, we must have $(rt+t)-t-m-(rt+t)$ is a 3-cycle in $Tof(\Gamma(M))$.

Case 3. $rm+m \neq t$ and $rt+t=m$. By an argument like that the Case 2, we conclude that $(rm+m)-m-t-(rm+m)$ is a 3-cycle in $Tof(\Gamma(M))$.

Case 4. $rm+m=t$ and $rt+t=m$. If $rs+s+m \in T(M)$, then $(rs+s)-s-m-(rs+s)$ is a 3-cycle in $Tof(\Gamma(M))$. If $rm+m+s \in T(M)$, then $(rm+m)-m-s-(rm+m)$ is a 3-cycle in $Tof(\Gamma(M))$. So we may assume that $rm+m+s, rs+s+m \notin T(M)$. Then $(rs+s)-s-m-(rm+m)-(rs+s)$ is a 4-cycle in $Tof(\Gamma(M))$.

(6) We may assume that $Tof(\Gamma(M))$ contains a cycle. So there is a path $m-m_1-m_2$ in $Tof(\Gamma(M))$. If m and m_2 are adjacent, then we have a 3-cycle in $Tof(\Gamma(M))$. So we may assume that $m+m_2 \notin T(M)$. It is clear that either $m+m_1 \neq 0$ or $m_1+m_2 \neq 0$ (otherwise $m=m_2$, a contradiction). Without loss of generality that we can assume that $m+m_1 \neq 0$. Then $m-m_1-(-m_1)-m-(-m)$ is a 4-cycle in $Tof(\Gamma(M))$, and the proof is complete. □

References

[1] D. F. Anderson and A. Badawi, The total graph of a commutative ring, J. Algebra, **320** (2008), 2706-2719.

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, **217** (1999), 434-447.

[3] I. Beck, Coloring of commutative rings, J. Algebra, **116** (1988), 208-226.

[4] B. Bollaboás, Graph Theory, An Introductory Course, Springer-Verlag, New York, 1979.

[5] S. Ebrahimi Atani, Indecomposable Weak multiplication modules over Dedekind domains, Demonstratio Mathematica, **41 (1)** (2008), 33-43.

[6] R.Y. Sharp, Steps in Commutative Algebra, London Mathematical Society, Student Texts Vol. 19, Cambridge University Press, Cambridge, 1990.

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